St Leonard's College

VCE Unit 4 Specialist Maths

EXAMINATION Paper 1

Practice Exam 2019

Question and Answer Booklet

STUDENT NAME: SOLUTIONS

TEACHER(S): Ms S Woolley

TIME ALLOWED: Reading time 15 minutes

Writing time 60 minutes

INSTRUCTIONS

All answers are to be written on the examination paper.
Write your answers clearly with relevant working shown.
A formula sheet is included with this examination paper.

Materials permitted
No reference materials are allowed to be used.
No calculators are allowed to be used.

STRUCTURE OF BOOKLET / MARKINGSCHEME

Exam Section	Number of questions to be answered	Total marks
A	10	40

Students are not permitted to bring mobile phones and / or any other unauthorized electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.
Unless otherwise indicated, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude $g\ \text{ms}^{-2}$, where $g = 9.8$

Question 1 (3 marks)

Find the equation of the tangent to the curve $3y^2 + 2xy = 7$ at the point (2, 1).

$$6y\frac{dy}{dx} + \underbrace{2y + 2x\frac{dy}{dx}}_{\boxed{\text{product rule}}} = 0 \quad \checkmark 1A$$

$\boxed{\text{sub } x = 2,\ y = 1 \text{ now}}$

$$6\frac{dy}{dx} + 2 + 4\frac{dy}{dx} = 0$$

$$10\frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = -\frac{1}{5} \quad \checkmark 1C$$

$\therefore y = -\frac{1}{5}x + c$ sub (2, 1)

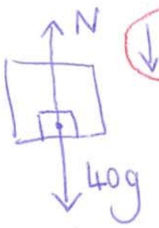
$1 = -\frac{1}{5}(2) + c$

$c = 7/5 \qquad \therefore y = -\frac{1}{5}x + \frac{7}{5} \quad \checkmark 1A$

Question 2 (4 marks)

A 40 kg trolley sits on the floor of a lift.

a. The lift accelerates downwards at the rate of 1.8 ms^{-2}. Find the reaction of the lift floor on the trolley in newtons. N — 2 marks



$\downarrow +$

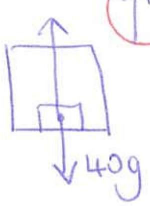
$$40g - N = 40 \times 1.8 \quad \checkmark 1M$$

I matched to accel direction

$$\therefore N = 40 \times (9.8 - 1.8)$$

$$= 320\ \text{N} \quad \checkmark 1A$$

b. The lift stops and then accelerates upwards so that the reaction of the lift floor on the trolley is 448 newtons. Find the acceleration of the lift upwards in ms^{-2}. N — 2 marks



$\uparrow +$

$$N - 40g = 40a \quad \checkmark 1M$$

$$448 - 40g = 40a$$

$$\therefore a = 11.2 - 9.8$$

$$= 1.4\ \text{ms}^{-2} \quad \checkmark 1A$$

Question 3 (4 marks)

The equation $z^3 - 3z^2 + 12z + 16 = 0$, $z \in C$, has one root given by $z = 4\text{cis}\left(\frac{\pi}{3}\right)$.

a. Find the other two roots of the equation in the form $a + bi$ where $a, b \in R$. 3 marks

$z = 4\,\text{cis}\left(\frac{\pi}{3}\right) = 4\cos\frac{\pi}{3} + 4i\sin\frac{\pi}{3}$

$= 2 + 2\sqrt{3}i$ ✓1A

By complex conjugate theorem, $2 - 2\sqrt{3}i$ also a root ✓1A

factor: $(z - 2 - 2\sqrt{3}i)(z - 2 + 2\sqrt{3}i)$

$= (z-2)^2 - (2\sqrt{3}i)^2$

$= z^2 - 4z + 4 + 12$

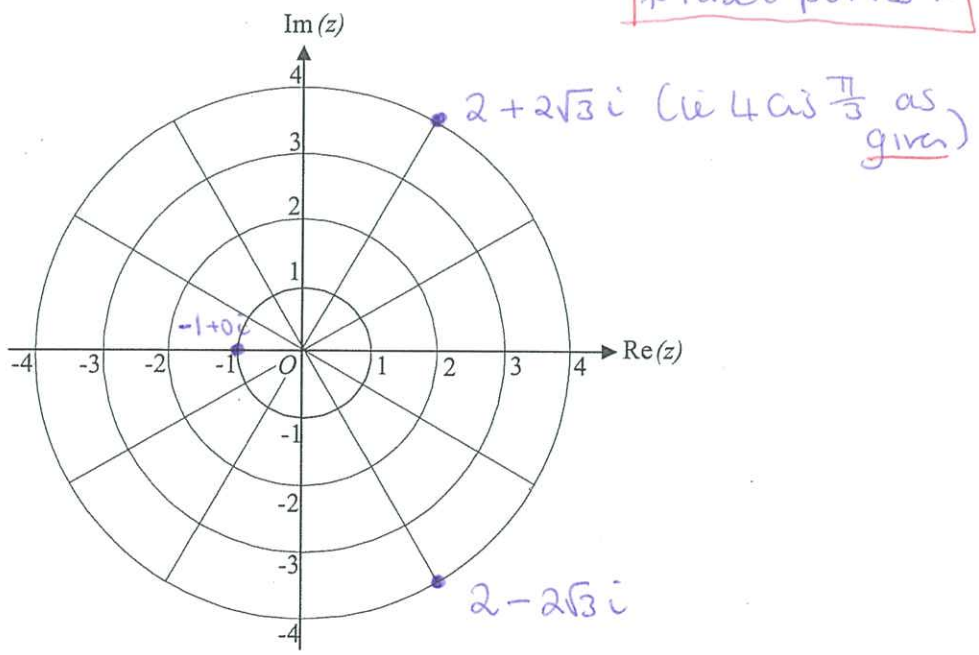
$= z^2 - 4z + 16$

$\therefore z + 1$ the other factor

$\therefore$ other 2 roots are $2 - 2\sqrt{3}i$, -1 ✓1A

b. Plot the roots of the equation on the Argand diagram below. 1 mark

$\boxed{\text{* label points *}}$



Question 4 (3 marks)

The mass, in grams, of mussels farmed in a bay, are normally distributed with a variance of 9.
The mussels are sold locally in bags of 100.
One such bag has a mass of 2400 grams.
Use this information, together with an integer multiple of the standard deviation, to calculate an approximate 95% confidence interval for the mean mass of mussels farmed in the bay.

$E(\bar{X}) = \frac{2400}{100} = 24$ ✓1A

$\sigma(\bar{X}) = \frac{3}{\sqrt{100}} = 0.3$ ✓1A

95% conf interval: $(\mu - 2\sigma_{\bar{x}}, \mu + 2\sigma_{\bar{x}})$

$= (24 - 0.6, 24 + 0.6)$

$= (23.4, 24.6)$ ✓1A

* decimals easier here *

Question 5 (4 marks)

The points M, N and P have position vectors, relative to a fixed origin, given respectively by

$$\underset{\sim}{m} = 2\underset{\sim}{i} + a\underset{\sim}{j},\quad \underset{\sim}{n} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} \text{ and } \underset{\sim}{p} = \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k},\quad \text{where } a \text{ is a real constant.}$$

The magnitude of angle MNP is $\frac{\pi}{4}$. Find the value of a. Give your answer in the form $\frac{b + c\sqrt{d}}{f}$, where b, c, d and f are integers.

$\boxed{\text{note both must point out, draw it}}$

$$\cos\frac{\pi}{4} = \frac{\overrightarrow{NM} \cdot \overrightarrow{NP}}{|\overrightarrow{NM}|\,|\overrightarrow{NP}|}$$

$\overrightarrow{NM} = \underset{\sim}{m} - \underset{\sim}{n} = \underset{\sim}{i} + (a-1)\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{NP} = \underset{\sim}{p} - \underset{\sim}{n} = -2\underset{\sim}{j} - \underset{\sim}{k}$

$$\frac{1}{\sqrt{2}} = \frac{1 \times 0 + (a-1) \times -2 + 1 \times -1}{\sqrt{2 + (a-1)^2}\ \sqrt{4+1}} \qquad \checkmark 1_M$$

$$\frac{1}{\sqrt{2}} = \frac{1 - 2a}{\sqrt{5}\sqrt{a^2 - 2a + 3}}$$

cross multiply

$$\sqrt{5}\sqrt{a^2-2a+3} = \sqrt{2}(1-2a) \qquad \checkmark 1_A$$

$\boxed{1 - 2a \geq 0 \quad \therefore a \leq \frac{1}{2}}$

square both sides

$$5(a^2 - 2a + 3) = 2(1-2a)^2$$

$$5a^2 - 10a + 15 = 2(1 - 4a + 4a^2)$$

$$5a^2 - 10a + 15 = 2 - 8a + 8a^2$$

$$\therefore 0 = 3a^2 + 2a - 13$$

$$a = \frac{-2 \pm \sqrt{2^2 - 4(3)(-13)}}{2(3)}$$

$$= \frac{-2 \pm \sqrt{160}}{6}$$

$$= \frac{-2 \pm 4\sqrt{10}}{6}$$

$$= \frac{-1 \pm 2\sqrt{10}}{3} \qquad \checkmark 1_A$$

$$\therefore a = \frac{-1 - 2\sqrt{10}}{3} \qquad \checkmark 1_A$$

Question 6 (4 marks)

Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{3+x}{x^2+3}\,dx$.

$= \displaystyle\int_0^{\sqrt{3}} \frac{3}{x^2+3}\,dx + \int_0^{\sqrt{3}} \frac{x}{x^2+3}\,dx$ ✓1M

$u = x^2+3$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$= \displaystyle\frac{3}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{\sqrt{3}}{x^2+3}\,dx + \frac{1}{2}\int_3^6 \frac{1}{u}\,du$

$= \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\Big]_0^{\sqrt{3}}$ ✓1A $+ \frac{1}{2}\log_e u\Big]_3^6$ ✓1A

$= \sqrt{3}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) + \frac{1}{2}\left(\log_e 6 - \log_e 3\right)$

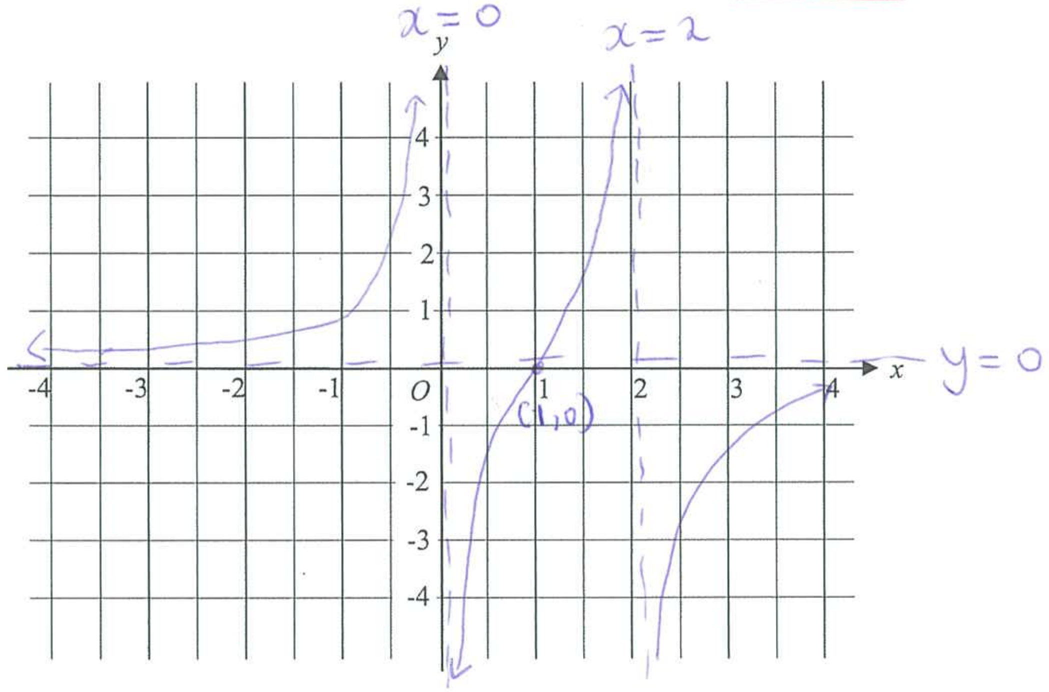
$= \frac{\sqrt{3}\pi}{4} + \frac{1}{2}\log_e 2$ ✓1A

1_A - correct asymptotes
1_A - correct x-int
1_A - middle branch, no stat pt
1_A - 2 x outer branches

Question 7 (4 marks)

Sketch the graph of $y = \dfrac{1-x}{x^2-2x}$ on the set of axes below.

Label any asymptotes with their equations and any intercepts with their coordinates.



$y = \dfrac{1-x}{x(x-2)}$ asymptotes $x=0$, $x=2$, $y=0$

x-intercept: $y=0$, $x = 1$

no y-intercepts

stat points: $\dfrac{dy}{dx} = \dfrac{-1(x^2-2x)-(1-x)(2x-2)}{(x^2-2x)^2}$

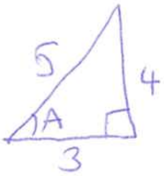
$= \dfrac{-x^2 + \cancel{2x} - \cancel{2x} + 2 + 2x^2 - 2x}{(x^2-2x)^2}$

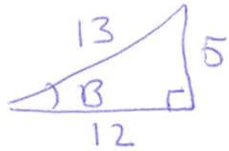
$= \dfrac{x^2-2x+2}{(x^2-2x)^2}$

Solve $= 0$, no solutions since $x^2-2x+2 \neq 0$

$\therefore$ no stat points

Question 8 (3 marks)





Find $\sec(x)$ given that $x = \arcsin\left(\frac{4}{5}\right) - \arctan\left(\frac{5}{12}\right)$.

$\sec x = \frac{1}{\cos x}$

$\cos x = \cos\left(\overset{A}{\arcsin\left(\frac{4}{5}\right)} - \overset{B}{\arctan\left(\frac{5}{12}\right)}\right)$

$= \cos A \cos B + \sin A \sin B$ ✓1M1

$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13}$ ✓1A

$= \frac{36+20}{65}$

$= \frac{56}{65}$

$\therefore \sec(x) = \frac{65}{56}$ ✓1A

Question 9 (4 marks)

Solve the differential equation $(1+x^2)\frac{dy}{dx} - \frac{1}{x} = 0$ for y, given that $x > 0$ and $y(1) = 2$.

$$(1+x^2)\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x(1+x^2)}$$

partial fractions

$$\frac{1}{x(1+x^2)} \equiv \frac{A}{\cancel{x}x} + \frac{Bx+C}{1+x^2}$$ ✓1M

$$1 \equiv A(1+x^2) + (Bx+C)x$$

let $x=0$: $1 = A$

$Cx = 0x$ $\quad \therefore C = 0$

terms in x^2: $0 = A + B$

$\therefore B = -1$

$$y = \int \frac{1}{x} - \frac{x}{1+x^2}\,dx$$ ✓1A

$$y = \log_e x - \frac{1}{2}\log_e(1+x^2) + c$$ ✓1A

$x > 0$

sub $x = 1$, $y = 2$

$$2 = \cancel{\log_e 1}^{0} - \frac{1}{2}\log_e 2 + c$$

$$c = 2 + \frac{1}{2}\log_e 2$$

$$y = \log_e x - \frac{1}{2}\log_e(1+x^2) + \frac{1}{2}\log_e 2 + 2$$ ✓1A

$$= \log_e\left(\frac{\sqrt{2}\,x}{\sqrt{1+x^2}}\right) + 2$$

Question 10 (7 marks)

Let $f(x) = \arcsin\left(\frac{x+1}{2}\right)$.

a. Find $f'(x)$. Express your answer in the form $\frac{a}{\sqrt{-(x+b)(x-a)}}$ where a and b are positive integers. 2 marks

$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x+1}{2}\right)^2}} \times \left(\frac{1}{2}\right)$ ✓1M

chain rule

derivative of $\frac{x+1}{2}$

* or from formula sheet (since deriv of $x+1 = 1$)

$f'(x) = \frac{1}{\sqrt{2^2-(x+1)^2}}$ *

$= \frac{1}{2\sqrt{1-\frac{(x+1)^2}{4}}}$

mult through by 2 ie $\sqrt{4}$ or common denominator and cancel

$= \frac{1}{\sqrt{4-(x+1)^2}}$

or DOPS

$= \frac{1}{\sqrt{4-(x^2+2x+1)}}$

$= \frac{1}{\sqrt{3-(x^2+2x-3)}} = \frac{1}{\sqrt{-(x+3)(x-1)}}$ ✓1A

b. Show that the rule of the inverse function of f, f^{-1}, is given by $f^{-1}(x) = 2\sin(x) - 1$. 1 mark

swap x and y for inverse

must write

$x = \arcsin\left(\frac{y+1}{2}\right)$

$\sin x = \frac{y+1}{2}$ ✓1M/A

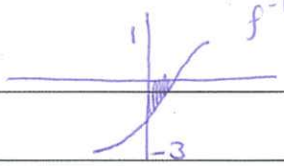
$2\sin x = y+1$

$y = 2\sin x - 1$

$\therefore f^{-1}(x) = 2\sin x - 1$ as required

must rewrite in function notation

c. Let S be the region enclosed by the graph of f^{-1} and the x and y-axes.
Find the volume of the solid of revolution that is generated when the region S is rotated about the x-axis. 4 marks



$2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ ✓1A

$\text{Vol} = \pi \int_0^{\frac{\pi}{6}} (2\sin x - 1)^2 \, dx$ ✓1A

$= \pi \int_0^{\frac{\pi}{6}} 4\sin^2 x - 4\sin x + 1 \, dx$

$\cos(2x) = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos(2x)$

$= \pi \int_0^{\frac{\pi}{6}} 2 - 2\cos 2x - 4\sin x + 1 \, dx$ ✓1M

$= \pi \left[3x - \sin(2x) + 4\cos x \right]_0^{\frac{\pi}{6}}$

$- \frac{\sqrt{3}}{2} \quad + 4 \times \frac{\sqrt{3}}{2}$

$= \pi \left[\left(\frac{3\pi}{6} - \sin\frac{\pi}{3} + 4\cos\frac{\pi}{6} \right) - (0 - 0 + 4) \right]$

$= \pi \left(\frac{\pi}{2} + \frac{3\sqrt{3}}{2} - 4 \right)$ units^3 ✓1A

END OF QUESTION AND ANSWER BOOK