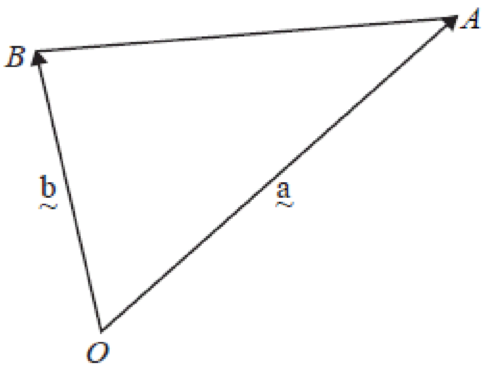
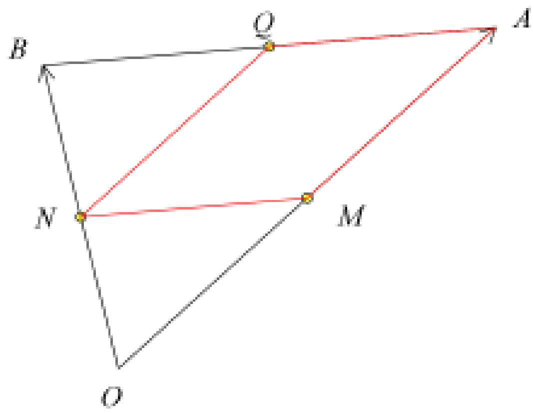
Review of VCAA 2010 Specialist Maths Exam 2 – Section 2

Question 1

The diagram below shows a triangle with vertices O, A and B. Let O be the origin, with vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.





a. Find the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

i. $\overrightarrow{MA}$, where M is the midpoint of the line segment OA

Possible solution

$\overrightarrow{MA} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\underset{\sim}{a}$ ...[A1]

Question 1ai.

Marks	0	1	Average
%	4	96	1

$\overrightarrow{MA} = \frac{1}{2}\underset{\sim}{a}$

This question was very well done. A small number of students had a sign error, quoting $\overrightarrow{MA} = -\frac{1}{2}\underset{\sim}{a}$.

ii. $\overrightarrow{BA}$

Possible solution

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \underset{\sim}{a} - \underset{\sim}{b}$ ...[A1]

Question 1aii.

Marks	0	1	Average
%	7	94	1

$\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b}$

This question was very well done, with the most common error being one of sign reversal: $\overrightarrow{BA} = \underset{\sim}{b} - \underset{\sim}{a}$.

iii. $\overrightarrow{AQ}$, where Q is the midpoint of the line segment AB.

Possible solution

$\overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{BA}$

$= -\frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b}) = \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$ ...[A1] either

Question 1aiii.

Marks	0	1	Average
%	20	80	0.8

$\overrightarrow{AQ} = \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

This question was reasonably well done. There were some sign errors and some students gave $\overrightarrow{AQ} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$.

b. Let N be the midpoint of the line segment OB. Use a vector method to prove that the quadrilateral $MNQA$ is a parallelogram.

Possible solution

Sufficient to prove that $\overrightarrow{MN} = \overrightarrow{AQ}$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$

$= \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$ ...[A1]

$= \overrightarrow{AQ}$ ...[M1]

$\Rightarrow \overrightarrow{MN} \parallel \overrightarrow{AQ}$ and $|\overrightarrow{MN}| = |\overrightarrow{AQ}|$

$\therefore MNQA$ is a parallelogram as one pair of opposite sides

are parallel and equal in length. ...[M1]

Could also prove both pairs of opposite sides are parallel, but more work.

Question 1b.

Marks	0	1	2	3	Average
%	30	8	18	44	1.8

$\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a} = \overrightarrow{AQ}$, $\overrightarrow{NQ} = \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b}) = \frac{1}{2}\underset{\sim}{a} = \overrightarrow{MA}$.

The most popular approach to this question was to show opposite sides to be parallel. A number of students attempted to show the diagonals intersected at right angles. Others attempted to show that opposite sides were equal using simplifying assumptions such as $\left|\frac{1}{2}\underset{\sim}{a}\right| = \frac{1}{2}$, believing that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ were orthogonal unit vectors. It was evident that some students did not read the question carefully enough and as a result these students worked with the wrong quadrilateral. Not all students understood clearly what they needed to show to prove that a given quadrilateral is a parallelogram.

Now consider the **particular** triangle OAB with $\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$ and $\overrightarrow{OB} = \alpha\,\underset{\sim}{i}$ where α, which is greater than zero, is chosen so that triangle OAB is isosceles, with $\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA}\right|$.

c. Show that $\alpha = 4$.

Possible solution

$\left|\overrightarrow{OA}\right| = \sqrt{9+4+3} = 4 \qquad \left|\overrightarrow{OB}\right| = \sqrt{\alpha^2} = \alpha$ as $\alpha > 0$

$\Rightarrow \alpha = 4 \qquad \ldots[\text{A1}]$

Question 1c.

Marks	0	1	Average
%	13	87	0.9

$\left|\overrightarrow{OA}\right| = \sqrt{9+4+3} = 4$ and so $\alpha = 4$.

This 'show that' question was well done; however, some students did not include enough connecting working.

d. i. Find $\overrightarrow{OQ}$, where Q is the midpoint of the line segment AB.

Possible solution

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$

$= \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} \qquad \ldots[\text{A1}]$ either here or cartesian

$= \frac{1}{2}\left(3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}\right) + \frac{1}{2}\left(4\underset{\sim}{i}\right)$

$= \frac{1}{2}\left(7\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}\right) \qquad \ldots[\text{A1}]$ either here or above

Question 1di.

Marks	0	1	Average
%	33	67	0.7

$\overrightarrow{OQ} = \frac{7}{2}\underset{\sim}{i} + \underset{\sim}{j} + \frac{\sqrt{3}}{2}\underset{\sim}{k}$ OR $\overrightarrow{OQ} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$. Both specific and general forms were allowed.

This question was moderately well done. The most common error was miscalculation of coefficients of the cartesian form of the vector.

ii. Use a vector method to show that $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$.

Possible solution

$\overrightarrow{OQ} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \neq \underset{\sim}{0} \qquad \overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} \neq \underset{\sim}{0}$

$\overrightarrow{OQ} \cdot \overrightarrow{AB} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$= \frac{1}{2}(\underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a})$ ...[M1]

$= \frac{1}{2}\left(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2\right)$

$= 0$

as $|\underset{\sim}{b}| = |\underset{\sim}{a}|$ as $\triangle OAB$ is isosceles ...[M1]

$\Rightarrow \overrightarrow{OQ} \perp \overrightarrow{AB}$ as $\overrightarrow{OQ}, \overrightarrow{AB} \neq \underset{\sim}{0}$ ...[M1]

OR using cartesian form

$\overrightarrow{OQ} = \frac{1}{2}\left(7\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}\right) \neq \underset{\sim}{0}$

$\overrightarrow{AB} = 4\underset{\sim}{i} - \left(3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}\right) = \underset{\sim}{i} - 2\underset{\sim}{j} - \sqrt{3}\underset{\sim}{k} \neq \underset{\sim}{0}$ ...[A1]

$\overrightarrow{OQ} \cdot \overrightarrow{AB} = \frac{1}{2}\left(7\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}\right) \cdot \left(\underset{\sim}{i} - 2\underset{\sim}{j} - \sqrt{3}\underset{\sim}{k}\right)$

$= \frac{1}{2}(7 - 4 - 3) = 0$ ...[M1]

$\Rightarrow \overrightarrow{OQ} \perp \overrightarrow{AB}$ as $\overrightarrow{OQ}, \overrightarrow{AB} \neq \underset{\sim}{0}$ ...[M1]

Question 1dii.

Marks	0	1	2	3	Average
%	24	19	8	49	1.8

$\overrightarrow{AB} = \underset{\sim}{i} - 2\underset{\sim}{j} - \sqrt{3}\underset{\sim}{k}$, $\overrightarrow{OQ}.\overrightarrow{AB} = \frac{7}{2} - 2 - \frac{3}{2} = 0$

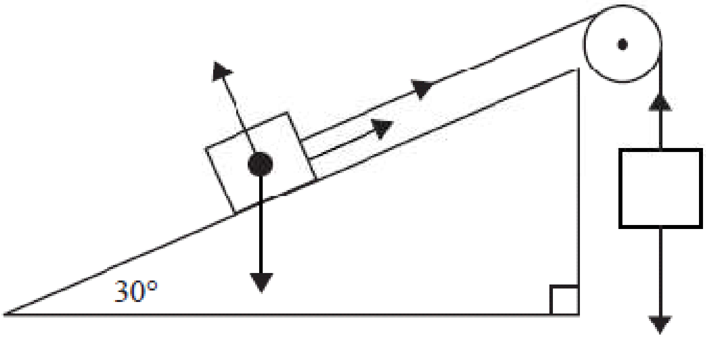
OR $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$, $\overrightarrow{OQ}.\overrightarrow{AB} = \frac{1}{2}(\underset{\sim}{b}.\underset{\sim}{b} - \underset{\sim}{a}.\underset{\sim}{a}) = \frac{1}{2}\left(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2\right) = 0$ (as triangle OAB is isosceles with $|\overrightarrow{OA}| = |\overrightarrow{OB}|$).

Both the specific and the general case solutions were accepted.

The question was reasonably well done. Students realised that they had to get a scalar product to be zero, and there were numerous instances of values being adjusted within the scalar product calculation to this end. For students working with the general case, it was common to see $|\underset{\sim}{b}| = |\underset{\sim}{a}| = 1$ used for their scalar product to give zero.

Question 2

A block of m kg sits on a rough plane which is inclined at 30° to the horizontal. The block is connected to a mass of 10 kg by a light inextensible string which passes over a frictionless light pulley.



a. If the block is on the point of moving **down the plane**, clearly label the forces which are shown in the diagram.

2 marks

Possible solution

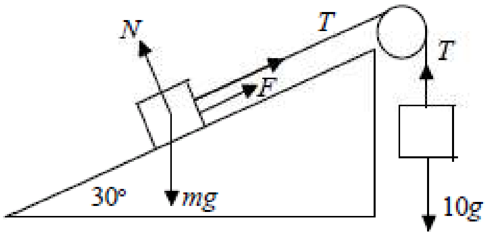
See diagram below

3 forces labelled correctly ...[A1]

Remaining 3 forces labelled correctly ...[A1]

Question 2a.

Marks	**0**	**1**	**2**	**Average**
%	5	11	84	**1.8**



This question was quite well done. Students could use their own symbols, but often the same symbol denoting the two different weight forces was used.

b. Write down, **but do not attempt to solve**, equations involving the forces acting on the block, and the forces acting on the 10 kg mass, for the situation in **part a.**

Possible solution

n.b. mg weight force makes an angle of 60° with the plane

Resolving $\perp$ to plane at m kg mass

$N - mg\sin(60°) = 0$ ...[A1]

Resolving $||$ to plane at m kg mass

$T + F - mg\cos(60°) = 0 \ldots(1)$ ...[A1]

Resolving vertically at 10 kg mass

$10g - T = 0 \quad \ldots(2)$ ...[A1]

Question 2b.

Marks	0	1	2	3	Average
%	17	9	18	56	2.2

$N - mg\cos 30^\circ = 0$, $T - 10g = 0$, $T + F - mg\sin 30^\circ = 0$

This question was reasonably well done. A popular response was to equate the third equation to ma or to simply write an expression for 'net force'. A number of students carried ma some way through the problem before realising that a was zero. Students who combined the three equations correctly into one or two equations obtained full marks.

c. Given that the coefficient of friction is $\frac{1}{4}$, show that the mass of the block in kg is given by $m = \frac{80}{4-\sqrt{3}}$.

Possible solution

Limiting equilibrium, so $F = \mu N = \frac{1}{4}N$

$N = mg\sin(60°) \qquad \Rightarrow F = \frac{1}{4}mg\sin(60°)$

$(1)+(2) \Rightarrow F + 10g - mg\cos(60°) = 0$

$\frac{1}{4}mg\sin(60°) + 10g - mg\cos(60°) = 0$ ...[M1]

$m\left(\frac{1}{4}g\sin(60°) - g\cos(60°)\right) = -10g$

$$m = \frac{-10g}{\left(\frac{1}{4}g\sin(60°) - g\cos(60°)\right)} = \frac{10}{\cos(60°) - \frac{1}{4}\sin(60°)}$$

$$= \frac{10}{\frac{1}{2} - \frac{\sqrt{3}}{8}} = \frac{10}{\frac{4-\sqrt{3}}{8}} \qquad \ldots[M1]$$

$$= \frac{80}{4-\sqrt{3}}$$

Question 2c.

Marks	0	1	2	Average
%	32	11	57	1.3

$10g + 0.25 \times mg\cos 30^\circ - mg\sin 30^\circ = 0$, $mg\left(\frac{1}{2} - 0.25 \times \frac{\sqrt{3}}{2}\right) = 10g$, $m = \frac{10}{\left(\frac{1}{2} - \frac{\sqrt{3}}{8}\right)} = \frac{80}{4-\sqrt{3}}$

Students performed reasonably well on this 'show that' question; however, some students did not show connecting steps.

In order to pull the block up the plane, the mass hanging from the string is replaced by a 20 kg mass, and a lubricant is spread on the inclined plane to decrease the coefficient of friction.

d. If the block is now on the point of moving **up the plane**, show that the new value of the coefficient of friction, correct to three decimal places, is $\mu_1 = 0.077$.

Possible solution

Don't forget to reverse the direction of the friction and to include a 20 kg mass.

Limiting equlibrium, so $F = \mu N = \mu mg\sin(60°)$

Resolve system in direction about to move.

$20g - T + T - F - mg\cos(60°) = 0$

$20g - \mu mg\sin(60°) - mg\cos(60°) = 0$ ...[M1]

$\mu mg\sin(60°) = 10g - mg\cos(60°)$

$\mu = \dfrac{20g - mg\cos(60°)}{mg\sin(60°)} = \dfrac{20 - m\cos(60°)}{m\sin(60°)}$ ...[M1]

$= \dfrac{20 - \dfrac{80}{4-\sqrt{3}} \times \dfrac{1}{2}}{\dfrac{80}{4-\sqrt{3}} \times \dfrac{\sqrt{3}}{2}} \approx 0.077$ as required

Question 2d.

Marks	0	1	2	Average
%	45	20	35	0.9

$20g - \mu mg\cos 30^{\circ} - mg\sin 30^{\circ} = 0$, $\mu = \dfrac{20g - 0.5mg}{\dfrac{mg\sqrt{3}}{2}}$ where $m = \dfrac{80}{4-\sqrt{3}}$ gives 0.077.

Common errors included failure to reverse the direction of friction and not changing the tension to $20g$.

Before the block actually starts to move up the lubricated plane where $\mu_1 = 0.077$, the string breaks and the block begins to slide down the plane.

e. Find the velocity of the block three seconds after the string breaks. Give your answer in ms^{-1} correct to one decimal place.

Possible solution

So friction is uphill again with $\mu = 0.077$ and with no tension in string.

Moving, so maximum friction, so $F = \mu N = 0.077mg\sin(60°)$

Resolve system in direction about to move i.e. downhill positive $\Rightarrow a > 0$

$mg\cos(60°) - F = ma$

$mg\cos(60°) - 0.077mg\sin(60°) = ma$

$g\cos(60°) - 0.077g\sin(60°) = a$ ...[M1]

$a \approx 4.246\ldots$ (constant) n.b. store accurate value to a

$\left.\begin{array}{l} u = 0,\ t = 3,\ a \approx 4.246\ldots, v = ? \\ v = u + at \end{array}\right\}$...[M1]

$= 0 + a \times 3$

≈ 12.7 ms^{-1} to 1 d.p. ...[A1]

Question 2e.

Marks	0	1	2	3	Average
%	39	21	4	36	**1.4**

$mg\sin 30^{\circ} - \mu mg\cos 30^{\circ} = ma$, $a = \frac{g}{2} - 0.077 \times g \times \frac{\sqrt{3}}{2}$, $V = \left(\frac{g}{2} - 0.077 \times g \times \frac{\sqrt{3}}{2}\right) \times 3 = 12.7$

Obtaining a correct expression for the acceleration proved to be difficult for many students, although most knew the correct constant acceleration formula to apply. Some students rounded off a correct acceleration to only one decimal place instead of a minimum of two, and hence did not get the velocity correct to one decimal place.

Question 3

The population of a town is initially 20 000 people. This population would increase at a rate of one per cent per year, except that there is a steady flow of people arriving at and leaving from the town. The population P after t years may be modelled by the differential equation

$$\frac{dP}{dt} = \frac{P}{100} - k \text{ with the initial condition, } P = 20\,000 \text{ when } t = 0,$$

where k is the number of people leaving per year minus the number of people arriving per year.

a. Verify by substitution that for $k = 800$,

$$P = 20\,000(4 - 3e^{0.01t})$$

satisfies both the differential equation and the initial condition.

Possible solution

Given $\frac{dP}{dt} = \frac{P}{100} - 800$

$P = 20000\left(4 - 3e^{0.01t}\right)$

$t = 0, \quad P = 20000(4-3) = 20000$ initial condition satisfied ...[A1]

$\frac{P}{100} = 200\left(4 - 3e^{0.01t}\right) = 800 - 600e^{0.01t}$

$\frac{P}{100} - 800 = -600e^{0.01t}$ = RHS of DE ...[M1]

$\frac{dP}{dt} = 20000\left(0 - 0.03e^{0.01t}\right)$

$= -600e^{0.01t}$ = LHS of DE ...[M1]

LHS of DE = RHS of DE DE satisfied

Question 3a.

Marks	0	1	2	3	Average
%	85	7	3	5	0.3

$\frac{dP}{dt} = -600e^{0.01t}$, $\frac{P}{100} - 800 = \frac{20\,000}{100} \times \left(4 - 3e^{0.01t}\right) - 800 = -600e^{0.01t}$

$P(0) = 20\,000 \times \left(4 - 3 \times e^{0}\right) = 20\,000$

Many students attempted to solve the differential equation, despite this being asked for in part c. of this question. Some students who solved instead of verifying the solution by substitution, verified the initial condition.

b. For $k = 800$, find the time taken for the population to decrease to zero. Give your answer correct to the nearest whole year.

Possible solution

Given $P = 20000\left(4 - 3e^{0.01t}\right)$

$P = 0, \quad 20000\left(4 - 3e^{0.01t}\right) = 0$ ...[M1]

$t = -100\log_e\left(\frac{3}{4}\right) = 100\log_e\left(\frac{4}{3}\right)$

$\approx 28.768\ldots$

≈ 29 years to nearest whole year ...[A1]

Question 3b.

Marks	0	1	2	Average
%	16	5	79	1.7

$0 = 20\,000 \times \left(4 - 3e^{0.01t}\right),\ t = 29$

This question was quite well done by students who attempted it. Some students did not evaluate their answer to the nearest year and gave it in exact form. Others rounded their result incorrectly.

The differential equation which models the population growth can be expressed as

$$\frac{dt}{dP}=\frac{100}{P-100k} \text{ with } P=20\,000 \text{ when } t=0.$$

c. Show by integration that for $k<0$, the solution of this differential equation is

$$P=(20\,000-100k)e^{0.01t}+100k.$$

Possible solution

$\frac{dt}{dP}=\frac{100}{P-100k}$

$\frac{dt}{dP}=\frac{100}{P-80000}$ $\quad P\neq 80000, \quad t=0,\ P=20000$

$t=\int\frac{100}{P-100k}dP \quad k<0$

$=100\log_e|P-100k|+c$

but $P-100k>0$ as $P\geq 0$ and $k<0$
$\Rightarrow t=100\log_e(P-100k)+c$ $\Big\}\ldots$[M1]

$t=0,\ P=20000$

$0=100\log_e(20000-100k)+c$

$c=-100\log_e(20000-100k)$

$t=100\log_e(P-100k)-100\log_e(20000-100k)$

$t=100\log_e\left(\frac{P-100k}{20000-100k}\right)$ $\quad\ldots$[M1]

$0.01t=\log_e\left(\frac{P-100k}{20000-100k}\right)$

$\frac{P-100k}{20000-100k}=e^{0.01t}$

$P=(20000-100k)e^{0.01t}+100k$ as required

Question 3c.

Marks	0	1	2	Average
%	23	9	68	1.5

$t=100\log_e(P-100k)+c$, $c=-100\log_e(20\,000-100k)$, $\frac{t}{100}=\log_e\left(\frac{P-100k}{20\,000-100k}\right)$

$e^{\frac{t}{100}}=\frac{P-100k}{20\,000-100k}$, from which the given result follows.

It can be shown that the solution given in **part c.** is valid for all real values of k.

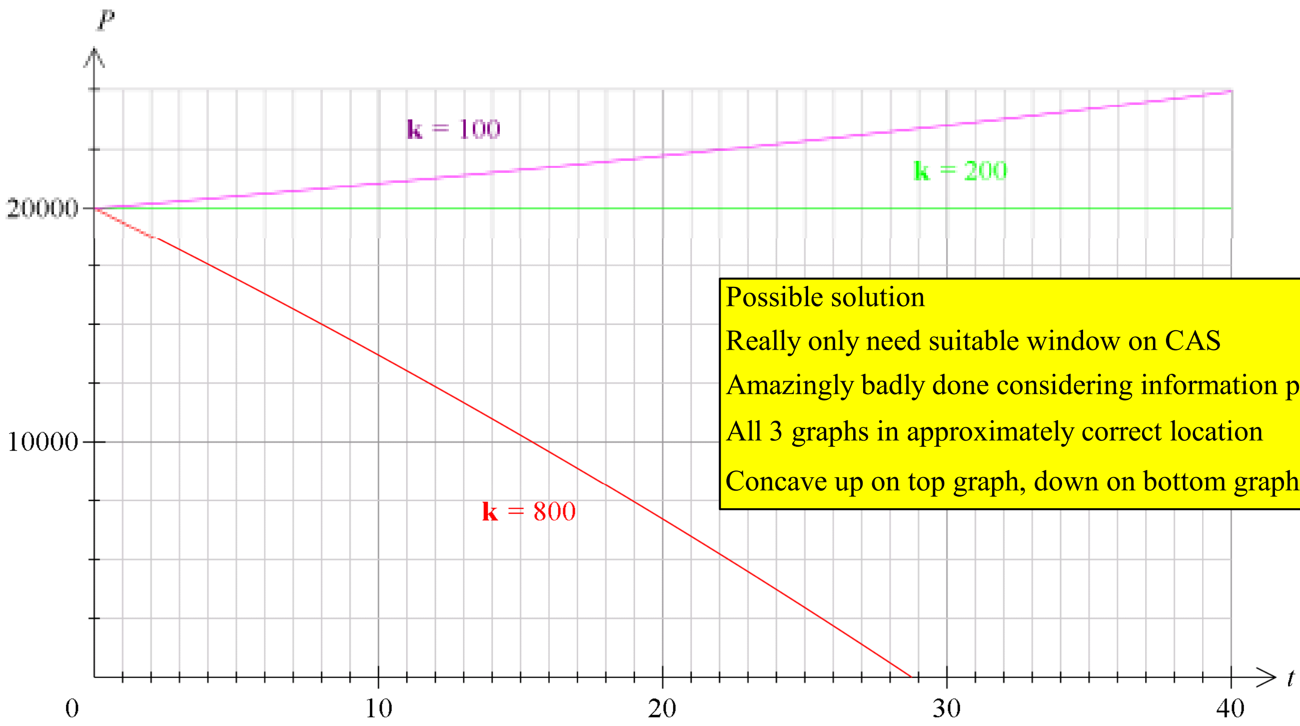
d. For each of the values

i. $k = 800$

ii. $k = 200$

iii. $k = 100$

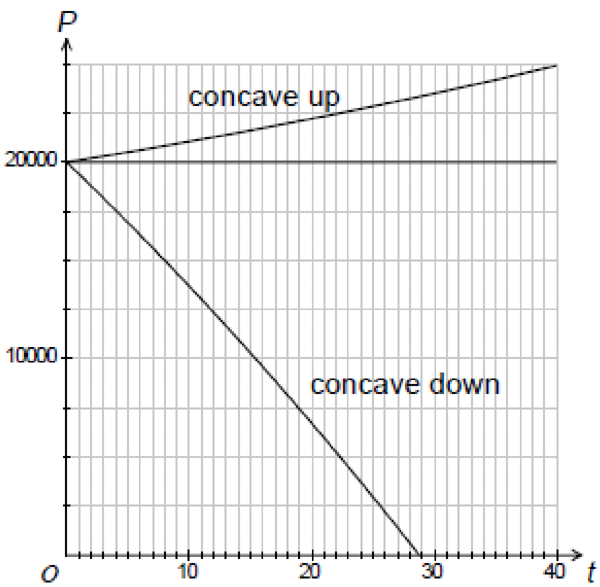
sketch the graph of P versus t on the set of axes below while the population exists, for $0 \le t \le 40$.



Possible solution
Really only need suitable window on CAS
Amazingly badly done considering information provided.
All 3 graphs in approximately correct location ...[A1]
Concave up on top graph, down on bottom graph...[A1]

Question 3di–iii.

Marks	0	1	2	3	Average
%	28	18	26	28	1.6



This question was not very well done. Common errors involved incorrect concavity, curves being drawn beyond the specified domain and, to a lesser extent, lack of use of the scale provided.

e. i. Find the value of k if the population has increased to 22 550 after twelve years.

Possible solution

$P = (20000 - 100k)e^{0.01t} + 100k$ with $P(12) = 22550$

CAS solve must surely be easy.

$k \approx -0.0049 = 0$ ...[A1] for either

n.b. No accuracy direction was given, so common sense should be used.
I'd think at least 2 s.f. would be appropriate for a decimal, but zero makes more sense given the definition of k

$\text{solve}(22550 = (20000 - 100 \cdot k) \cdot e^{0.12} + 100 \cdot k, k)$

$k = -0.004938820937$

Question 3ei.

Marks	0	1	Average
%	43	57	0.6

$22\,500 = (20\,000 - 100k) \times e^{0.12} + 100k$, $k = -0.0049$. $k = 0$ was also accepted.

This question was done reasonably well by those who attempted it.

ii. Use the definition of k to interpret your answer to **part i.** in the context of the population model.

Possible solution

k is defined as "the number of people leaving per year minus the number of people arriving per year"

This is most sensibly interpreted as either

the arrivals slightly exceeding departures in the long term $(k \approx -0.0049)$

or

the number of arrivals and departures being the same $(k = 0)$

...[A1] either

Question 3eii.

Marks	0	1	Average
%	83	17	0.2

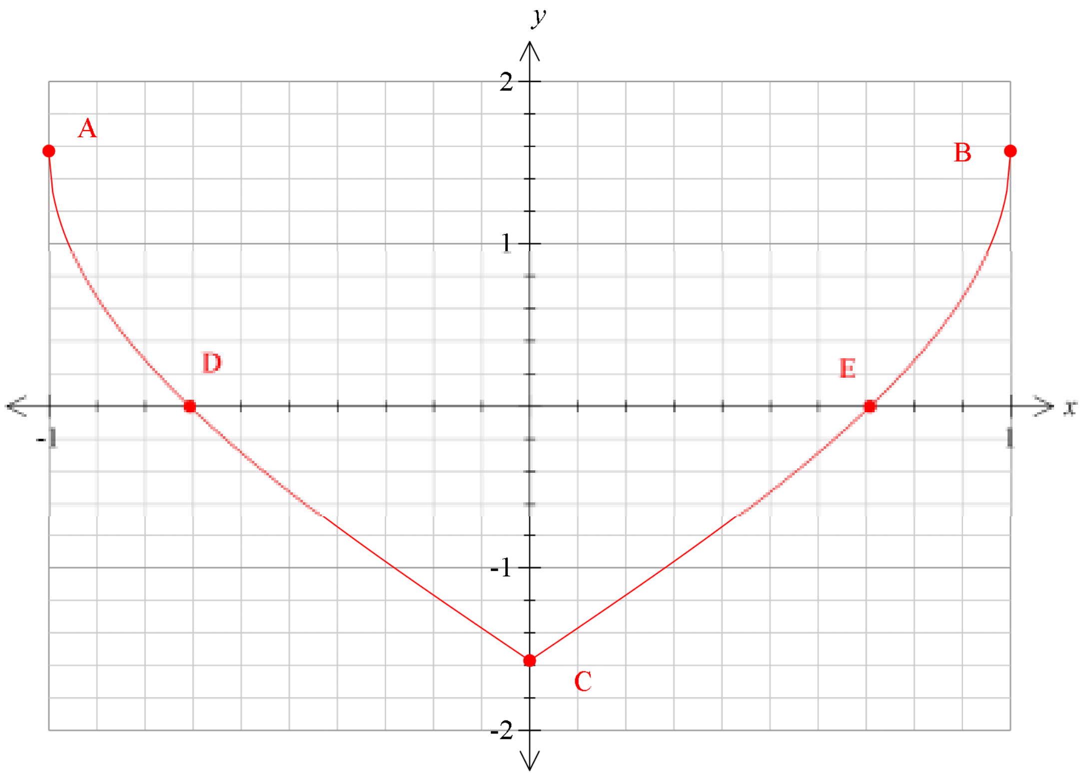
For students who obtained $k = -0.0049$, the idea that 'arrivals exceeded departures' needed to be articulated.
For students who rounded to $k = 0$, the idea that 'arrivals equalled departures' needed to be articulated.

This question was not well answered and a range of incorrect interpretations was given. Many students ignored the instruction to interpret their answer to Question 3ei.

Question 4

Consider the function f with rule $f(x) = \sin^{-1}(2x^2 - 1)$.

a. Sketch the graph of the relation $y = f(x)$ on the axes below. Label the endpoints with their **exact coordinates**, and label the x and y intercepts with their exact values.



Possible solution

$f(x) = \sin^{-1}(2x^2 - 1)$

$x = 0, y = \sin^{-1}(-1) = -\frac{\pi}{2} \qquad \Rightarrow C\left(0, -\frac{\pi}{2}\right)$

$y = 0, \sin^{-1}(2x^2 - 1) = 0$

$2x^2 - 1 = 0$

$x = \pm\frac{1}{\sqrt{2}} \qquad D\left(-\frac{1}{\sqrt{2}}, 0\right)$ & $E\left(\frac{1}{\sqrt{2}}, 0\right)$

Domain extremes

$2x^2 - 1 = -1 \qquad 2x^2 - 1 = 1$

$2x^2 = 0 \qquad 2x^2 = 2$

$x = 0 \qquad x = \pm 1$

$C\left(0, -\frac{\pi}{2}\right) \qquad A\left(-1, \frac{\pi}{2}\right)$ & $B\left(1, \frac{\pi}{2}\right)$

Shape ...[A1]

3 points correct ...[A1]

Remaining 2 points correct ...[A1]

Question 4a.

Marks	0	1	2	3	Average
%	16	14	25	44	2

This question was reasonably well done. Common errors included incorrect shape, omission of intercept and end point labels, not using the scale provided, and using decimal approximations for the exact values requested. The behaviour of the curve at $x = 0$ was the most elusive feature for students. Students are reminded to take care when transferring graphical information from technology to a graph with a given scale.

b. **i.** Write down a **definite integral** in terms of y, which when evaluated will give the volume of the solid of revolution formed by rotating the graph drawn above about the y-axis.

Possible solution

$$V = \int_{y=-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi R_y^{\,2}\, dy \qquad y = \sin^{-1}\left(2x^2 - 1\right)$$

$$= \int_{y=-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi x^2\, dy \qquad 2x^2 - 1 = \sin(y) \Rightarrow x^2 = \frac{1}{2}\left(\sin(y) + 1\right) \quad \ldots[\text{M1}]$$

$$= \int_{y=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2}\left(\sin(y) + 1\right) dy \qquad \ldots[\text{A1}]$$

Question 4bi.

Marks	0	1	2	Average
%	27	26	47	1.2

$$V = \int_{-\pi/2}^{\pi/2} \pi \times \frac{1}{2}\left(\sin y + 1\right) dy$$

This question was moderately well done. Common errors involved omission of brackets, incorrect terminals, and integrands involving x. A number of students left out π and others started with $V = 2\pi\int x^2 dy$ when setting up their volume integral.

ii. Find the exact value of the definite integral in part **i.**

Possible solution

$$V = \int_{y=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2}\left(\sin(y) + 1\right) dy$$

Use CAS

$$= \frac{\pi^2}{2} \qquad \ldots[\text{A1}]$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi}{2}\cdot(\sin(y)+1)\right) dy \qquad \frac{\pi^2}{2}$$

Question 4bii.

Marks	0	1	Average
%	57	43	0.5

$$\frac{\pi^2}{2}$$

This question was reasonably well done by students who were able to formulate the required integral in Question 4bi.

c. Use calculus to show that

$$f'(x) = \frac{2}{\sqrt{1-x^2}}, \text{ for } x \in (0, a) \text{ and find the value of } a.$$

Possible solution

$f(x) = \sin^{-1}(2x^2 - 1) = \sin^{-1}(u)$ $\quad u = 2x^2 - 1 \quad \frac{du}{dx} = 4x$

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \times 4x$$

$$= \frac{1}{\sqrt{1-(2x^2-1)^2}} \times 4x \qquad \ldots[\text{M1}]$$

$$= \frac{4x}{\sqrt{1-4x^4+4x^2-1}}$$

$$= \frac{4x}{\sqrt{4x^2-4x^4}}$$

$$= \frac{4x}{\sqrt{4x^2}\sqrt{1-x^2}}$$

$$= \frac{4x}{2|x|\sqrt{1-x^2}} \qquad \ldots[\text{M1}]$$

$$= \frac{2}{\sqrt{1-x^2}} \qquad \text{if } 0 < x < 1 \text{ so that } |x| = x \text{ and } 1 - x^2 > 0$$

$\Rightarrow a = 1 \qquad \ldots[\text{A1}]$

Question 4c.

Marks	0	1	2	3	Average
%	27	19	17	37	1.7

$f'(x) = \frac{1}{\sqrt{1-(2x^2-1)^2}} \times 4x = \frac{1}{\sqrt{4x^2-4x^4}} \times 4x = \frac{4x}{\sqrt{4x^2(1-x^2)}}$, which gives the stated result. $a = 1$

Some students did not attempt to find the value of a.

d. Complete the following to specify $f'(x)$ as a hybrid function over the maximal domain of f'.

Possible solution

$$f'(x)=\frac{4x}{2|x|\sqrt{1-x^2}}$$

n.b. $x=0$ is a cusp on graph, so $f'(0)$ does not exist

$$=\begin{cases}\dfrac{2}{\sqrt{1-x^2}} & \text{for } x\in(0,1) & \ldots[\text{A1}]\\[2ex] \dfrac{-2}{\sqrt{1-x^2}} & \text{for } x\in(-1,0) & \ldots[\text{A1}]\end{cases}$$

as $|x|=-x$ for $x\in(-1,0)$

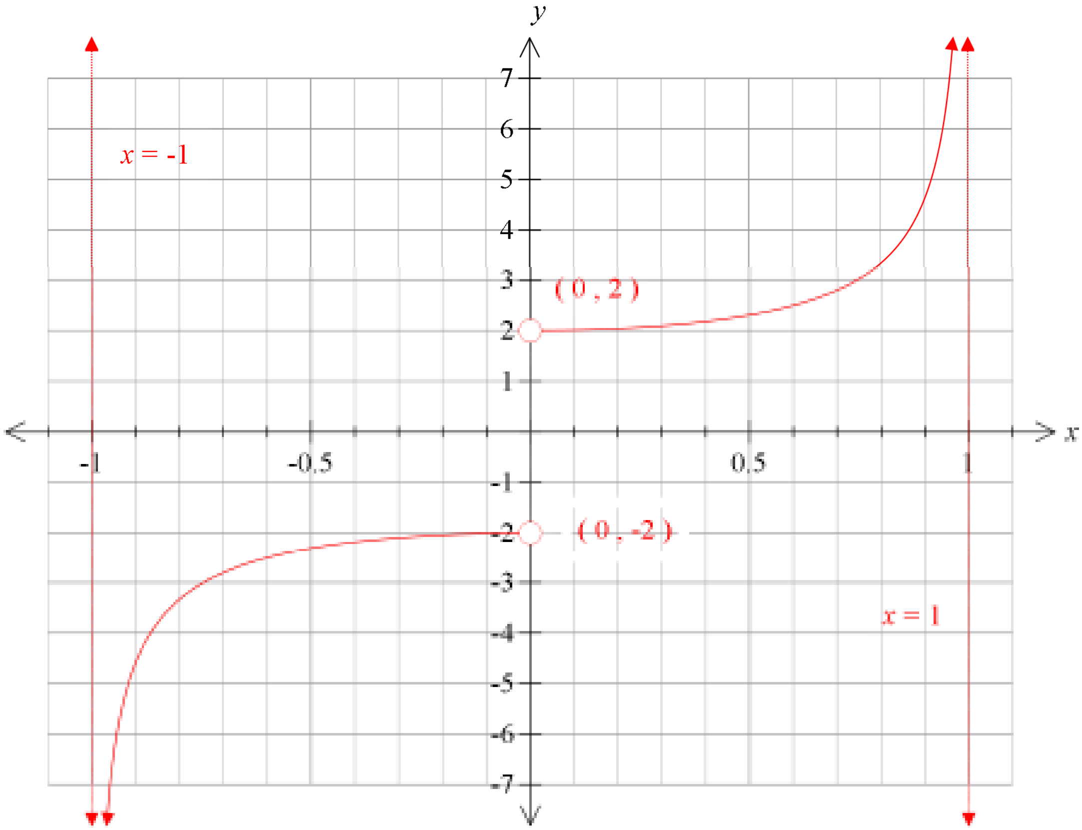
Question 4d.

Marks	0	1	2	Average
%	48	22	30	0.8

$f'(x)=\dfrac{2}{\sqrt{1-x^2}}$, for $x\in(0,1)$, $f'(x)=-\dfrac{2}{\sqrt{1-x^2}}$, for $x\in(-1,0)$

Common errors included the use of $\dfrac{2}{\sqrt{1-x^2}}$ for both parts of $f'(x)$ and errors involving domain endpoints such as $x\in(-1,0]$. Many students could complete only the first part of the hybrid function specification.

e. Sketch the graph of the hybrid function f' on the set of axes below, showing any asymptotes.



Possible solution

Use calculator with appropriate window and either

two restricted domain functions OR

a hybrid function by defining say $g(x)$ using when

Right branch with endpoint and vertical asymptote ...[A1]

Left branch with endpoint and vertical asymptote...[A1]

Question 4e.

Marks	0	1	2	Average
%	66	15	19	0.6

Students did not do well on this question. The most popular response was a 'U shape' curve with a vertex at (0, 2), sometimes with (0, 2) removed, and sometimes with the correct asymptotes. These students ignored the graph in Question 4a., which clearly had negative gradients to the left of the y-axis. Some students had the correct curves and asymptotes, but did not exclude the y-intercept points.

Question 5

Let $u = 6 - 2i$ and $w = 1 + 3i$ where $u, w \in C$.

a. Given that $z_1 = \dfrac{(u+w)\overline{u}}{iw}$, show that $|z_1| = 10\sqrt{2}$.

Possible solution - by hand

$u = 6 - 2i \qquad w = 1 + 3i \qquad u, w \in C$

$$z_1 = \frac{(u+w)\overline{u}}{iw}$$

$$= \frac{((6-2i)+(1+3i))(6+2i)}{i(1+3i)}$$

$$= \frac{(7+i)(6+2i)}{-3+i}$$

$$= \frac{40+20i}{-3+i} \times \frac{-3-i}{-3-i}$$

$$= \frac{-120-40i-60i+20}{10}$$

$$\left.\begin{aligned} &= -10-10i \\ &|z_1| = \sqrt{100+100} \end{aligned}\right\}\ldots[\text{M1}]$$

$= 10\sqrt{2}$ as required

Possible solution - by CAS

$u = 6 - 2i \qquad w = 1 + 3i \qquad u, w \in C$

$z_1 = \dfrac{(u+w)\overline{u}}{iw}$ use CAS now

$$\left.\begin{aligned} &= -10-10i \\ &|z_1| = \sqrt{100+100} \end{aligned}\right\}\ldots[\text{M1}]$$

$= 10\sqrt{2}$ as required

Question 5a.

Marks	0	1	Average
%	38	62	0.6

$z_1 = -10 - 10i$, $|z_1| = \sqrt{10^2 + 10^2} = 10\sqrt{2}$

Other acceptable methods included $|z_1| = \dfrac{|7+i||6+2i|}{|i||1+3i|} = \dfrac{\sqrt{50} \times \sqrt{40}}{\sqrt{10}} = 2\sqrt{50} = 10\sqrt{2}$.

Students performed reasonably well on this question; however, a large number of students got an incorrect answer for z_1

b. The complex number z_1 can be expressed in polar form as

$$z_1 = 200^{\frac{1}{2}} \operatorname{cis}\left(-\frac{3\pi}{4}\right).$$

Find all distinct complex numbers z such that $z^3 = z_1$.

Give your answers in the form $a^{\frac{1}{n}} \operatorname{cis}\left(\frac{b\pi}{c}\right)$, where a, b, c and n are integers.

Possible solution

Let $z = r\operatorname{cis}(\theta)$

$z^3 = r^3 \operatorname{cis}(3\theta)$

$\Rightarrow r^3 \operatorname{cis}(3\theta) = 200^{\frac{1}{2}} \operatorname{cis}\left(-\frac{3\pi}{4}\right)$ ...[M1]

$r^3 = 200^{\frac{1}{2}}$ $\quad 3\theta = -\frac{3\pi}{4} + 2k\pi$

$r = 200^{\frac{1}{6}}$ $\quad 3\theta = \frac{(8k-3)\pi}{4}$

$r = 200^{\frac{1}{6}}$ $\quad \theta = \frac{(8k-3)\pi}{12}$ ...[M1]

$k = -1 \quad z_\alpha = 200^{\frac{1}{6}} \operatorname{cis}\left(\frac{-11\pi}{12}\right)$

$k = 0 \quad z_\beta = 200^{\frac{1}{6}} \operatorname{cis}\left(\frac{-3\pi}{12}\right) = 200^{\frac{1}{6}} \operatorname{cis}\left(\frac{-\pi}{4}\right)$ ...[A1]

$k = 1 \quad z_\gamma = 200^{\frac{1}{6}} \operatorname{cis}\left(\frac{5\pi}{12}\right)$

Question 5b.

Marks	0	1	2	3	Average
%	32	15	8	45	1.6

$200^{\frac{1}{6}} \operatorname{cis}\left(-\frac{\pi}{4}\right)$, $200^{\frac{1}{6}} \operatorname{cis}\left(-\frac{\pi}{4}\right)$, $200^{\frac{1}{6}} \operatorname{cis}\left(\frac{13\pi}{12}\right)$ or $200^{\frac{1}{6}} \operatorname{cis}\left(-\frac{11\pi}{12}\right)$

Most students were able to find at least one of the cube roots of z_1; however, a number of students misinterpreted the question and found the cube of z_1. Some simplified $200^{\frac{1}{6}} \operatorname{cis}\left(\frac{13\pi}{12}\right)$ to $200^{\frac{1}{6}} \operatorname{cis}\left(\frac{-\pi}{12}\right)$ and others did not fully apply DeMoivre's theorem to the modulus and argument of z_1. A few students did not use the required form for $\alpha^{\frac{1}{n}}$, writing $5^{1/3}\sqrt{2}$ instead of $200^{\frac{1}{6}}$.

Let the argument of u be given by $\text{Arg}(u) = -\alpha$. **(You are not required to find α.)**

c. By expressing iw in polar form in terms of α, show that

$$\frac{\overline{u}}{iw} = 2\,\text{cis}(2\alpha - \pi).$$

Possible solution

$u = 6 - 2i = 2\sqrt{10}\,\text{cis}(-\alpha)$ $\qquad \overline{u} = 6 + 2i = 2\sqrt{10}\,\text{cis}(\alpha)$

$iw = i(1+3i) = -3 + i = -\frac{1}{2}u$

$= \frac{1}{2}i^2 u$ so rotate by π and divide by 2

$= \frac{1}{2}\left(2\sqrt{10}\,\text{cis}(-\alpha + \pi)\right)$

$= \sqrt{10}\,\text{cis}(\pi - \alpha)$ …[A1]

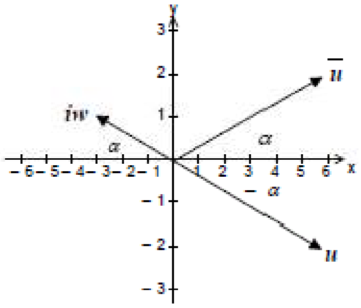
$\frac{\overline{u}}{iw} = \frac{2\sqrt{10}\,\text{cis}(\alpha)}{\sqrt{10}\,\text{cis}(\pi - \alpha)}$ …[M1]

$= 2\,\text{cis}(\alpha - (\pi - \alpha))$ …[M1]

$= 2\,\text{cis}(2\alpha - \pi)$ as required

Question 5c.

Marks	0	1	2	3	Average
%	66	12	4	18	0.8



$iw = \sqrt{10}\text{cis}(\pi\text{-}\alpha)$, $\overline{u} = \sqrt{40}\text{cis}(\alpha)$

$$\frac{\overline{u}}{iw} = \frac{\sqrt{40}\,\text{cis}\,\alpha}{\sqrt{10}\,\text{cis}(\pi - \alpha)} = \frac{2\sqrt{10}}{\sqrt{10}}\text{cis}(\alpha - (\pi - \alpha)) = 2\text{cis}(2\alpha - \pi)$$

A significant number of students did not attempt this question. Some students found a diagram useful in order to proceed with this question, and others attempted to work in terms of $\tan^{-1}\left(\frac{1}{3}\right)$ rather than α. Quite a few students did a substantial amount of work in cartesian form to little avail. A number of students managed to get $\overline{u} = \sqrt{40}\text{cis}(\alpha)$, but could not express iw in terms of α. As this was a 'show that' question, it was important that all connecting steps were shown.

d. Use the relation given in part **a.** to find $\text{Arg}(u+w)$ in terms of α.

Possible solution

It's important here to connect the dots. Why were you given all the previous results to find?

$z_1 = \dfrac{(u+w)\overline{u}}{iw} = (u+w)\times\dfrac{\overline{u}}{iw}$ already know the red bits

$200^{\frac{1}{2}}\text{cis}\left(-\dfrac{3\pi}{4}\right) = (u+w)\times 2\,\text{cis}(2\alpha-\pi)$ ...[M1]

$(u+w) = \dfrac{200^{\frac{1}{2}}\text{cis}\left(-\dfrac{3\pi}{4}\right)}{2\,\text{cis}(2\alpha-\pi)} = \dfrac{10\sqrt{2}\,\text{cis}\left(-\dfrac{3\pi}{4}\right)}{2\,\text{cis}(2\alpha-\pi)}$

$(u+w) = 5\sqrt{2}\,\text{cis}\left(-\dfrac{3\pi}{4}-(2\alpha-\pi)\right)$

$(u+w) = 5\sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}-2\alpha\right)$ ...[M1]

$\arg(u+w) = \dfrac{\pi}{4}-2\alpha$ now is this a principal form?

$\overline{u} = 6+2i = 2\sqrt{10}\,\text{cis}(\alpha)$

$\Rightarrow \alpha = \tan^{-1}\left(\dfrac{2}{6}\right) = \tan^{-1}\left(\dfrac{1}{3}\right)$

$\Rightarrow 0 < \alpha < \dfrac{\pi}{4}$

$\Rightarrow 0 < 2\alpha < \dfrac{\pi}{2}$

$\Rightarrow -\dfrac{\pi}{4} < \dfrac{\pi}{4} - 2\alpha < \dfrac{\pi}{4}$

$\text{Arg}(u+w) = \dfrac{\pi}{4}-2\alpha$ ...[A1]

Question 5d.

Marks	0	1	2	3	Average
%	82	3	2	13	0.5

$z_1 = \dfrac{(u+w)\overline{u}}{iw}$, $\text{Arg}(z_1) = \text{Arg}(u+w) + \text{Arg}\left(\dfrac{\overline{u}}{iw}\right)$, $-\dfrac{3\pi}{4} = \text{Arg}(u+w) + 2\alpha - \pi$, $\text{Arg}(u+w) = \dfrac{\pi}{4} - 2\alpha$

This question proved to be very difficult for most students and a significant number did not attempt this question. Some students who did attempt the question used approaches such as $\text{Arg}(u+w) = \text{Arg}(u) + \text{Arg}(w)$, which were incorrect.