VCE Specialist Mathematics Sample Exam Questions - 2023

These sample questions are intended to demonstrate how new aspects of Units 3 and 4 of VCE Specialist Mathematics written examination (1 & 2) may be examined. They do not constitute a full examination paper

Exam 1

Question 1 (4 marks)

Proof by induction (new in study design) for a series - and the question really holds the students' hand!

Consider the statement $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \cdots\ +\frac{1}{2^n}=1-\frac{1}{2^n}$, where $n \in \mathbb{N}$.

a. Show that if $n = 1$, the statement is true. 1 mark

Base case: $n = 1$

$LHS = \frac{1}{2}$, $RHS = 1-\frac{1}{2^1}=\frac{1}{2}$

$\implies LHS = RHS \quad \square$

b. Assume that the statement is true for $n = k$

Write down the assumption in terms of k 1 mark

Induction Hypothesis: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \cdots\ +\frac{1}{2^k}=1-\frac{1}{2^k}$ for some $k \in \mathbb{N}$

c. Hence, prove by mathematical induction that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \cdots\ +\frac{1}{2^n}=1-\frac{1}{2^n}$, for $n \in \mathbb{N}$ 2 marks

$$LHS = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \cdots\ +\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$$= \left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}} \text{ by the inductive hypothesis}$$

$$= 1-\frac{1}{2^k}\left(1-\frac{1}{2}\right) = 1-\frac{1}{2^{k+1}} = RHS$$

Given the base case in part a and the above induction step, it follows from induction that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \cdots\ +\frac{1}{2^n}=1-\frac{1}{2^n}$, for all $n \in \mathbb{N}$..

Question 2 (4 marks)

Proof by induction (new in study design) for an inequality - less hand holding

a. Consider the inequality $2^n > n^2$ for $n \geq n_0$, where $n \in \mathbb{N}$. Show that $n_0 = 5$. 1 mark

When $n = 4:\ 2^4 = 16 \not> 4^2 = 16$
When $n = 5:\ 2^5 = 32 > 5^2 = 25$

It's clear what they want... the inequality is true for $n = 0, 1$ but not true for $n = 2, 3, 4$.
But strange wording! Maybe it should be something like

> Consider the inequality $2^n > n^2$, where $n \in \mathbb{N}$.
> This inequality is true for all $n \geqslant n_0$. Show that the smallest such n_0 is $n_0 = 5$.

b. Prove by mathematical induction that $2^n > n^2$ for $n \geq 5$, where $n \in \mathbb{N}$. 3 marks

Let $P(n)$ be the statement that $2^n > n^2$.
Already have the base case, $P(5)$ is true.

Assume that $P(k)$ is true for some $n = k$: i.e., $2^k > k^2$
What to show that this implies $P(k+1)$, i.e. the inequality $2^{k+1} > (k+1)^2$ is true.

$2^{k+1} = 2^k \times 2 > 2k^2$ by the induction hypothesis
We need to show that $2k^2 > (k+1)^2 \iff k^2 - 2k > 1 \iff (k-1)^2 > 2 \iff k \geqslant 3$
As $k \geqslant 5 > 3$, we have
$2^{k+1} > (k+1)^2$ and so $P(k) \implies P(k+1)$

So, by the principle of induction, $2^n > n^2$ for all integer $n \geq 5$

Question 3 (4 marks)

Proof by induction (new in study design) for divisibility - even less hand holding

Prove by mathematical induction that the number $9^n - 5^n$ is divisible by 4 for all $n \in \mathbb{N}$

Let $D(n)$ be the statement that $9^n - 5^n$ is divisible by 4
Check the base case: $n = 0,\ D(0) = 9^0 - 5^0 = 1 - 1 = 0$, which is divisible by 4.
For monsters that say $0 \notin \mathbb{N}$, could also check $D(1) = 9^1 - 5^1 = 4$, which is also divisible by 4.

Assume that $D(k)$ is true for some $k \in \mathbb{N}$, i.e., $9^k - 5^k = 4m$ for some $m \in \mathbb{N}$

Then for $n = k + 1$:

$$\begin{aligned} 9^{k+1} - 5^{k+1} &= 9\left(9^k - 5^k\right) + 9 \times 5^k - 5^{k+1} \\ &= 9(4m) + (9-5)5^k \text{, by the induction hypothesis} \\ &= 4\left(9m + 5^k\right) \text{, which is divisible by 4} \end{aligned}$$

So, $D(k) \implies D(k+1)$
So, by the principle of induction, $9^n - 5^n$ is divisible by 4 for all $n \in \mathbb{N}$

Question 4 (3 marks)

Proof by contradiction is new in the study design.

Use proof by contradiction to prove that if n is odd, where $n \in \mathbb{N}$, then $n^3 + 1$ is even.

This is kind of annoying, as a direct proof would be cleaner. Let's see that first...
Given n is odd $\iff$ $n = 2m + 1$ for $m \in \mathbb{N}$
Then $n^3 + 1 = (2m+1)^3 + 1 = \left(8m^3 + 12m^2 + 6m + 1\right) + 1 = 2\left(4m^3 + 6m^2 + 3m + 1\right) = 2k$
where $k = 4m^3 + 6m^2 + 3m + 1 \in \mathbb{N}$. So $n^3 + 1 = 2k$ is even for any odd n.

Let's be careful, the original statement is $P \implies Q$, with $P =$ "n is odd", $Q =$ "$n^3 + 1$ is even"
To use a proof by contradiction, we first assume the negation of what is to be proved, i.e., $P \implies \neg Q$

Assume, for contradiction, that there exists an odd n such that $n^3 + 1$ is odd (as odd $\Leftrightarrow$ not even).
If n is odd, then $n = 2m + 1$ for $m \in \mathbb{N}$
Then $n^3 + 1 = (2m+1)^3 + 1 = \left(8m^3 + 12m^2 + 6m + 1\right) + 1 = 2\left(4m^3 + 6m^2 + 3m + 1\right) = 2k$
So no matter what odd n we start with, $n^3 + 1$ is even, contradicting our assumption.
So, for all odd n, $n^3 + 1$ is even.

Question 5 (3 marks)

Proof by contradiction is new in the study design.

Use proof by contradiction to prove that $\sqrt{3} + \sqrt{5} > \sqrt{11}$

Assume the negation: $\sqrt{3} + \sqrt{5} \leqslant \sqrt{11}$
$\iff \left(\sqrt{3} + \sqrt{5}\right)^2 \leqslant 11$ (because both sides above are positive)
$\iff 8 + 2\sqrt{15} \leqslant 11$
$\iff 2\sqrt{15} \leqslant 3$
$\iff 4 \times 15 \leqslant 3^2 = 9$
But clearly $60 > 9$, which contradicts our assumption

So $\sqrt{3} + \sqrt{5} > \sqrt{11}$

Again, it feels like the same argument could be run directly, without using proof by contradiction.

Question 6 (4 marks)

Surface area of a solid of revolution are new in the study design

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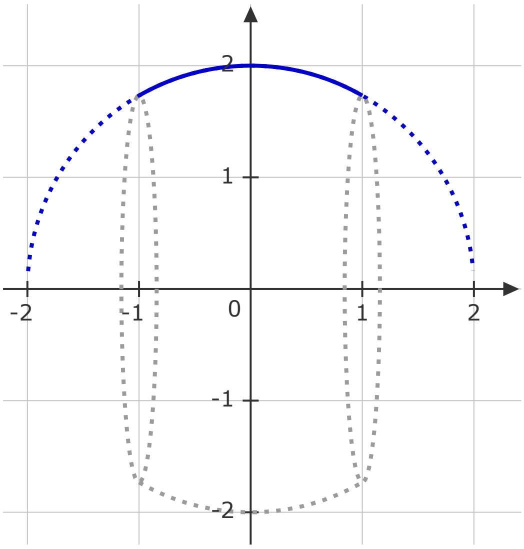
The curve given by $y = \sqrt{4 - x^2}$, where $x \in [-1, 1]$ is rotated about the x-axis to form a solid of revolution. Find the surface area of this solid of revolution



This is like the outside surface of a napkin ring.

Area is the sum of frustrums $= \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

Need $\frac{dy}{dx} = \frac{1}{2} \frac{-2x}{\sqrt{4 - x^2}} = \frac{-x}{y}$

Area $= 2\pi \int_{-1}^{1} y \sqrt{1 + \frac{x^2}{y^2}} dx = 2\pi \int_{-1}^{1} \sqrt{y^2 + x^2} dx$

Given the curve is an arc of a circle centered at the origin, $x^2 + y^2 =$

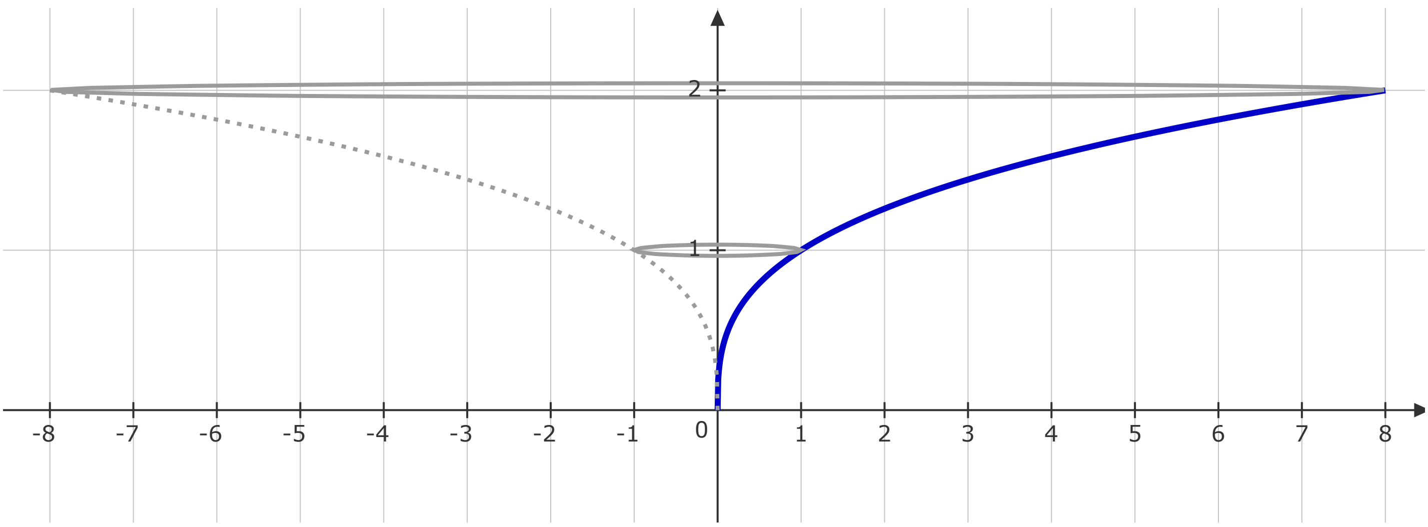
Area $= 2\pi \int_{-1}^{1} 2 dx = 4\pi \int_{-1}^{1} 1 dx = 8\pi$

Check: The whole sphere would have area $4\pi(2)^2 = 16\pi$, which is 2 times more, seems ok - and is what we'd get if we increase the domain to $[-2, 2]$

Question 7 (5 marks)

Another surface area of a solid of revolution - around y-axis this time

The curve given by $y = \sqrt[3]{x}$ is rotated about the y-axis to form a solid of revolution.
Find the surface area of the part of this solid of revolution where $x \in [0, 8]$.



$$\text{area} = \int_0^2 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy, \quad \text{note: } x = y^3,\ \frac{dx}{dy} = 3y^2$$

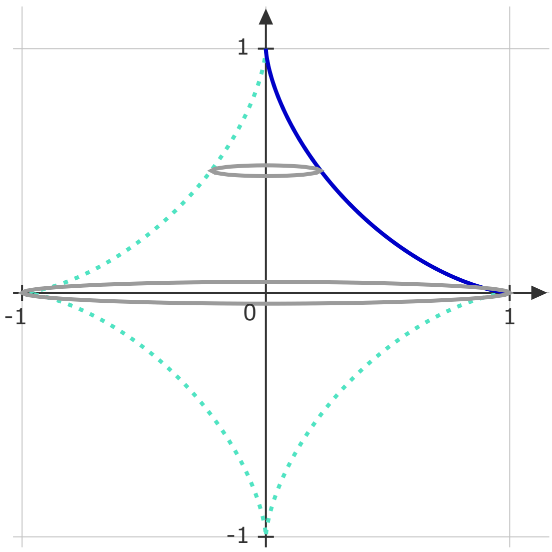
$$= 2\pi\int_0^2 y^3\sqrt{1+9y^4}\,dy, \quad \text{let } u = 1+9y^4,\ du = 36y^3dy$$

$$= \frac{\pi}{18}\int_1^{1+9\times 16}\sqrt{u}\,du = \frac{\pi}{18}\left[\frac{2}{3}y^{3/2}\right]_1^{145} = \frac{\pi}{27}\left[145\sqrt{145}-1\right]$$

Question 8 (4 marks)

Another surface area of a solid of revolution - but parametric this time.

Determine the surface area obtained by rotating the curve defined by the parametric equations $x = \sin^3\theta,\ y = \cos^3\theta$, where $\theta \in \left[0, \frac{\pi}{2}\right]$, about the y-axis.



This is a concave-curved conic shape.
$\theta = 0$ is the top (vertex), $(x, y) = (0, 1)$
$\theta = \pi/2$ is the point $(x, y) = (1, 0)$
We're in the 1st quadrant, so sine and cosine are positive.

$$\text{area} = \int_0^{\pi/2} 2\pi x\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= 2\pi\int_0^{\pi/2}\sin^3\theta\sqrt{\left(3\sin^2\theta\cos\theta\right)^2+\left(-3\cos^2\theta\sin\theta\right)^2}$$

$$= 6\pi\int_0^{\pi/2}\sin^3\theta\sin\theta\cos\theta\sqrt{(\sin\theta)^2+(\cos\theta)^2}\,d\theta$$

$$= 6\pi\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta, \quad \text{let } u = \sin\theta,\ du = \cos\theta\,d\theta$$

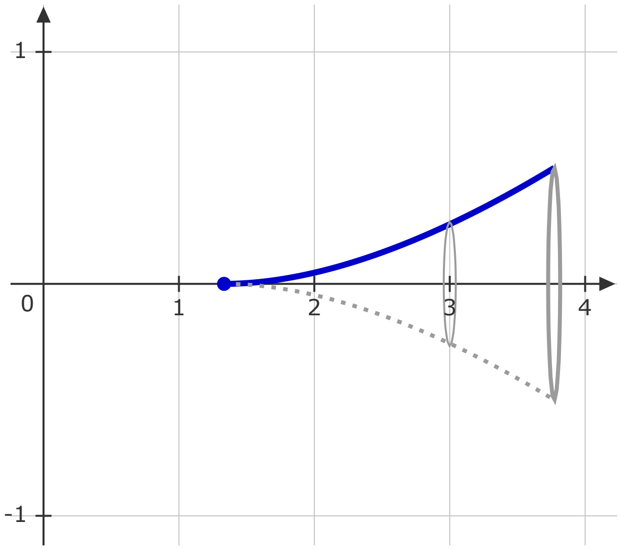
$$= 6\pi\int_0^1 u^4du = 6\pi\left[\frac{1}{5}u^5\right]_0^1 = \frac{6\pi}{5} = 1.2\pi$$

Check: This should be a bit less than the area of a cone radius 1

Area of cone $= \pi rs = \pi(1)\sqrt{2} \approx 1.414\pi$

Question 9 (3 marks)

Find the surface area of revolution formed when the curve defined by the parametric equations $x = \frac{4}{3}\sqrt{(t+1)^3},\ y = \frac{1}{2}t^2$, where $0 \le t \le 1$, is rotated about the x-axis.



$$\text{area} = \int_0^1 2\pi y\sqrt{\dot{x}^2+\dot{y}^2}dt$$
$$= \pi\int_0^1 t^2\sqrt{\left(2(t+1)^{1/2}\right)^2+(t)^2}\,dt$$
$$= \pi\int_0^1 t^2\sqrt{t^2+4t+4}\,dt$$
$$= \pi\int_0^1 t^2\sqrt{(t+2)^2}\,dt$$
$$= \pi\int_0^1 t^2(t+2)\,dt$$
$$= \pi\left[\frac{1}{4}t^4+\frac{2}{3}t^3\right]_0^1 = \pi\left(\frac{1}{4}+\frac{2}{3}\right) = \frac{11\pi}{12}$$

Question 10 (7 marks)

The logistic differential equation is new in the study design.

The population of bacteria, $P(t)$, in a Petri dish satisfies the logistic differential equation

$$\frac{dP}{dt} = 2P\left(6-\frac{P}{8000}\right) = 12P\left(1-\frac{P}{48000}\right)$$

where t is measured in hours and the initial population is 4000 bacteria.

a. Find the maximum number of bacteria predicted by this model. 1 mark

$$0 = \frac{dP}{dt} \implies P = 0 \text{ or } P = 48000$$

So, max number of bacteria is 48000.

Note that this is the horizontal asymptote $P = 48000$ as $t \to \infty$, but given it's a continuous model for a discrete population, it will round up to the whole number eventually...

b. Find the number of bacteria when the population is growing at its fastest rate. 2 marks

$$\frac{d^2P}{dt^2} = 12 - \frac{P}{2000} = 0 \implies P = 24000$$

So, the max growth occurs at half the max population.

Note that as $P'(t)$ is a negative quadratic in P, you can also see the max growth occurs halfway between the zeros $P = 0$ and $P = 48000$, so at 24000.

c. Solve the differential equation to find P as a function of t. 4 marks

Separable equation:

$$\frac{dP}{2P\left(6-\frac{P}{8000}\right)} = dt$$

$$\Longrightarrow t = 4000\int \frac{1}{(48000-P)P}dP = \frac{1}{12}\int \frac{1}{48000-P} + \frac{1}{P}\,dP$$

$$= \frac{1}{12}(\log_e(P) - \log_e(48000-P)) + C,$$ nb $0 < P < 48000$ so the arguments of logs are positive

When $t = 0,\ P = 4000$, so

$$0 = \frac{1}{12}\log_e\left(\frac{4000}{48000-4000}\right) + C = \frac{1}{12}\log_e\left(\frac{1}{11}\right) + C \Longrightarrow C = \frac{1}{12}\log_e(11)$$

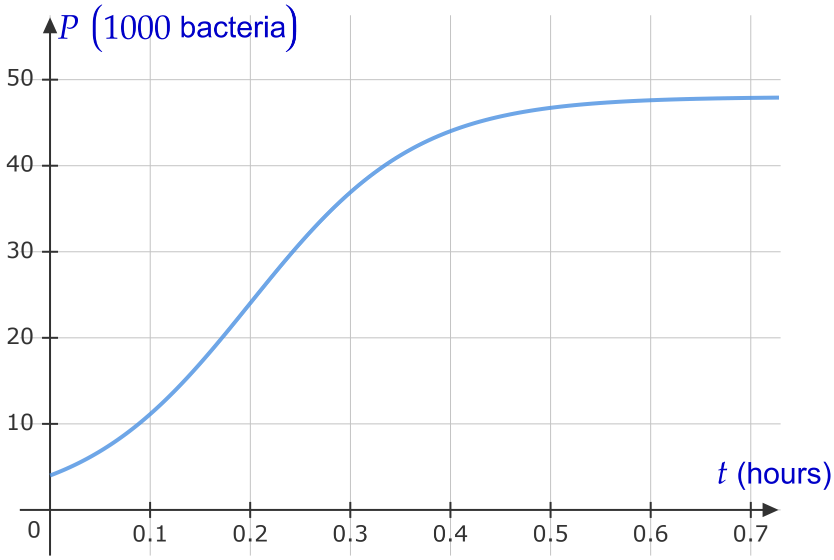
$$\Longrightarrow 12\,t = \log_e\left(\frac{11P}{48000-P}\right)$$

$$\Longrightarrow \frac{11P}{48000-P} = e^{12t}$$

$$\Longrightarrow 11P = e^{12t}(48000-P)$$

$$\Longrightarrow P = 48000\,\frac{e^{12t}}{11+e^{12t}}$$

$$\Longrightarrow P = \frac{48000}{1+11e^{-12t}}$$



Question 11 (4 marks)

Integration by parts - without the hand holding "hey, look at this derivative" stuff from the last study design

Find $\int x^2\cos(2x)dx$

Plan, hit the x^2 with derivatives....

$$I = \int x^2\cos(2x)dx$$

$$= \frac{1}{2}\int x^2\frac{d}{dx}\sin(2x)dx = \frac{1}{2}x^2\sin(2x) - \int x\sin(2x)dx$$

$$= \frac{1}{2}x^2\sin(2x) + \frac{1}{2}\int x\frac{d}{dx}\cos(2x)dx = \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{2}\int\cos(2x)dx$$

$$= \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C$$

Check: $\frac{d}{dx}I = \frac{d}{dx}\left(\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C\right)$

$$= x\sin 2x + x^2\cos 2x + \frac{1}{2}\cos 2x - x\sin 2x - \frac{1}{2}\cos 2x = x^2\cos 2x$$ ☺

Alternatively, as it did not explicitly require integration by parts, define $I(a) = \int\cos(ax)dx$

$$\Longrightarrow I''(a) = \frac{d}{da}\int -x\sin(ax)dx = -\int x^2\cos(ax)dx \Longrightarrow I = -I''(2)$$

So, $I(a) = \frac{1}{a}\sin(ax) \implies I'(a) = -\frac{1}{a^2}\sin(ax) + \frac{x}{a}\cos(ax)$

$$\implies I''(a) = \frac{2}{a^3}\sin(ax) - \frac{x}{a^2}\cos(ax) - \frac{x}{a^2}\cos(ax) - \frac{x^2}{a}\sin(ax)$$

$$\implies -I''(2) = -\frac{1}{4}\sin 2x + \frac{x}{2}\cos 2x + \frac{x^2}{2}\sin 2x + C \quad ☺$$

Question 12 (3 marks)

Vector equation of a plane is new in study design

The vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + k$, $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ lie in a plane that passes through the point $(3,\ 2,\ 1)$.
Find the Cartesian equation of this plane.

Let $\mathbf{r}_0 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ be the position vector of the point
and $\mathbf{r}$ be the position vector of any point on the plane.
Then $\mathbf{r} - \mathbf{r}_0$ is in the span of {**a, b**}, i.e., all points in the plane are parameterised
as $\mathbf{r}(s,t) = \mathbf{r}_0 + s\,\mathbf{a} + t\,\mathbf{b}$.
To get the Cartesian equation, dot this with the normal $\mathbf{n} = \mathbf{a} \times \mathbf{b}$
& use $\mathbf{a}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} = 0 \implies \boxed{\mathbf{r}\cdot\mathbf{n} = \mathbf{r}_0\cdot\mathbf{n}}$

The normal vector (not normalised) is

$$\begin{aligned}\mathbf{n} &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\\ &= (9-2)\mathbf{i} + (4+6)\mathbf{j} + (4+12)\mathbf{k}\\ &= 7\mathbf{i} + 10\mathbf{j} + 16\mathbf{k}\end{aligned}$$

[A good check when calculating by hand is $\mathbf{a}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} = 0$]

We need: $\mathbf{r}_0\cdot\mathbf{n} = 3\times7 + 2\times10 + 1\times16 = 57$

So the Cartesian equation is
$7x + 10y + 16z = 57$

Question 13 (6 marks)

Equation of a plane again - and intersection with a line - both new in the SD

a. Find the equation of the plane that passes through the points $P(3,3,6)$, $Q(1,-1,2)$ and $R(5,2,0)$. 4 marks

It doesn't specify whether it wants Cartesian or Parametric form.
But 4 marks is a bit of work, so probably wants the Cartesian equation...

Let $\mathbf{a} = \overrightarrow{OR} - \overrightarrow{OP} = 2\mathbf{i} - \mathbf{j} - 6\mathbf{k}$, $\mathbf{b} = \overrightarrow{OP} - \overrightarrow{OQ} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{r}_0 = \overrightarrow{OQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

The Parametric equation of the plane is
$\mathbf{r}(s,t) = \mathbf{r}_0 + s\,\mathbf{a} + t\,\mathbf{b} = (1 + 2s + 2t)\mathbf{i} + (-1 - s + 4t)\mathbf{j} + (2 - 6s + 4t)\mathbf{k}$

The Cartesian equation of the plane requires the normal vector (choose simple coefficients)

$\mathbf{n} \propto \mathbf{a} \times \mathbf{b} = (-4+24)\mathbf{i} + (-12-8)\mathbf{j} + (8+2)\mathbf{k} = 20\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}$

$\implies \mathbf{n} = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

Calculating $\mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}$ gives: $2x - 2y + z = 6$

b. Find the point of intersection of the line given by $\mathbf{r} = 2\mathbf{i} + 5\mathbf{k} + t(2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$, where $t \in \mathbb{R}$ with the plane given by $2x - 2y + z = 6$. 2 marks

Substitute the parametric form of the line into the plane:

$2(2+2t) - 2(0-4t) + (5-3t) = 4 + 4t + 8t + 5 - 3t = 9 + 9t = 6$

$\implies 9t = -3$

$\implies t = -\dfrac{1}{3}$

Sub back into the equation for the line to get the point of intersection:

$$\mathbf{r} = 2\mathbf{i} + 5\mathbf{k} - \frac{1}{3}(2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}) = \frac{4}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + 6\mathbf{k} \implies \left(\frac{4}{3}, \frac{4}{3}, 6\right)$$

Question 14 (3 marks)

Find the angle between the plane given by $2x + y + z = 7$ and the line given by $\mathbf{r} = 11\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + t\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, where $t \in \mathbb{R}$.

We need the complementary angle of the angle between the direction of the line and the normal to the plane.

The direction of the line is $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

The normal to the plane is $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

So the complementary angle is

$$\theta_C = \arccos(\widehat{\mathbf{v}} \cdot \widehat{\mathbf{n}}) = \arccos\left(\frac{2+2-1}{\sqrt{1+4+1}\sqrt{4+1+1}}\right) = \arccos\left(\frac{3}{6}\right) = \frac{\pi}{3}$$

and the angle between the plane and the line is $\dfrac{\pi}{2} - \dfrac{\pi}{3} = \dfrac{\pi}{6}$

Question 15 (5 marks)

a. Find the vector equation of the line through the points $A(3, 1, -1)$ and $B(5, 2, -6)$. 2 marks

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \overrightarrow{OA} + t\,(\overrightarrow{BA}) = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 5-3 \\ 2-1 \\ -6+1 \end{bmatrix} = \begin{bmatrix} 3+2t \\ 1+t \\ -1-5t \end{bmatrix}$$

In i-j-k notation, that's

$\mathbf{r}(t) = 3\mathbf{i} + \mathbf{j} - \mathbf{k} + t\,(2\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = (3+2t)\mathbf{i} + (1+t)\,\mathbf{j} - (1+5t)\mathbf{k}$

Other choices of parametrisation are possible!

b. Find the sine of the angle that this line makes with the plane given by $x + 2y - z = 9$ 3 marks

The angle the line makes to the normal of the plane is
$\cos(\theta_C) = \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}$, where $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$
The angle the line makes with the plane is $\theta = \frac{\pi}{2} - \theta_C$

$$\sin(\theta) = \cos(\theta_C) = \hat{\mathbf{v}} \cdot \hat{\mathbf{n}} = \frac{2+2+5}{\sqrt{4+1+25}\sqrt{1+4+1}} = \frac{9}{6\sqrt{5}} = \frac{3}{2\sqrt{5}}$$

Question 16 (4 marks)

The position of a particle after t seconds is given by $\mathbf{r} = t^2\mathbf{i} + 5t\mathbf{j} + \left(t^2 - 16t\right)\mathbf{k}$, where $t \geq 0$ and components are measured in metres.
Find the time at which the minimum speed occurs and calculate the minimum speed.
Give your answer in ms^{-1}.

Speed: $|\mathbf{v}| = |\dot{\mathbf{r}}| = |2t\mathbf{i} + 5\mathbf{j} + 2(t-8)\mathbf{k}| = \sqrt{4t^2 + 25 + 4(t-8)^2} = \sqrt{8t^2 - 64t + 281}$

Want $\frac{d}{dt}|\mathbf{v}| = 0 \iff \frac{d}{dt}|\mathbf{v}|^2 = 0 \quad (|\mathbf{v}| \neq 0)$

So, $\frac{d}{dt}|\mathbf{v}|^2 = 16t - 64 = 0 \implies \boxed{t = 4}$, it is a minimum as v is the square root of a positive quadratic.
[Could also complete the square to see $|v|^2 = 8(t-4)^2 + 153$, so min speed is at $t = 4$]

When $t = 4$, we find the minimum speed: $|\mathbf{v}| = \sqrt{153} = 3\sqrt{17}$ ms^{-1}

Question 17 (3 marks)

Two planes have equations $x + y - z = 3$ and $2x - y - 2z = 4$.
Given that the angle between the two planes is θ, find $\sec(\theta)$

The angle between two planes is equal to the angle between their normal vectors, which are $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ respectively.

$$\cos(\theta) = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = \frac{2-1+2}{\sqrt{1+1+1}\sqrt{4+1+4}} = \frac{3}{\sqrt{3}\sqrt{9}} = \frac{1}{\sqrt{3}}$$

So, $\sec(\theta) = \sqrt{3}$

Question 18 (3 marks)

The position vectors $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ form two sides of a triangle.
Find the area of the triangle in the form $c\sqrt{d}$, where $c, d \in \mathbb{N}$.

Area of triangle = half the area of the parallelogram = $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

$$\text{Area} = \frac{1}{2}|(-4\times3 + 2\times2)\mathbf{i} + (2\times1 - 2\times3)\mathbf{j} + (-2\times2 + 4\times1)\mathbf{k}|$$
$$= \frac{1}{2}|-8\mathbf{i} - 4\mathbf{j} + 0\mathbf{k}| = \frac{1}{2}\sqrt{8^2 + 4^2} = 2\sqrt{4+1} = 2\sqrt{5}$$

Question 19 (4 marks)

A parallelogram, $OABC$, has vertices at $O(0,\ 0,\ 0)$, $A(1,\ 2,\ -1)$ and $C(3,\ m,\ 1)$, where $m \in \mathbb{R}$.

Find the value(s) of m if the area of the parallelogram is $4\sqrt{5}$.

area $= \left|\overrightarrow{OA} \times \overrightarrow{OC}\right| = \left|(2 \times 1 + 1 \times m)\mathbf{i} + (-1 \times 3 - 1 \times 1)\mathbf{j} + (1 \times m - 2 \times 3)\mathbf{k}\right|$

$= \left|(2 + m)\mathbf{i} - 4\mathbf{j} + (m - 6)\mathbf{k}\right|$

$= \sqrt{(2 + m)^2 + 4^2 + (m - 6)^2}$

$= \sqrt{2m^2 - 8m + 56}$

Given area $= 4\sqrt{5} \implies 4^2 \times 5 = 2m^2 - 8m + 56$

$\Longrightarrow m^2 - 4m - 12 = 0$

$\Longrightarrow (m - 6)(m + 2) = 0$

So $m = 6$, $m = -2$

Exam 2: Section A -- MCQs

Question 1 (D)

Consider the following statement.

'For all integers n, if n^2 is even, then n is even.'

The contrapositive is...

In general, the contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$

For all integers n, if n is odd, then n^2 is odd.

Note that some other options (A & E) are true statements, just not the contrapositive.

Question 2 (C)

Pseudocode and algorithms are new in the study design. However, this does not follow the pseudocode rules outlined a [VCAA:specialistmathematics/Pages/PseudoCode.aspx]() or in [VCAA:professionallearning/2023/MathematicsPseudocodePresentation.pdf]()
The use of **declare** and **set ... to ...**, the **repeat** loop, the lack of **bold** keywords and most importantly the use of = instead of $\leftarrow$ for assignment, all not part of the VCAA specified pseudocode. At least it has indentation...

I also don't understand why t1 and n are variables, or why have both f and t2 in the code when they serve the same purpose. The code is just the first order difference equation $t_{n+1} = 2 + 2t_n,\ t_0 = 3$
This code doesn't even pretend to be useful...

Here is the same code on [Python Tutor]() implemented in Python - showing the line-by-line equivalent code

The procedure below has been written in pseudocode.

```
declare integer n
declare integer f
declare integer t1
declare integer t2
set f to 0                0
set t1 to 2               2
set t2 to 3               3
set n to 3
repeat n times
    f = t1 + 2 × t2       2+6=8,   2+16=18,   2+36=38
    t2 = f                8,       18,        38
    print f               8        18,        38
end loop
```

The output of the pseudocode is a list of numbers. What is the final number?

$\boxed{38}$

Question 3 (E)

A vector perpendicular to both of the lines represented by $\mathbf{r}_1 = 2\mathbf{i} + 3\mathbf{j} + t\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r}_2 = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} + t\,(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ is given by

It's just testing the determinant way of calculating the cross product of the direction vectors of the lines:

$\mathbf{n} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$ --- the correct answer is E

Question 4 (A)

Consider two points with coordinates $(5,\ -6,\ 4)$ and $(-3,\ -1,\ -10)$.
Which one of the following is the equation of the straight line that passes through these two points?

Parametric vector equation of a line: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, we have choices in what to call $\mathbf{r}_0$ and $\mathbf{v}$ -- and more. So there is no "the" equation...

Call the first point A and the second point B.
Let's try the obvious choice: $\mathbf{r}_0 = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{AB}$

$$\mathbf{r} = \begin{bmatrix} 5 \\ -6 \\ 4 \end{bmatrix} + t \begin{bmatrix} -3-5 \\ -1+6 \\ -10-4 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 4 \end{bmatrix} + t \begin{bmatrix} -8 \\ 5 \\ -14 \end{bmatrix}$$

nope, but note that only one option has the correct direction vector v up to sign & scale, which is option **A**, corresponding to the choice $\mathbf{r}_0 = \overrightarrow{OB}$ and $\mathbf{v} = \overrightarrow{BA}$

Question 5 (B)

A plane is perpendicular to the vector $\mathbf{n} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and passes through the point $(3,\ 2,\ -4)$.
The Cartesian equation of this plane is

Cartesian equation: $\mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n} \implies x - y + 3z = 3 - 2 - 12 = -11$ -- option **B**

Question 6 (D)

The shortest distance between the planes given by $5x - 4y - 12z = 10$ and $-15x + 12y + 36z = 20$ is

These two planes are parallel, so the question is just looking at the distance between them in the direction of the normal. The planes are

Plane 1: $5x - 4y - 12z = \begin{bmatrix} 5 \\ -4 \\ -12 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 10 \implies \mathbf{n} \cdot \mathbf{r} = 10 = d_1,$

Plane 2: $-15x + 12y + 36z = -3\begin{bmatrix} 5 \\ -4 \\ -12 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 20 \implies \mathbf{n} \cdot \mathbf{r} = -\dfrac{20}{3} = d_2$

where the normal vector (for both planes) is $\mathbf{n} = 5\mathbf{i} - 4\mathbf{j} - 12\mathbf{k}$.

Can just use the general formula: distance $= \dfrac{d_2 - d_1}{|\mathbf{n}|} = \dfrac{10-(-20/3)}{\sqrt{5^2+4^2+12^2}} = \dfrac{50}{3\sqrt{185}} = \dfrac{10\sqrt{5}}{3\sqrt{37}}$

If you don't know the formula, the easiest method (to derive it) is to take a point on each plane and construct the vector between them, then take the scalar resolute in the direction of the normal

A point on the first plane is $\mathbf{u} = (2,\ 0,\ 0)$ and on the second plane $\mathbf{v} = \left(-\frac{4}{3}, 0, 0\right)$.

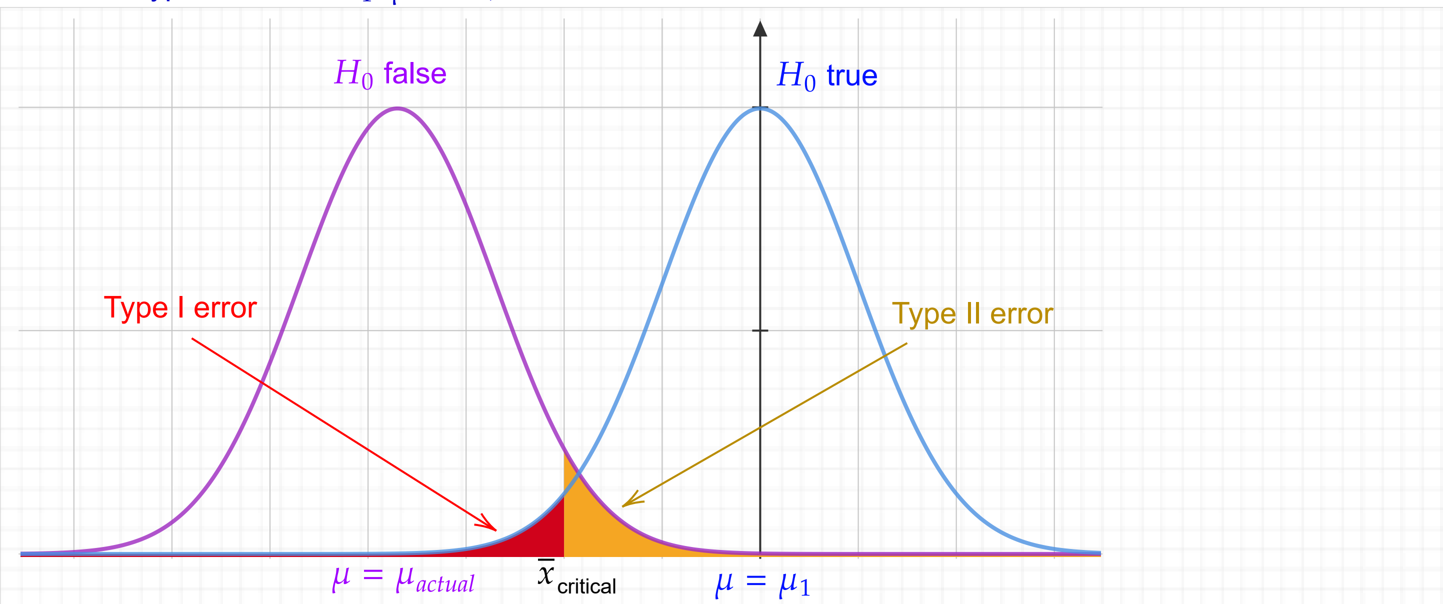
So the distance is $\left|(\mathbf{v}-\mathbf{u})\cdot\hat{\mathbf{n}}\right| = \dfrac{10}{3}\cdot\dfrac{5}{\sqrt{5^2+4^2+12^2}} = \dfrac{10\sqrt{5}}{3\sqrt{37}}$

Alternatively, choose a point on plane 2, then use the formula for distance between a point $\mathbf{r}_2 = (a, b, c)$ and a plane $\mathbf{n}\cdot\mathbf{r} = d_1$, distance $= \dfrac{\mathbf{n}\cdot\mathbf{r}_2 + d}{|\mathbf{n}|} = \dfrac{20\,(-4/3)+10}{\sqrt{5^2+4^2+12^2}} = \dfrac{-50}{3\sqrt{185}}$, which matches up to sign.

Question 7 (A)

The time taken by a machine to make electronic components varies normally with a mean of 20 seconds and a standard deviation of 2 seconds. After the machine is serviced, it is believed that the mean time taken has been reduced to 18.5 seconds with the standard deviation remaining the same.
A statistical test is proposed to check whether there is any evidence of a 1.5 second reduction in the mean time taken to make components. The test statistic will be the mean time taken to make a random sample of 16 such components. The type I error for the test will be α = 5% with a critical sample mean of 19.2 seconds. The type II error (β) for the test is closest to

One-tailed hypothesis test with sample size $n = 16$, and significance level $\alpha = 0.05$
Null Hypothesis H_0: $\mu = 20,\ \sigma = 2$
Alternate Hypothesis H_1: $\mu < 20,\ \sigma = 2$



$\alpha = 0.05$ gives the probability of Type I errors, which occur when H_0 is true, but the sample mean falls below the critical sample mean just by pure chance.

To find the critical sample mean, solve $\Pr(\overline{X} \leqslant \overline{x}_{\text{crit}} \mid H_0) = \alpha$ where $\overline{X} \sim \mathrm{N}\left(\mu = 20,\ \sigma_{\overline{X}} = \dfrac{2}{\sqrt{16}} = \dfrac{1}{2}\right)$, this [`invNorm(0.05,20,0.5)`] gives $\overline{x}_{\text{crit}} \approx 19.178$, matching the question's $\overline{x}_{\text{crit}} = 19.2$.

β is the expected rate of Type II errors given the suspected alternate population mean, which in this case is $\beta = \Pr(\overline{X} > \overline{x}_{\text{crit}} |\ \mu = 18.5)$ where where $\overline{X} \sim \mathrm{N}\left(\mu = 18.5,\ \sigma_{\overline{X}} = \frac{1}{2}\right)$.

We calculate [`normCdf(19.2,∞,18.5,0.5)`] $\beta \approx 0.080757$

Exam 2: Section B -- Extended Response

Question 1 (10 marks)

Why is this question in here? What is the new content?

a. Express $\left\{z : |z| = \left|z - 2\text{cis}\left(\frac{\pi}{4}\right)\right|, z \in \mathbb{C}\right\}$ in the form $y = ax + b$, where $a, b \in \mathbb{R}$ 2 marks

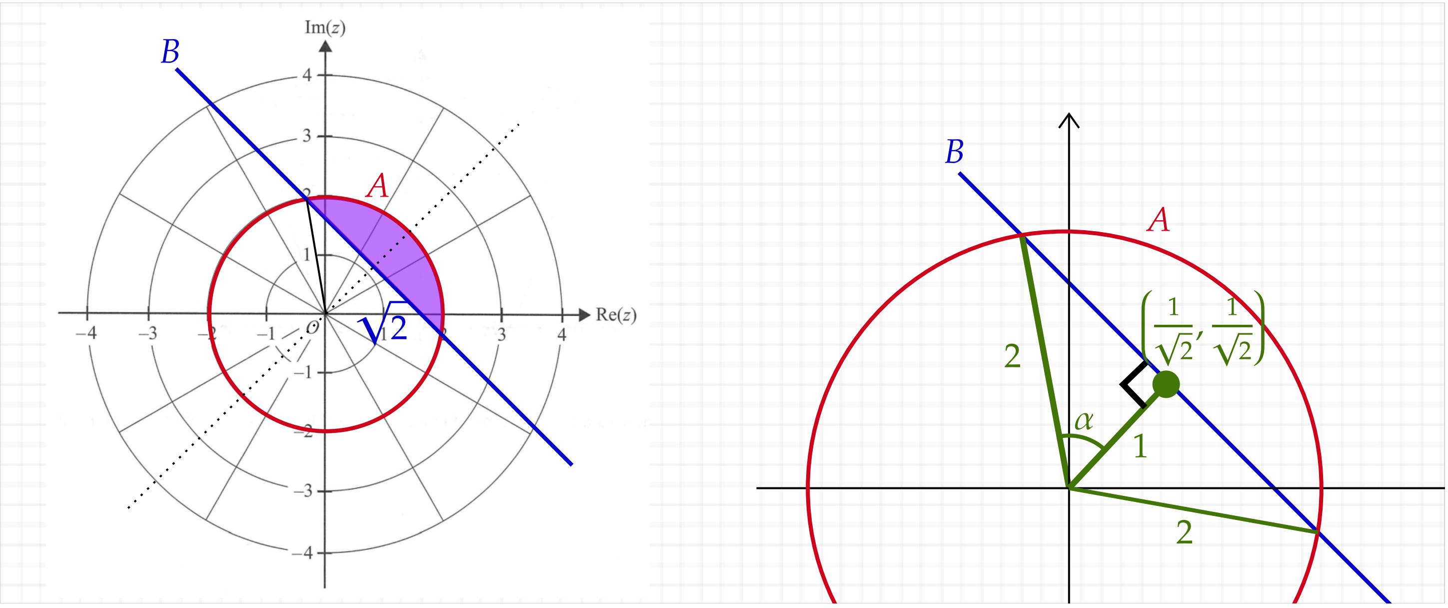
This is the perpendicular bisector of the line segment between 0 and $2\text{cis}\left(\frac{\pi}{4}\right) = \sqrt{2} + \sqrt{2}i$

So it has gradient -1 and goes through $\text{cis}\left(\frac{\pi}{4}\right)$: $y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$

$\implies y = (-1)x + \sqrt{2}$

Alternatively, $x^2 + y^2 = \left(x - \sqrt{2}\right)^2 + \left(y - \sqrt{2}\right)^2 \implies 0 = 4 - 2\sqrt{2}(x + y) \implies y = -x + \sqrt{2}$

b. On the Argand diagram below, sketch and label $A = \{z : z\bar{z} = 4, z \in \mathbb{C}\}$ and sketch and label $B = \left\{z : |z| = \left|z - 2\text{cis}\left(\frac{\pi}{4}\right)\right|, z \in \mathbb{C}\right\}$. Label the axis intercepts of the graph of B. 3 marks



c. On the Argand diagram in part **b.**, shade the region defined by

$\{z : z\bar{z} = 4, z \in \mathbb{C}\} \cap \left\{z : \text{Re}(z) + \text{Im}(z) \geqslant \sqrt{2}, z \in \mathbb{C}\right\}$ 1 mark

Note that $\text{Re}(z) + \text{Im}(z) = x + y = \sqrt{2}$ is just the line from part **a.**
Area is shaded in purple.

d. Find the area of the shaded region in part **c.** 2 marks

From the green triangle above, see that the angle subtended by the chord B makes on A

is $2\alpha = 2\arccos\left(\frac{1}{2}\right) = \frac{2\pi}{3}$. So the area of the segment is

$$\text{sector} - \text{triangle} = \frac{1}{2}\left(\frac{2\pi}{3}\right)(2)^2 - \frac{1}{2}\left(2^2\right)\sin\left(\frac{2\pi}{3}\right) = \boxed{\frac{4\pi}{3} - \sqrt{3}} \approx 2.45674$$

I didn't see this triangle earlier (thanks to MG for pointing it out in her solutions).

So here are some other, definitely not 2 mark approaches!

We can find the intersection points

$x + y = \sqrt{2} \Longrightarrow x^2 + 2xy + y^2 = 2$, sub in $x^2 + y^2 = 4 \Longrightarrow 4 + 2xy = 2 \Longrightarrow xy = -1$

$$\Longrightarrow x - \frac{1}{x} = \sqrt{2} \Longrightarrow x^2 - \sqrt{2}x - 1 = 0 \Longrightarrow x = \frac{\sqrt{2} \pm \sqrt{2+4}}{2} = \frac{1 \pm \sqrt{3}}{\sqrt{2}} = x_\pm$$

$$\Longrightarrow y_\pm = -\frac{\sqrt{2}}{1 \pm \sqrt{3}} \times \frac{1 \mp \sqrt{3}}{1 \mp \sqrt{3}} = \frac{\sqrt{2}\left(1 \mp \sqrt{3}\right)}{2} = \frac{1 \mp \sqrt{3}}{\sqrt{2}}$$

Then can integrate $\int_{x_-}^{2} \left(\sqrt{4 - x^2} - \left(\sqrt{2} - x\right)\right) dx + \int_{2}^{x_+} \left(-\sqrt{4 - x^2} - \left(\sqrt{2} - x\right)\right) dx \approx 2.45674$

(the exact result is horrible, but should be able to be simplified!)

Instead of the integration, use geometry. Leveraging the results above, find

The complex form for the intersection points: $z_\pm = x_\pm + iy_\pm = 2\text{cis}\left(-\frac{\pi}{12}\right)$ or $2\text{cis}\left(\frac{7\pi}{12}\right)$

Note: can do this in one step using the CAS

$$\left(\text{cSolve}\left(|z| = \left|z - 2\cdot e^{\frac{i\cdot\pi}{4}}\right| \text{ and } |z| = 2, z\right)\right) \blacktriangleright \text{Polar}$$

$$z = e^{\frac{-i\cdot\pi}{12}} \cdot 2 \text{ or } z = e^{\frac{7\cdot i\cdot\pi}{12}} \cdot 2$$

So the the angle subtended by the chord is $\frac{7\pi}{12} + \frac{\pi}{12} = \frac{2\pi}{3}$. Then can find the area of the segment.

Easier (?) way: the points of intersection satisfy $|z| = 2 = \left|z - 2\text{cis}\left(\frac{\pi}{4}\right)\right|$

$$\Longrightarrow z = 2\,\text{cis}\,\theta \Longrightarrow \left|\text{cis}\,\theta - \text{cis}\left(\frac{\pi}{4}\right)\right| = 1 \Longrightarrow 1 - \text{cis}\left(\theta - \frac{\pi}{4}\right) - \text{cis}\left(-\theta + \frac{\pi}{4}\right) + 1 = 1$$

$$\Longrightarrow \text{cis}\left(\theta - \frac{\pi}{4}\right) + \text{cis}\left(-\theta + \frac{\pi}{4}\right) = 1$$

$\Longrightarrow 2\cos\left(\theta - \frac{\pi}{4}\right) = 1 \Longrightarrow \theta - \frac{\pi}{4} = \pm\frac{\pi}{3} \Longrightarrow \theta = \frac{7\pi}{12}, -\frac{\pi}{12}$, then can find area of the segment again.

e. The elements of $\{z : z\bar{z} = 4, z \in \mathbb{C}\} \cap \left\{z : |z| = \left|z - 2\text{cis}\left(\frac{\pi}{4}\right)\right|, z \in \mathbb{C}\right\}$ provide two of the cube roots of w, where $w \in \mathbb{C}$.
Write down all three cube roots of w in the form $r\,\text{cis}(\theta)$ and find w in the form $a + ib$, where $a, b \in \mathbb{R}$. 2 marks

Why did they switch the description of B back to its original form again?

The two intersections labelled $z_\pm$ above are $\frac{2\pi}{3}$ radians apart, so are indeed the cube roots of some w.

Just cube any one of them to get $w = \left(2\text{cis}\left(-\frac{\pi}{12}\right)\right)^3 = 8\,\text{cis}\left(-\frac{\pi}{4}\right) = 8 \times \frac{1 - i}{\sqrt{2}} = 4\sqrt{2}\,(1 - i)$

So, $\boxed{w = 4\sqrt{2} - 4\sqrt{2}i}$

de Moivre's theorem says that the cube roots are

$$z = 2\text{cis}\left(-\frac{\pi}{12} + \frac{2\pi n}{3}\right),\ n = 0, 1, 2$$

$$\boxed{z = 2\ \text{cis}\left(-\frac{\pi}{12}\right),\ 2\ \text{cis}\left(\frac{7\pi}{12}\right),\ 2\ \text{cis}\left(\frac{5\pi}{4}\right) = 2\ \text{cis}\left(\frac{-3\pi}{4}\right)}$$

$$z = \sqrt{2}\left(\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i\right), \sqrt{2}\left(\frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i\right),\ -\sqrt{2}(1+i)\ \text{(don't req this last line)}$$

Which is also what the CAS says! `cSolve(z^3=4*√2-4*√2*i,z)`

Question 2 (10 marks)

logistic equation is new to the study design

In a certain region, 500 rare butterflies are released to maintain the species.
It is believed that the region can support a maximum of 30 000 such butterflies.
The butterfly population, P, t years after release can be modelled by the logistic differential equation $\frac{dP}{dt} = r\,P\left(1 - \frac{P}{30000}\right)$, where r is the growth rate of the population.

a. Use an integration technique and partial fractions to solve the differential equation above to find P in terms of r and t. 3 marks

It specifies we have to use partial fractions, as though that is also not an integration technique.
I guess it prevents us from completing the square and using arctan...

$$\int r\,dt = \int \frac{dP}{P\left(1 - \frac{P}{30000}\right)} = \int \frac{1}{P} + \frac{1}{30000 - P}dP$$

$$\Longrightarrow\ rt = \log_e\left(\frac{P}{30000 - P}\right) + \log_e(C),\ C = \text{constant of integration}$$

When $t = 0,\ P = 500 \Longrightarrow 0 = \log_e\left(\frac{500}{30000 - 500}\right) + \log_e(C) = \log_e\left(\frac{C}{59}\right) \Longrightarrow C = 59$

$$\Longrightarrow e^{rt} = \frac{59P}{30000 - P} \Longrightarrow 30000e^{rt} = 59P + Pe^{rt}$$

$$\Longrightarrow \boxed{P = \frac{30000e^{rt}}{59 + e^{rt}} = \frac{30000}{1 + 59e^{-rt}}}$$

b. Given that after 10 years there are 1930 butterflies in the population, find the value of r correct to two decimal places. 2 marks

$$r = \frac{1}{t}\log_e\left(\frac{59P}{30000 - P}\right) = \frac{1}{10}\log_e\left(\frac{59 \times 1930}{30000 - 1930}\right) \approx 0.14$$

c. What is the initial rate of increase of the population, correct to one decimal place? 1 mark

Initially $P(0) = 500$, so just sub into the original DE, $\dfrac{dP}{dt} = r\,P\left(1 - \dfrac{P}{30000}\right)$

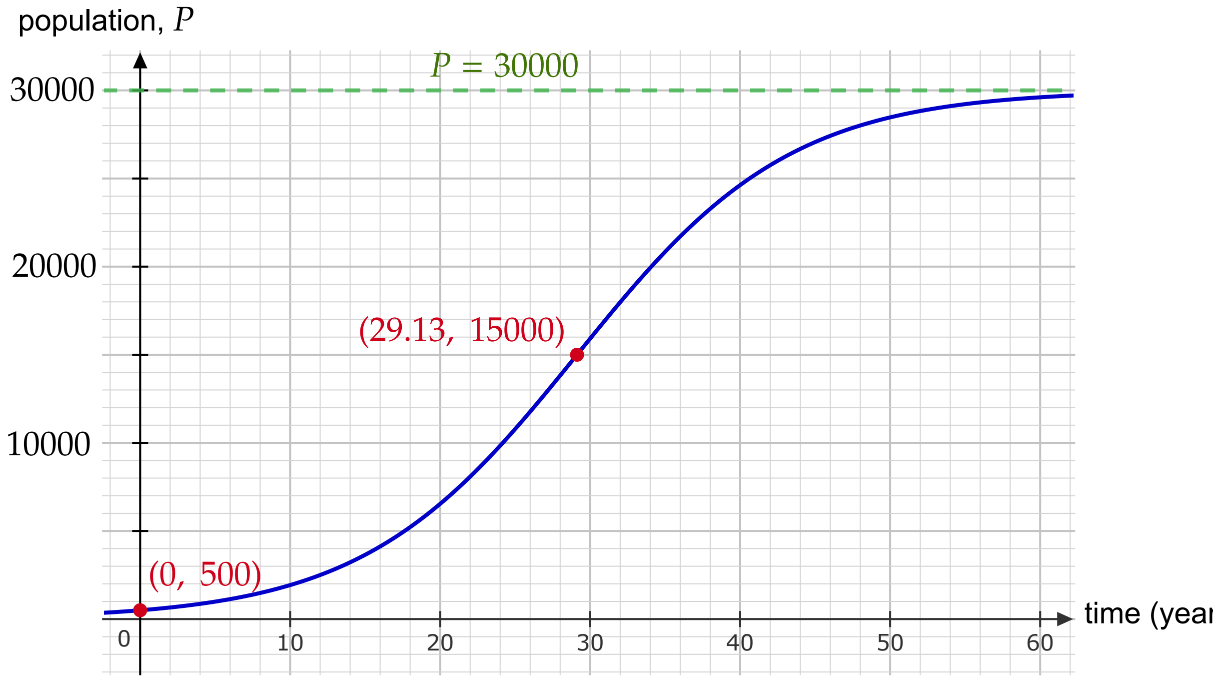
$$P'(0) = 0.1400356 \times 500\left(1 - \frac{500}{30000}\right) \approx 68.8508 \approx 68.9$$

Note, if you use $r = 0.14$, then you get $P'(0) \approx 68.83 \approx 68.8$ 🤷🏽‍♀️

d. After how many years will the population reach 10 000 butterflies? Give your answer correct to one decimal place. 1 mark

$$P(t) = 10000 \implies t = \frac{\log_e(29.5)}{r} \approx 24.2$$

e. Sketch the graph of P versus t on the axes below, showing the value of the vertical intercept. Label the point of fastest population growth as a coordinate pair (t, P), with t labelled correct to two decimal places, and label the asymptote with its equation. 3 marks



Question 3 (10 marks)

Parametric and vector equations of planes are new to the study design

A plane, Π_1, is described by the parametric equations

$$x = 1 + 2s + 3t$$
$$y = -2 - s - 2t$$
$$z = 2 - s + t$$

A second plane, Π_2, contains the point $P(1, 0, 3)$ and is parallel to the plane Π_1 .

a. Find a vector equation of the plane Π_1 in the form $\mathbf{r} = \mathbf{a} + s\,\mathbf{b} + t\,\mathbf{c}$ 2 marks

Can just read it directly off the question (why is it two marks?)

$$\mathbf{r}(s,t) = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} s + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} t$$

b. Hence, find a Cartesian equation of the plane Π_1. 2 marks

We need a normal to the plane, go with $\mathbf{b}\times\mathbf{c} = -\begin{bmatrix}2\\-1\\-1\end{bmatrix}\times\begin{bmatrix}3\\-2\\1\end{bmatrix} = -\begin{bmatrix}-1-2\\-3-2\\-4+3\end{bmatrix} = -\begin{bmatrix}-3\\-5\\-1\end{bmatrix}$

Actually, $\mathbf{n} = \mathbf{c}\times\mathbf{b} = -\mathbf{b}\times\mathbf{c} = \begin{bmatrix}3\\5\\1\end{bmatrix}$ is more convenient!

$\mathbf{r}\cdot\mathbf{n} = (\mathbf{a}+s\mathbf{b}+t\mathbf{c})\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$

$3x+5y+z = 3-10+2 = -5$ $\qquad \Pi_1 : \boxed{3x+5y+z=-5}$

c. Find a Cartesian equation of the plane Π_2. 1 mark

It's parallel, so $3x+5y+z=d$

Substitute the point: $3\times1+5\times0+1\times3=6$

$\Pi_2 : \boxed{3x+5y+z=6}$

d. **i.** Find the shortest distance between the planes Π_1 and Π_2 . 2 marks

$$\text{distance} = \frac{d_2-d_1}{|\mathbf{n}|} = \frac{6+5}{\sqrt{3^2+5^2+1^2}} = \frac{11}{\sqrt{35}} = \frac{11\sqrt{35}}{35}$$

Check: Choose a point on each plane, construct the vector between them and project in the direction of $\mathbf{n}$

$$\mathbf{a} = \begin{bmatrix}1\\-2\\2\end{bmatrix},\ \overrightarrow{OP} = \mathbf{p} = \begin{bmatrix}1\\0\\3\end{bmatrix},\ \mathbf{n} = \begin{bmatrix}3\\5\\1\end{bmatrix},\quad (\mathbf{a}-\mathbf{p})\cdot\hat{\mathbf{n}} = \frac{-11\sqrt{35}}{35}\ \checkmark$$

ii. Hence, find the coordinates of point Q, which is the reflection of point P in the plane Π_1 , as shown in the diagram above (not shown). 3 marks

To get to Q, start at P (on plane Π_2) then travel twice the distance from P to Π_1

$\overrightarrow{OQ} = \mathbf{p} + 2((\mathbf{a}-\mathbf{p})\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} = -\frac{31}{35}\mathbf{i} - \frac{22}{7}\mathbf{j} + \frac{83}{35}\mathbf{k}$, so the coordinates are $\left(-\frac{31}{35}, -\frac{22}{7}, \frac{83}{35}\right)$

This seems messy...

Question 4 (10 marks)

a. Find the shortest distance between the two parallel lines given by

$\mathbf{r}(t) = 4\mathbf{i}+2\mathbf{j}+\mathbf{k}+t\,(-\mathbf{i}+\mathbf{j}+3\mathbf{k})$, where $t\in\mathbb{R}$, and

$\mathbf{r}(s) = 5\mathbf{i}+4\mathbf{j}-2\mathbf{k}+s\,(-\mathbf{i}+\mathbf{j}+3\mathbf{k})$, where $s\in\mathbb{R}$. 3 marks

Parallel lines, along vector $\mathbf{v} = -\mathbf{i}+\mathbf{j}+3\mathbf{k}$

just choose a point on each line and take the perpendicular projection....

Choose $s=t=0$ points, vector between them is $\mathbf{d} = (5\mathbf{i}+4\mathbf{j}-2\mathbf{k})-(4\mathbf{i}+2\mathbf{j}+\mathbf{k}) = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$

$$(\mathbf{d}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} = \frac{(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}+3\mathbf{k})}{(-\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}+3\mathbf{k})}(-\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

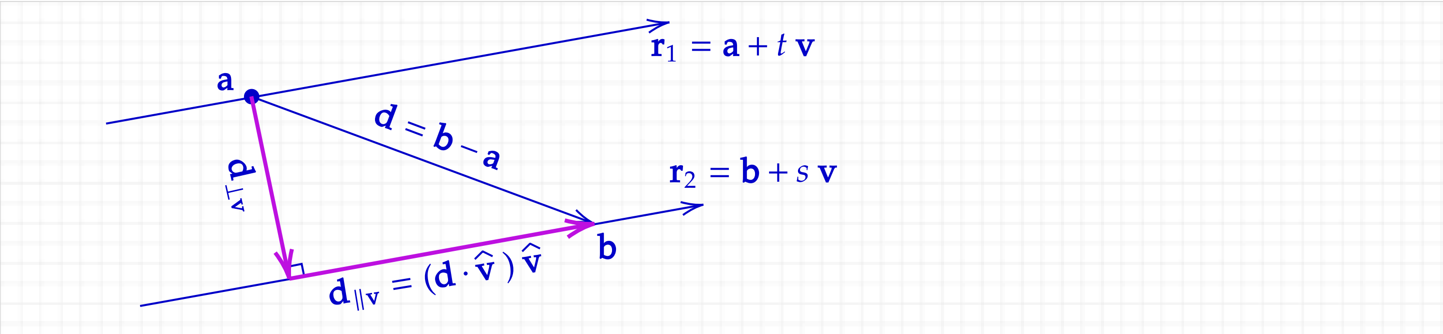
$$= \frac{-1+2-9}{1+1+9}(-\mathbf{i}+\mathbf{j}+3\mathbf{k}) = \frac{-8}{11}(-\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

$$\mathbf{d}-(\mathbf{d}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}+\frac{8}{11}(-\mathbf{i}+\mathbf{j}+3\mathbf{k}) = \frac{3}{11}\mathbf{i}+\frac{30}{11}\mathbf{j}-\frac{9}{11}\mathbf{k}$$

$$\text{distance} = \left|\frac{3}{11}\mathbf{i}+\frac{30}{11}\mathbf{j}-\frac{9}{11}\mathbf{k}\right| = \frac{1}{11}\sqrt{9+900+81} = \frac{\sqrt{990}}{11} = 3\sqrt{\frac{10}{11}}$$

In general, the distance between two parallel lines is

$\mathbf{r}_1 = \mathbf{a}+t\,\mathbf{v},\ \mathbf{r}_2 = \mathbf{b}+s\,\mathbf{v},\quad \text{dist}(\mathbf{r}_1,\mathbf{r}_2) = |(\mathbf{b}-\mathbf{a})-((\mathbf{b}-\mathbf{a})\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}|$



Alternatively: note that $|(\mathbf{b}-\mathbf{a})\times\hat{\mathbf{v}}|$ is the area of the parallelogram with "base" length one and height equal to the perpendicular distance.
So,

$$\text{dist} = \left|(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\times(-\mathbf{i}+\mathbf{j}+3\mathbf{k})\frac{1}{\sqrt{11}}\right| = \frac{1}{\sqrt{11}}|(6+3)\mathbf{i}+(3-3)\mathbf{j}+(1+2)\mathbf{k}| = \frac{\sqrt{90}}{\sqrt{11}} = 3\sqrt{\frac{10}{11}}\ \checkmark$$

This is not that much of a surprise, as in general we always have $|d-(d\cdot\hat{v})\hat{v}| = |d\times\hat{v}|$

$|d-(d\cdot\hat{v})\hat{v}|^2 = |d|^2-(d\cdot\hat{v})^2,\ \ |d\times\hat{v}|^2 = (d\cdot d)(\hat{v}\cdot\hat{v})-(d\cdot\hat{v})(d\cdot\hat{v}) = |d|^2-(d\cdot\hat{v})^2$

b. Given that the lines with equations $\mathbf{r}_1(t) = \mathbf{i}-3\mathbf{j}+6\mathbf{k}+t\,(3\mathbf{i}+5\mathbf{j}-a\mathbf{k})$, where $t\in\mathbb{R}$, and $\mathbf{r}_2(s) = -6\mathbf{i}+2\mathbf{j}+\mathbf{k}+s\,(4\mathbf{i}-10\mathbf{j}+6\mathbf{k})$, where $s\in\mathbb{R}$, intersect, find the value of a and the point of intersection. 4 marks

Setting $r_1(t) = r_2(s)$ gives three equations for three unknowns $\Rightarrow s = 1,\ t = -1,\ \boxed{a = 1}$

So the point of intersection is $(-2,\ -8,\ 7)$

An easy 4 CAS marks...
Maybe worth 4 if solving by hand. Solve first two for s, t and then sub to find a from the third.

Alternatively: Minimum distance between skew lines occurs between two points P_1 and P_2 s.t. $\overrightarrow{P_1P_2}$ is at right angles to both lines. Let $\mathbf{r}_1 = \mathbf{u}_1+t\,\mathbf{v}_1,\ \mathbf{r}_2 = \mathbf{u}_2+s\,\mathbf{v}_2$, then $\overrightarrow{P_1P_2}\parallel\mathbf{v}_1\times\mathbf{v}_2$ and so $\text{distance}_{1,2} = P_1P_2 = \overrightarrow{P_1P_2}\cdot\hat{\mathbf{n}}$ where $\mathbf{n} = \mathbf{v}_1\times\mathbf{v}_2$.

Note that $\overrightarrow{OP_1} = \mathbf{r}_1(t_*)$ for some t_* and similarly $\overrightarrow{OP_2} = \mathbf{r}_2(s_*)$ for some s_*.

Then distance $= \left(\overrightarrow{OP_2}-\overrightarrow{OP_1}\right)\cdot\hat{\mathbf{n}} = (\mathbf{u}_2+s_*\,\mathbf{v}_2-\mathbf{u}_1-t_*\,\mathbf{v}_1)\cdot\hat{\mathbf{n}} = (\mathbf{u}_2-\mathbf{u}_1)\cdot\hat{\mathbf{n}}$

If the two lines intersect, then their distance is zero: $(\mathbf{u}_2-\mathbf{u}_1)\cdot\hat{\mathbf{n}} = (\mathbf{u}_2-\mathbf{u}_1)\cdot\mathbf{n} = 0$

Let's apply that to our question. $\mathbf{n} = (3\mathbf{i}+5\mathbf{j}-a\mathbf{k})\times(4\mathbf{i}-10\mathbf{j}+6\mathbf{k}) = (30-10a,\ -4a-18,\ -50)$

$(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{n} = 50(a-1) \implies a = 1$
Can then find the intersection as normal.

c. The line with equation $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} - 5\mathbf{k} + t(4\mathbf{i} + b\mathbf{j} + 2\mathbf{k})$, where $t, b, \in \mathbb{R}$, is parallel to the plane with equation $2x - 3y - z = 2$.
Find the value of b and the shortest distance of the line from the plane. 3 marks

If the line is parallel to the plane, then it is perpendicular to the plane's normal: $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$
$0 = (4\mathbf{i} + b\mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 8 - 3b - 2 = 6 - 3b \implies \boxed{b = 2}$

The distance is then the usual projection of the displacement vector of any point on the line to any point on the plane:

distance $= ((\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - (\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})) \cdot \hat{\mathbf{n}} = (\mathbf{j} - 5\mathbf{k}) \cdot \dfrac{(2\mathbf{i} - 3\mathbf{j} - \mathbf{k})}{\sqrt{14}} = \dfrac{-3+5}{\sqrt{14}} = \dfrac{2}{\sqrt{14}}$ units

Question 5 (10 marks)

The whole shebang - cross products, planes, lines, distances...

a. Given the points $A(1,0,2)$, $B(2,3,0)$ and $C(1,2,1)$

i. find the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ 1 mark

$\overrightarrow{AB} = (2-1)\mathbf{i} + (3-0)\mathbf{j} + (0-2)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
$\overrightarrow{AC} = (1-1)\mathbf{i} + (2-0)\mathbf{j} + (1-2)\mathbf{k} = 0\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (2\mathbf{j} - \mathbf{k}) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

ii. show that the Cartesian equation of the plane Π_1, containing the points A, B and C, is $x + y + 2z = 5$. 1 mark

Equation of the plane: $\mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}$. Choose $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, $\mathbf{r}_0 = \overrightarrow{OA}$

$$\implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \implies x + y + 2z = 5$$

b. A second plane, Π_2 , has the Cartesian equation $x - y - z = 0$.
L is the line of intersection of the planes Π_1 and Π_2 .

i. Find the coordinates of the point P, where L crosses the y-z plane. 1 mark

$$x = 0 \implies \begin{cases} y + 2z = 5 \\ y + z = 0 \end{cases} \implies \begin{cases} z = 5 \\ y = -5 \end{cases} \implies P = (0, -5, 5)$$

ii. Hence, find the vector equation of the line L. 2 marks

The line is perpendicular to normal vectors of both planes, so it is parallel to

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

and it goes through the point P, so its equation is

$$\begin{aligned} L:\ r(t) &= \overrightarrow{OP} + t\,\mathbf{v} \\ &= 0\mathbf{i} - 5\mathbf{j} + 5\mathbf{k} + t\,(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \\ &= t\,\mathbf{i} + (3t - 5)\mathbf{j} + (5 - 2t)\mathbf{k} \end{aligned}$$

iii. Find the distance from the point A to the plane Π_2 . 2 marks

Choose the origin as a point on the plane Π_2: $O(0,\ 0,\ 0)$

Then a vector from the plane to A is $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{k}$

A normal vector for Π_2 is $\mathbf{n}_2 = \mathbf{i} - \mathbf{j} - \mathbf{k}$

Then dist$(A, \Pi_2) = |(\overrightarrow{OA} \cdot \hat{\mathbf{n}}_2)\,\hat{\mathbf{n}}_2| = \dfrac{1}{\sqrt{3}}$

iv. Find the distance from the point A to the line L. 3 marks

dist$(A, L) = \min(|r(t) - \overrightarrow{OA}|)$

$|r(t) - \overrightarrow{OA}|^2 = 14t^2 - 44t + 35 = 14\left(t - \frac{11}{7}\right)^2 + \frac{3}{7}$

So the minimum value is $\sqrt{\dfrac{3}{7}}$ and it occurs when $t = \frac{11}{7}$.

Alternatively: Choose a point on the line; $\mathbf{r}(0) = \overrightarrow{OP}$, then the distance is

$$|\overrightarrow{AP} \times \hat{\mathbf{v}}| = \frac{1}{\sqrt{14}} \left| \left(\begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right) \times \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right| = \frac{1}{\sqrt{14}} \left| \begin{bmatrix} -1 \\ -5 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right| = \frac{1}{\sqrt{14}} \left| \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right| = \sqrt{\frac{3}{7}} \quad \checkmark$$

This can also be obtained as the length of the perpendicular projection $|\overrightarrow{AP} - (\overrightarrow{AP} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}}|$.

Question 6 (11 marks)

The position vector $\mathbf{r}_S(t)$, from an origin O, of a sparrow t seconds after being sighted is modelled by $\mathbf{r}_S(t) = 23t\,\mathbf{i} + 5t\,\mathbf{j} + \left(4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2}\right)\mathbf{k},\ t \geqslant 0$, where $\mathbf{i}$ is a unit vector in the forward direction, $\mathbf{j}$ is a unit vector to the left and $\mathbf{k}$ is a unit vector vertically up. Displacement components are measured in centimetres.

a. Find the value of t when the sparrow first lands on the ground. 2 marks

$$0 = 4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2} \Longrightarrow \sin\left(\frac{\pi t}{2}\right) = -1 \Longrightarrow \frac{\pi t}{2} = \frac{3\pi}{2} + 2n\pi,\ n \in \{0, 1, 2,\ \dots\}$$

So the sparrow first lands on the ground when 3 seconds.

b. Find the distance of the sparrow from O when it first lands. Give your answer correct to one decimal place. 2 marks

$$|\mathbf{r}_S(2)| = \sqrt{(3\times 23)^2 + (3\times 5)^2 + 0^2} = 3\sqrt{23^2 + 5^2} = 3\sqrt{554} \approx 70.6 \text{ cm}$$

c. Find the maximum flight speed, in centimetres per second, of the sparrow. Give your answer correct to one decimal place. 2 marks

$$\mathbf{v}_S(t) = \frac{d}{dt}\mathbf{r}_S(t) = 23\,\mathbf{i} + 5\,\mathbf{j} + 2\pi\sqrt{2}\cos\left(\frac{\pi t}{2}\right)\mathbf{k}$$

This has maximum magnitude when $\cos(\pi t/2) = \pm 1$

max speed $= |23\,\mathbf{i} + 5\,\mathbf{j} + 2\pi\sqrt{2}\,\mathbf{k}| \approx 25.1586 \approx 25.2$ cm s^{-1}

(Could of course, also maximise $|\mathbf{v}_S(t)|$ using calculus)

A second bird, a miner, flies such that its velocity vector $\mathbf{v}_M(t)$, relative to the same origin O, is modelled by $\mathbf{v}_M(t) = 6\mathbf{i} + \mathbf{j} + \left(\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)\right)\mathbf{k}$, $t \geqslant 0$, where velocity components are measured in centimetres per second.

Note: Indian Myna and Noisy Miner birds are not related (https://austraflora.com/2016/05/19/indian-myna-noisy-miner/)

d. Given that the miner has an initial position vector of $10\,\mathbf{i} + 4\,\mathbf{j} + 4\sqrt{2}\,\mathbf{k}$, show that its position vector at time t seconds is given by $\mathbf{r}_M(t) = (6t+10)\mathbf{i} + (t+4)\mathbf{j} + \left(\sin\left(\frac{\pi t}{6}\right) + 4\sqrt{2}\right)\mathbf{k}$ 2 marks

Fairly straight forward integration:

$$\mathbf{r}_M(t) = \mathbf{r}_M(0) + \int_0^t \mathbf{v}_M(\tau)d\tau = 10\,\mathbf{i} + 4\,\mathbf{j} + 4\sqrt{2}\,\mathbf{k} + \left[6\tau\mathbf{i} + \tau\mathbf{j} + \sin\left(\frac{\pi\tau}{6}\right)\mathbf{k}\right]_0^t$$

$$= (10+6t)\mathbf{i} + (4+t)\mathbf{j} + \left(4\sqrt{2} + \sin\left(\frac{\pi t}{6}\right)\right)\mathbf{k}$$

e. The sparrow and the miner are at the same position at different times.
Find the coordinates of this position and the times at which each bird is at this position. 3 marks

Need to find $s, t \geqslant 0$ such that $\mathbf{r}_S(s) = \mathbf{r}_M(t)$

can just solve the vector equation directly in the CAS, but if you want to do it by hand:

$$\implies \begin{cases} 23s = 10 + 6t & \textcircled{1} \\ 5s = 4 + t & \textcircled{2} \\ 4\sqrt{2}\sin\left(\frac{\pi s}{2}\right) + 4\sqrt{2} = 4\sqrt{2} + \sin\left(\frac{\pi t}{6}\right) & \textcircled{3} \end{cases}$$

$6\textcircled{2} - \textcircled{1} \implies 7s = 14 \implies s = 2$

Sub into $\textcircled{2} \implies t = 5(2) - 4 = 6$

Check it's actually a solution by substituting into $\textcircled{3}$: $4\sqrt{2}\sin(\pi) = \sin(\pi) = 0$ ✓

So, at time = 2 for the sparrow and 6 for the miner, the birds are at the same position:

$$\mathbf{r}_S(2) = \mathbf{r}_M(6) = 46\mathbf{i} + 10\mathbf{j} + 4\sqrt{2}\,\mathbf{k} \sim \left(46,\ 10,\ 4\sqrt{2}\right)$$