SPECIALIST MATHEMATICS

Written examination 1

TSSM™
Creating VCE Success

2023 Trial Examination

SOLUTIONS

Question 1 (6 marks)

a. (1 mark)

Answer:

$\underset{\sim}{a} = -\underset{\sim}{\text{i}} + \text{c}\,\underset{\sim}{\text{j}} + 3\,\underset{\sim}{\text{k}}$

$|\underset{\sim}{a}| = \sqrt{1 + c^2 + 9} = \sqrt{14}$

$c^2 + 10 = 14$
$c^2 = 4$
$c = \pm 2$

1 A

b. (2 marks)

Answer:
$-\underset{\sim}{i} + c\underset{\sim}{j} + 3\underset{\sim}{k} = n(3\underset{\sim}{i} + 6\underset{\sim}{j} + d\underset{\sim}{k})$

1 W

Equating coefficients:
$-1 = 3n \ldots (1)$
$c = 6n \ldots (2)$
$3 = nd \ldots (3)$
$n = -\frac{1}{3}$
$c = -2$
$d = -9$

1 A

c. (3 marks)

Answer:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & c & 3 \\ 3 & 6 & d \end{vmatrix}$$

$(cd - 18)\boldsymbol{i} - (-d - 9)\boldsymbol{j} + (-6 - 3c)\boldsymbol{k} = -2\boldsymbol{i} + \boldsymbol{j} + f\boldsymbol{k}$

1 W

$cd - 18 = -2 \ldots (1)$

$d + 9 = 1 \ldots (2)$

$-6 - 3c = f \ldots (3)$

1 W

$c = -2, d = -8, f = 0$

1 A

Question 2 (3 marks)

Answer:

$f(x) = x \tan^{-1}(\frac{x}{2})$

$f'(x) = \tan^{-1}(\frac{x}{2}) + \frac{2x}{4 + x^2}$

2 W

$f'(-2) = \tan^{-1}(-1) + \frac{-4}{4 + 4}$

$f'(-2) = -\frac{\pi}{4} - \frac{1}{2}$

1 A

Question 3 (3 marks)

Answer:

$W = x^2 + x, x \in N.$
$W(1) = 1^2 + 1 = 2$
So, W is even for the smallest Natural number.

1 W

Assume the proposition is true for $x = k,\ k \in N.$
So, $k^2 + k, k \in N$ is an even number.
$k^2 + k = 2n, n \in N$

1 W

Test the proposition for $x = k + 1,\ k \in N.$
So, $W = (k + 1)^2 + (k + 1)$

$W = (k^2 + 2k + 1) + (k + 1) = k^2 + 3k + 2$
$k^2 + 3k + 2 = (k^2 + k) + 2k + 2$
$(k^2 + k) + 2k + 2 = 2n + 2k + 2 = 2(n + k + 1), n, k \in N$
Since W has a common factor of 2, and $n, k \in N$, W must be an even number.

1 A

Question 4 (6 marks)

a. (2 marks)

$\sec^2(2x) = 2$

$\cos 2x = \pm\frac{1}{\sqrt{2}}$

1 W

$2x = \frac{\pi}{4}, \frac{3\pi}{4}$

$x = \frac{\pi}{8}, \frac{3\pi}{8}$

1 A

b. (2 marks)

$\sin 4x = 2 \times \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right)$ or $\sin 4x = 2 \times \left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right)$

$= \pm 1$

1 W

$\text{cosec}\, 4x = \frac{1}{\pm 1} = \pm 1$
(Or use $cosec\ 4x = \frac{1}{\sin\frac{\pi}{2}}$ or $\frac{1}{\sin\frac{3\pi}{2}}$)

1 A

c. (2 marks)

$1 + \cot^2 4x = cosec^2 4x$

$1 + \cot^2 4x = (-1)^2$

1 W

$\cot^2 4x = 0$

$\cot 4x = 0$

(Or use $cot\ 4x = \frac{1}{\tan\frac{3\pi}{2}}$)

1 A

Question 5 (6 marks)

a. (3 marks)

$\int_0^1 (x \tan^{-1} x)dx$

Integration by parts

$\int (u)dv = uv - \int (v)\ du$

Let $u = \tan^{-1} x\,, dv = x\ dx$

1 W

$\frac{du}{dx} = \frac{1}{1+x^2}\,, \frac{dv}{dx} = x, v = \frac{1}{2}x^2$

$\int (x \tan^{-1} x)dx = \frac{1}{2}x^2.\tan^{-1} x - \int \left(\frac{1}{2}x^2\right)\ du$

$= \frac{1}{2}x^2.\tan^{-1} x - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{1+x^2}\right)dx$

$= \frac{1}{2}x^2.\tan^{-1} x - \frac{1}{2}\int \left(\frac{x^2}{1+x^2}\right)dx$

$= \frac{1}{2}x^2.\tan^{-1} x - \frac{1}{2}\int \left(\frac{1+x^2-1}{1+x^2}\right)dx$

$= \frac{1}{2}x^2.\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$

$= \frac{1}{2}x^2.\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x$

1 W

$= \left[\frac{1}{2}x^2.\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x\right]_0^1 = \frac{1}{2}\tan^{-1}(1) - \frac{1}{2} + \frac{1}{2}\tan^{-1}(1) - 0 + 0 - \frac{1}{2}\tan^{-1}(0)$

$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$

1 A

b. (3 marks)

$$\int_0^{\frac{1}{2}} \left(\frac{2x+1}{\sqrt{1-x^2}}\right)dx$$

$$= \int_0^{\frac{1}{2}} \left(\frac{2x}{\sqrt{1-x^2}}\right)dx + \int_0^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

1 W

Let $u = 1 - x^2$

$$\frac{du}{dx} = -2x$$

$$= \int_1^{\frac{3}{4}} \left(\frac{-\frac{du}{dx}}{\sqrt{u}}\right)dx + \int_0^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \int_{\frac{3}{4}}^{1} \left(u^{-\frac{1}{2}}\right)du + \int_0^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \left[2u^{\frac{1}{2}}\right]_{0.75}^{1} + \left[\sin^{-1} x\right]_0^{0.5}$$

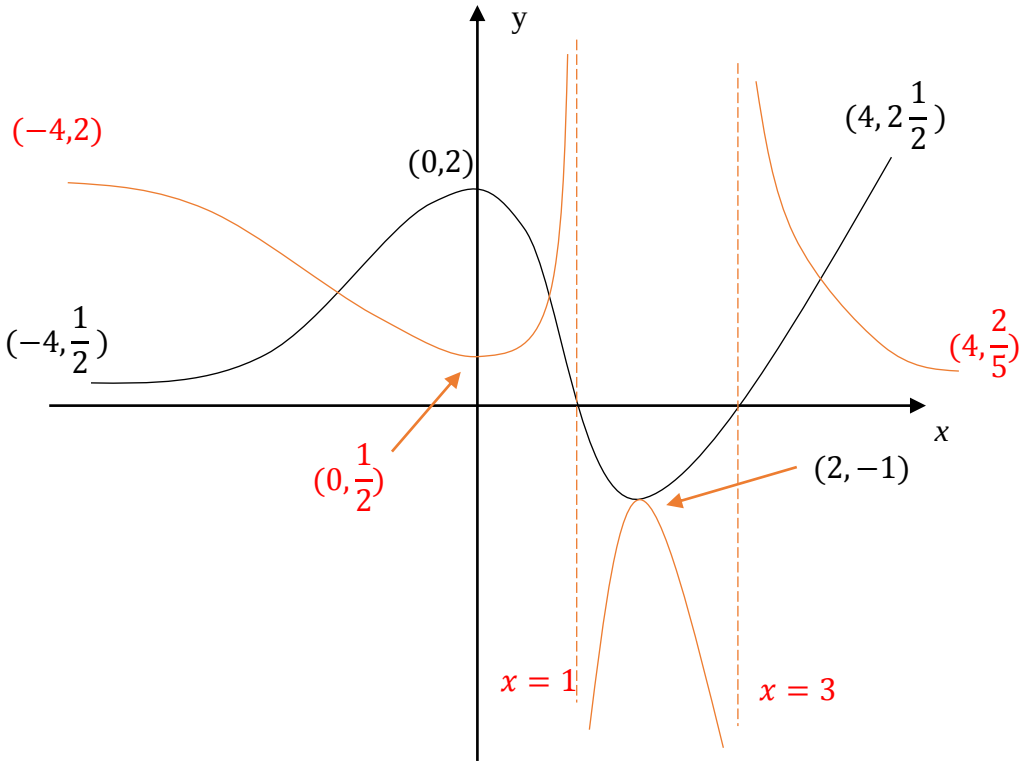
1 W

$$= 2 - \sqrt{3} + \sin^{-1}(0.5) - 0$$

$$= 2 - \sqrt{3} + \frac{\pi}{6}$$

1 A

Question 6 (3 marks)



Shape 1, Asymptotes 1, Points 1

Question 7 (6 marks)

a. (2 marks)

$z = -2\sqrt{2} + 2\sqrt{2}i$

$|z| = \sqrt{8+8} = 4$

1 W

$\tan\theta = -1\ (Quadrant\ 2)$

$\theta = \frac{3\pi}{4}$

$z = 4\ cis(\frac{3\pi}{4})$

1 A

b. (2 marks)

$z^2 = 4\ cis(\frac{3\pi}{4})$

$z_1 = 2\ cis(\frac{3\pi}{8})$

1 W

$z_2 = -2\ cis\left(\frac{3\pi}{8}\right) = 2\ cis(-\frac{5\pi}{8})$

1 A

c. (2 marks)

$\cos 2\theta = 2cos^2\theta - 1$

$\cos\left(\frac{3\pi}{4}\right) = 2\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 - 1$

$-\frac{\sqrt{2}}{2} = 2\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 - 1$

$2\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$

$\cos\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$

1 W

$\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 + \left(\sin\left(\frac{3\pi}{8}\right)\right)^2 = 1$

$\frac{2-\sqrt{2}}{4} + \left(\sin\left(\frac{3\pi}{8}\right)\right)^2 = 1$

$\left(\sin\left(\frac{3\pi}{8}\right)\right)^2 = 1 - \frac{2-\sqrt{2}}{4} = \frac{2+\sqrt{2}}{4}$

$\sin\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$

$z_1 = 2\ cis\left(\frac{3\pi}{8}\right) = \left(\sqrt{2-\sqrt{2}}\right) + \left(\sqrt{2+\sqrt{2}}\right)i$

1 A

Question 8 (4 marks)

$2x^3 - xy^2 = y$

$\frac{d}{dx}(2x^3 - xy^2) = \frac{d}{dx}(y)$

$6x^2 - y^2 - x\frac{d}{dx}(y^2) - \frac{dy}{dx} = 0$

$6x^2 - y^2 - 2xy\frac{dy}{dx} - \frac{dy}{dx} = 0$

1 W

$\frac{dy}{dx}(-2yx - 1) = y^2 - 6x^2$

$\frac{dy}{dx} = \frac{6x^2 - y^2}{2yx + 1}$

1 W

When $x = -1$

$-2 + y^2 = y$

$y^2 - y - 2 = 0$

$(y - 2)(y + 1) = 0$

$y = 2, y = -1$

$y = 2$

1 W

$\frac{dy}{dx} = \frac{6 - 4}{-4 + 1} = -\frac{2}{5}$

$y = -1$

$\frac{dy}{dx} = \frac{6 - 1}{2 + 1} = \frac{5}{3}$

1 A

Question 9 (3 marks)

Answer:

$E(\bar{x}) = 200$

$sd(\bar{x}) = \frac{2.5}{\sqrt{25}} = 0.5$

1 W

$\Pr(Z \leq 3.00) \approx 0.999$

$\Pr(Z \geq 3.00) \approx 0.001$

1 W

$\Pr(X \geq 200 + 3.00 \times 0.5) \approx 0.001$

So, 0.1% **1 A**