Student Name: ____

SPECIALIST MATHEMATICS 2023

Unit 4
Key Topic Test 4 – Differential equations
Technology Active

Recommended writing time*: 45 minutes
Total number of marks available: 30 marks

QUESTION BOOK

* The recommended writing time is a guide to the time students should take to complete this test. Teachers may wish to alter this time and can do so at their own discretion.

Conditions and restrictions
- Students are permitted to bring into the room for this test: pens, pencils, highlighters, erasers, sharpeners and rulers, one CAS and bound reference book
- Students are NOT permitted to bring into the room for this test: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 9 pages.

Instructions
- Print your name in the space provided on the top of the front page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic communication devices into the room for this test.

Instructions for Section A

- All questions are worth one mark.
- Answer all questions by circling the correct response.
- Marks are not deducted for incorrect answers.
- No marks will be awarded if more than one answer is completed for any question.

Question 1

Euler's method is used to find an approximate solution to the differential equation $\frac{dy}{dx} = 3x^2$. Given that $x_0 = 1,\ y_0 = 1\ \ and\ y_2 = 3.421,$ the value of the step size h is

A. 0.1

B. 0.2

C. 0.3

D. 0.4

E. 0.5

Question 2

The solution of the differential equation $\frac{dy}{dx} = y^2,$ where $y = 2$ when $x = 0$, is

A. $y = -x$

B. $y = 1 + 2x$

C. $y = \frac{1}{2x-1}$

D. $y = \frac{2}{1-2x}$

E. $y = x$

Question 3

Water is being heated in a kettle.

The rate of increase of temperature of water at any time t is modelled by the differential equation $\frac{d\theta}{dt} = k(120 - \theta), where\ k\ is\ a\ positive\ constant.$

The temperature at any time *t* can be modelled by the equation

A. $\theta = -ae^{-kt},\ a\epsilon R$

B. $\theta = 120a - e^{-kt},\ a\epsilon R$

C. $\theta = ae^{-kt} - 120,\ a\epsilon R$

D. $\theta = 120 - ae^{kt},\ a\epsilon R$

E. $\theta = 120 - ae^{-kt},\ a\epsilon R$

Question 4

A solution to the differential equation $\frac{dy}{dx} = \frac{1}{y\sin^2(x)}$ can be obtained from

A. $\int y\,dy = \int cosec^2(x)\,dx$

B. $\int y\,dy = \int \sin^2(x)\,dx$

C. $\int \frac{1}{y}dy = \int \sin^2(x)\,dx$

D. $\int \frac{1}{y}dy = \int cosec^2(x)\,dx$

E. $\int y\,dy = \int \frac{1}{cosec^2\,(x)}dx$

Question 5

Using Euler's rule, the differential equation $\frac{dy}{dx} = \cos(x)\,,\ y = 2\ when\ x = \frac{\pi}{3}$ with a step size of 0.1, gives y_1 as

A. 2.10

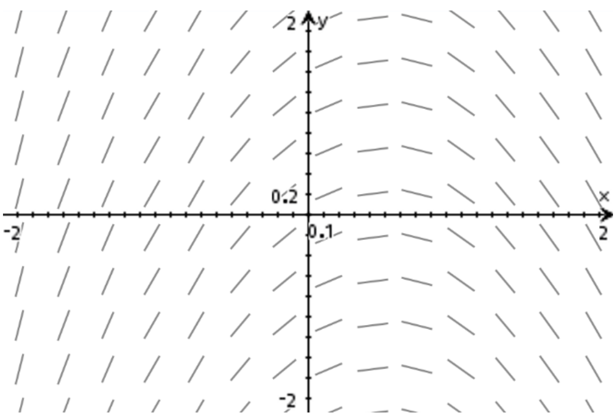
B. 2.05

C. 1.25

D. 1.20

E. 0.40

Question 6



The differential equation that best represents the direction field above is

A. $\frac{dy}{dx} = -\frac{2y}{x}$

B. $\frac{dy}{dx} = 2x$

C. $\frac{dy}{dx} = \frac{y}{2x}$

D. $\frac{dy}{dx} = 1 - 2x$

E. $\frac{dy}{dx} = 2x + 1$

Question 7

The differential equation that relates $\frac{dh}{dt}$ with depth h of a container is given by $\frac{dh}{dt} = -\frac{1}{50h}$.

The family of solutions for this differential equation is

A. $h = -25t + k$

B. $h = -\frac{t}{25} + k$

C. $h^2 = \frac{t}{25} + k$

D. $h^2 = -\frac{t}{25}$

E. $h^2 = -\frac{t}{25} + k$

Question 8

The amount of a salt Q g in a tank at time t minutes is given by the differential equation

$$\frac{dQ}{dt} = 6 - \frac{3}{3-t}.$$

The amount of salt, Q g in the tank after t minutes can be calculated by

A. $Q = \int \frac{-3}{3-t}\,dt$

B. $Q = \int \frac{21-6t}{3-t}\,dt$

C. $Q = \int \frac{15-6t}{3-t}\,dt$

D. $Q = \int \frac{2t}{3-t}\,dt$

E. $Q = \int \frac{15+6t}{3-t}\,dt$

SECTION B

Instructions for Section B

- Answer each question in the space provided.
- Please provide appropriate workings and use exact answers unless otherwise specified.

Question 1 (6 marks)

Consider the differential equation

$\frac{dy}{dx} = \frac{x+1}{2y}$

a. Calculate the slope at $(0.5, -1)$.

1 mark

b. Find the equation of the solution curve that contains the point (0.5, -1).

3 marks

c. Find the exact value(s) of y when $x = 1.5$.

2 marks

Question 2 (9 marks)

A particle moves in a straight line so that its distance, *x* metres, from a fixed origin *O* after time *t* seconds is given by the differential equation

$$\frac{dx}{dt} = \frac{e^{-x}}{1 + 81t^2}$$

where $x = 0 \ when\ t = 0$.

a. Solve the differential equation to show that

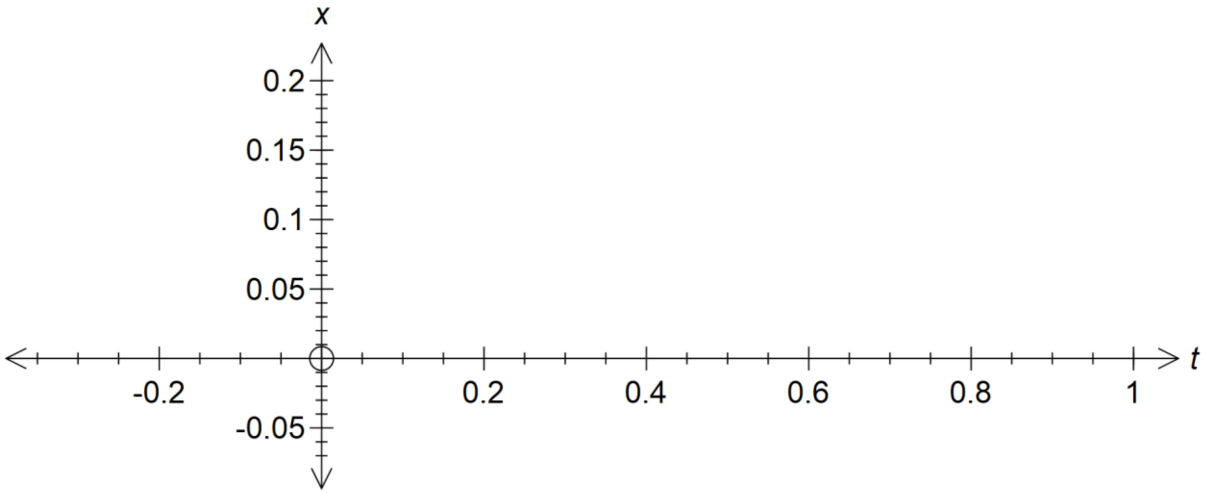
$$x = \ln\left|\frac{1}{9}tan^{-1}(9t) + 1\right|$$

4 marks

b. How much distance does the particle cover in 10 seconds?
Give your answer o the nearest centimetre.

2 marks

c. Sketch the graph of $x = \ln\left|\frac{1}{9}tan^{-1}(9t) + 1\right|$ identifying all key features.



3 marks

Question 3 (7 marks)

Water is poured into the vessel.

Due to a crack at the base, water leaks out at a rate proportional to the square root of the depth h of water in the vessel given by $\frac{dV}{dt} = -2\sqrt{h}$, where V is the volume of water remaining in the vessel, in cubic centimetres, after t minutes.

The volume of the vessel in terms of h is given by $V = \frac{\pi}{5}\left((h+6)^{\frac{5}{3}} - 16\right)$

a. Show that $\frac{dh}{dt} = -\frac{6}{\pi}\frac{\sqrt{h}}{(h+6)^{\frac{2}{3}}}$

4 marks

b. Find the maximum rate, in centimetres per minute, at which the depth of water in the vessel decreases, correct to two decimal places, and find the corresponding depth in centimetres.

3 marks

END OF KEY TOPIC TEST