



SPECIALIST MATHEMATICS 2023

Unit 4

Key Topic Test 3 – Differential equations

Technology Free

Recommended writing time: 45 minutes

Total number of marks available: 30 marks

SOLUTIONS

Question 1

a. $\frac{dy}{dx} = \frac{x-1}{2y}, y(0) = \frac{1}{2}$

$$\int 2y \, dy = \int (x-1) \, dx$$

$$y^2 = \frac{x^2}{2} - x + c$$

$$\frac{1}{4} = c$$

$$y^2 = \frac{x^2}{2} - x + \frac{1}{4}$$

3 marks

b. $\frac{dy}{dx} = (1-2y)(1-y)$

$$\int \frac{dy}{(1-2y)(1-y)} = \int dx$$

$$\int \left(\frac{2}{1-2y} - \frac{1}{1-y} \right) dy = \int dx$$

$$-\ln|1-2y| + \ln|y-1| = x + c$$

$$\ln \left| \frac{y-1}{1-2y} \right| = x + c$$

$$x = 0, y = 2 \rightarrow c = 0$$

$$\frac{y-1}{1-2y} = e^x$$

$$y-1 = e^x - 2ye^x$$

$$y(1+2e^x) = e^x + 1$$

$$y = \frac{e^x+1}{1+2e^x}$$

4 marks

Question 2

$$y = (1+2x)e^{-x}$$

$$\frac{dy}{dx} = e^{-x}(-1-2x+2) = e^{-x}(1-2x)$$

$$\frac{d^2y}{dx^2} = e^{-x}(-1+2x-2) = e^{-x}(2x-3)$$

$$\begin{aligned} \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y &= e^{-x}(2x-3) + 2e^{-x}(1-2x) + (1+2x)e^{-x} \\ &= e^{-x}(2x-3+2-4x+1+2x) = 0 \end{aligned}$$

3 marks

Question 3

a. $\frac{dh}{dt} = 0.6 - 0.04\sqrt{h}$

$$t = \int_0^{100} \frac{dh}{0.6 - 0.04\sqrt{h}}$$

$$t = \int_0^{100} \frac{100}{60 - 4\sqrt{h}} dh$$

$$t = \int_0^{100} \frac{25}{15 - \sqrt{h}} dh$$

3 marks

b. Let $u = \sqrt{h}$

$$\frac{du}{dh} = \frac{1}{2\sqrt{h}}$$

$$t = \int_0^{100} \frac{25}{15 - \sqrt{h}} dh$$

$$t = 25 \int_0^{10} \frac{2u}{15 - u} du$$

$$t = 50 \int_0^{10} \frac{u}{15 - u} du$$

$$t = 50 \int_0^{10} \frac{u - 15 + 15}{15 - u} du$$

$$t = 50 \int_0^{10} \left(-1 + \frac{15}{15 - u}\right) du$$

$$t = 50(-u - 15 \ln|15 - u|) \Big|_0^{10}$$

$$t = 50(-10 - 15 \ln(5) + 15 \ln(15))$$

$$t = 50(-10 + 15 \ln(3))$$

$$t = 250(-2 + 3 \ln(3))$$

4 marks

Question 4

a. 6000

1 mark

b. Fastest rate at $\frac{d^2P}{dt^2} = 0$

$$2P \left(-\frac{1}{6000}\right) + 2 \left(4 - \frac{P}{6000}\right) = 0$$

$$-2P + 48000 - 2P = 0$$

$$48000 - 4P = 0$$

$$P = 12000$$

2 marks

$$\begin{aligned} \text{c. } \frac{dP}{dt} &= 2P \left(4 - \frac{P}{6000} \right) \\ \frac{dP}{dt} &= \frac{1}{3000} P(24000 - P) \\ \int \frac{dP}{P(24000 - P)} &= \int \frac{1}{3000} dt \end{aligned}$$

$$\frac{1}{P(24000 - P)} = \frac{a}{P} + \frac{b}{24000 - P}$$

$$1 = a(24000 - P) + bP$$

$$P = 0 \rightarrow 1 = 24000a \rightarrow a = \frac{1}{24000}$$

$$P = 24000 \rightarrow 1 = 24000b \rightarrow b = \frac{1}{24000}$$

$$\frac{1}{24000} \int \left(\frac{1}{P} + \frac{1}{24000 - P} \right) dP = \int \frac{1}{3000} dt$$

$$\frac{1}{8} \int \left(\frac{1}{P} + \frac{1}{24000 - P} \right) dP = \int dt$$

$$t = \frac{1}{8} (\ln|P| - \ln|24000 - P|) + c$$

$$t = \frac{1}{8} \ln \left| \frac{P}{24000 - P} \right| + c$$

$$t = 0, P = 5000 \rightarrow 0 = \frac{1}{8} \ln \left| \frac{5}{19} \right| + c \rightarrow c = -\frac{1}{8} \ln \left| \frac{5}{19} \right|$$

$$t = \frac{1}{8} \ln \left| \frac{P}{24000 - P} \right| - \frac{1}{8} \ln \left| \frac{5}{19} \right|$$

$$t = \frac{1}{8} \ln \left| \frac{19P}{5(24000 - P)} \right|$$

$$\frac{19P}{5(24000 - P)} = e^{8t}$$

$$19P = (120000 - 5P)e^{8t}$$

$$19P + 5Pe^{8t} = 120000e^{8t}$$

$$P(19 + 5e^{8t}) = 120000e^{8t}$$

$$P = \frac{120000e^{8t}}{19 + 5e^{8t}}$$

5 marks

Question 5

$$\text{Let } g(x) = \frac{1}{x}$$

$$x_0 = 1, y_0 = \frac{1}{4}$$

$$x_1 = 1 + 0.1 = 1.1, y_1 = \frac{1}{4} + 0.1 \times 1 = 0.25 + 0.1 = 0.35$$

$$x_2 = 1.1 + 0.1 = 1.2, y_2 = 0.35 + 0.1 \times \frac{1}{1.1} = 0.35 + \frac{1}{11} = \frac{7}{20} + \frac{1}{11}$$

$$y_2 = \frac{97}{270}$$

3 marks

Question 6

$$\frac{dx}{dt} = -\frac{x}{40}$$

$$\int \frac{dx}{x} = \int -\frac{1}{40} dt$$

$$\ln|x| = -\frac{1}{40}t + c$$

$$x = e^{-\frac{1}{40}t+c}, c \in \mathbb{R}$$

2 marks

END OF KEY TOPIC TEST SOLUTIONS