SPECIALIST MATHEMATICS 2023

Unit 3

Key Topic Test 16 – Antidifferentiation Techniques

Technology Active

Recommended writing time*: 45 minutes
Total number of marks available: 30 marks

SOLUTIONS

Section A: Multiple-choice questions

Question 1

Answer: **D**

Explanation:

$$\int_0^1 \frac{\sqrt{1+x^2}}{x-2}\,dx \qquad -0.813186807464$$

Question 2

Answer: **B**

Explanation:

$Let\ x-1=u \rightarrow \frac{du}{dx}=1 \rightarrow \int_0^3 x^2\sqrt{x-1}\,dx=\int_{-1}^2 (u+1)^2\sqrt{u}\,du=\int_{-1}^2 \left(u^{\frac{5}{2}}+2u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du$

Question 3

Answer: **E**

Explanation:

$\int x\sin(x)\,dx$ by parts

$Let\ u=x,\ \frac{dv}{dx}=\sin(x)$

$\frac{du}{dx}=1,\ v=-\cos(x)$

$\int x\sin(x)\,dx=-x\cos(x)-\int -\cos(x)\times 1\,dx$

$\int x\sin(x)\,dx=-x\cos(x)+\int \cos(x)\,dx$

Question 4

Answer: **C**

Explanation:

$$\int_{\frac{\pi}{4}}^{k} \frac{1}{\cos^2(x)\tan(x)}\,dx = \int_{\frac{\pi}{4}}^{k} \frac{\sec^2(x)}{\tan(x)}\,dx = [\ln(\tan(x))]_{\frac{\pi}{4}}^{k}$$

$$= \ln|tan(k)| - \ln\left|tan\left(\frac{\pi}{4}\right)\right| = \ln|\tan(k)|$$

$$\tan(k) = \frac{5}{4}$$

Question 5

Answer: **A**

Explanation:

$$\int_a^b \cos(2x)\sin(2x)\,dx = \frac{1}{2}\int_a^b \sin(4x)\,dx = \frac{1}{2}\int_a^b \sin(4u)\,du$$

Question 6

Answer: **B**

Explanation:

$$\text{expand}\left(\frac{2}{(x^2-1)\cdot(x+2)}\right)$$

$$\frac{2}{3\cdot(x+2)} - \frac{1}{x+1} + \frac{1}{3\cdot(x-1)}$$

Question 7

Answer: **E**

Explanation:

$$Let\ f(x) = u,\ \frac{du}{dx} = f'(x) \rightarrow \int f'(x)\cos(f(x))\,dx = \int \cos(u)\,du = \sin(f(x)) + c$$

Section B: Short-answer questions

Question 1

a. $Let\ x = \tan(\theta) \rightarrow \frac{dx}{d\theta} = \sec^2(\theta)$

$\int_0^1 \frac{x^2}{(1+x^2)^{\frac{5}{2}}} dx = \int_0^{\frac{\pi}{4}} \frac{\tan^2(\theta)}{(1+\tan^2(\theta))^{\frac{5}{2}}} \sec^2(\theta)\, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{tan^2(\theta)}{sec^3(\theta)} d\theta$

$= \int_0^{\frac{\pi}{4}} \sin^2(\theta)\cos(\theta)\, d\theta$

$= \left[\frac{\sin^3(\theta)}{3}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3}\left(\sin^3\left(\frac{\pi}{4}\right) - \sin^3(0)\right)$

$= \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3$

$= \frac{\sqrt{2}}{12}$

4 marks

b. $\int_0^1 \frac{x^2}{(1+x^2)^{\frac{5}{2}}}\ x\, dx$

Let $u = x,\ \frac{dv}{dx} = \frac{x^2}{(1+x^2)^{\frac{5}{2}}}$

$\frac{du}{dx} = 1,\ v = \frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3$

$\int_0^1 \frac{x^2}{(1+x^2)^{\frac{5}{2}}}\ dx = \left[\frac{1}{3}x\left(\frac{x}{\sqrt{1+x^2}}\right)^3\right]_0^1 - \int_0^1 \frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3 dx$

$= \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{1}{3}\int_0^1 \frac{x^3}{(1+x^2)^{\frac{3}{2}}} dx$

$= \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{1}{6}\int_1^2 \frac{u-1}{u^{\frac{3}{2}}} du \quad (1+x^2 = u)$

$= \frac{1}{6\sqrt{2}} - \frac{1}{6}\int_1^2 \left(u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\right) du$

$= \frac{\sqrt{2}}{12} - \frac{1}{6}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^2$

$= \frac{\sqrt{2}}{12} - \frac{1}{6}\left[2\sqrt{2} + \sqrt{2} - 2 - 2\right]$

$= \frac{\sqrt{2}}{12} - \frac{1}{6}\left(3\sqrt{2} - 4\right)$

$= \frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{2} + \frac{2}{3}$

$= \frac{-5\sqrt{2}}{12} + \frac{2}{3}$

4 marks

Question 2

a. $f(x) = \frac{x^2-5x+5}{x^2-5x+8} = \frac{x^2-5x+8-3}{x^2-5x+8} = 1 - \frac{3}{x^2-5x+8}$

2 marks

b. $\int f(x)\,dx = \int \left(1 - \frac{3}{x^2-5x+8}\right) dx$

$= \int \left(1 - \frac{3}{\left(x-\frac{5}{2}\right)^2+\frac{7}{4}}\right) dx$

$= x - 3 \times \sqrt{\frac{4}{7}} \tan^{-1}\left(\frac{x-\frac{5}{2}}{\sqrt{\frac{7}{4}}}\right)$

$= x - \frac{6}{\sqrt{7}} \tan^{-1}\left(\frac{2x-5}{\sqrt{7}}\right)$

3 marks

c. $k = 3.08$

$$\text{solve}\left(\int_0^k f(2\cdot x)\,dx = \int_0^1 f(x)\,dx, k\right)$$

$$k=3.08226242456$$

1 mark

Question 3

a. $x^2 = \cos(y)$

$2x = -\sin(y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-2x}{\sin(y)}$

$\frac{dy}{dx} = -\frac{2x}{\sqrt{1-\cos^2(y)}}$

$\frac{dy}{dx} = -\frac{2x}{\sqrt{1-x^4}}$

2 marks

b. $Let\ u = \cos^{-1}(x^2), \frac{dv}{dx} = x$

$\frac{du}{dx} = -\frac{2x}{\sqrt{1-x^4}}, \ \ v = \frac{x^2}{2}$

$\int x\cos^{-1}(x^2)\,dx = \frac{x^2}{2}\cos^{-1}(x^2) - \int \frac{x^2}{2} \times -\frac{2x}{\sqrt{1-x^4}}\ dx$

$= \frac{x^2}{2}\cos^{-1}(x^2) + \int \frac{x^3}{\sqrt{1-x^4}}\ dx$

$= \frac{x^2}{2}\cos^{-1}(x^2) - \frac{1}{4}\int p^{-\frac{1}{2}}\ dp$ (Let $1 - x^4 = p$)

$= \frac{x^2}{2}\cos^{-1}(x^2) - \frac{1}{4}\left(2\sqrt{p}\right) + c$

$= \frac{x^2}{2}\cos^{-1}(x^2) - \frac{1}{2}\sqrt{1-x^4} + c$

4 marks

c. $\left[\frac{x^2}{2}\cos^{-1}(x^2) - \frac{1}{2}\sqrt{1-x^4}\right]_0^1$

$= \frac{1}{2}\cos^{-1}(1) - \frac{1}{2}(0) - 0 + \frac{1}{2}$

$= \frac{1}{2} \times 0 + \frac{1}{2}$

$= \frac{1}{2}$

1 mark

d. $\int_0^1 (x\cos^{-1}(x^2) - kx)\ dx = \frac{1}{4}$

$\int_0^1 x\cos^{-1}(x^2)\ dx - \int_0^1 kx\ dx = \frac{1}{4}$

$\frac{1}{2} - \left[\frac{kx^2}{2}\right]_0^1 = \frac{1}{4}$

$\frac{1}{2} - \frac{k}{2} = \frac{1}{4}$

$2 - 2k = 1 \rightarrow k = \frac{1}{2}$

2 marks