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SPECIALIST MATHEMATICS 2023

Unit 3

Key Topic Test 15 – Antidifferentiation Techniques

Technology Free

Recommended writing time*: 45 minutes
Total number of marks available: 30 marks

SOLUTIONS

Question 1

a. $\frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$3 = A(x+1) + B(x-2)$

$x = -1 \rightarrow B = -1, \;\; x = 2 \rightarrow A = 1$

$\int \frac{3}{(x-2)(x+1)} dx = \int \left(\frac{1}{x-2} - \frac{1}{x+1}\right) dx = \ln|x-2| - \ln|x+1| + \ln|c| = \ln\left|\frac{c(x-2)}{x+1}\right|$

3 marks

b. Let $\cos^2(2x) = u \rightarrow \frac{du}{dx} = -2\cos(2x)\sin(2x) = -\sin(4x)$

$\int \cos^2(2x)\sin(4x)\, dx = -\int u\, du = -\frac{u^2}{2} + c = -\frac{1}{2}\cos^4(2x) + c$

2 marks

c. $\int x^2 \ln(x)\; dx$

Let $u = \ln(x)$ and $\frac{dv}{dx} = x^2$

$\frac{du}{dx} = \frac{1}{x}$ *and* $v = \frac{1}{3}x^3$

$\int x^2 \ln(x)\; dx = uv - \int v \frac{du}{dx}\; dx$

$\int x^2 \ln(x)\; dx = \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3}x^3 \times \frac{1}{x}\; dx$

$\int x^2 \ln(x)\; dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2\; dx$

$\int x^2 \ln(x)\; dx = \frac{1}{3}x^3 \ln(x) - \frac{x^3}{9} + c$

3 marks

Question 2

$\int_1^2 \frac{2x+1}{x^2-2x+2} dx$

$= \int_1^2 \frac{2x-2+3}{x^2-2x+2} dx$

$= \int_1^2 \left(\frac{2x-2}{x^2-2x+2} + \frac{3}{x^2-2x+2}\right) dx$

$= \int_1^2 \left(\frac{2x-2}{x^2-2x+2} + \frac{3}{(x-1)^2+1}\right) dx$

$= \left[\ln|x^2-2x+2| + 3\tan^{-1}(x-1)\right]_1^2$

$= \ln|2| + 3\tan^{-1}(1) - \ln|1| - 3\tan^{-1}(0)$

$= \ln(2) + \frac{3\pi}{4}$

4 marks

Question 3

a. Let $x^4 - 9 = u \rightarrow \frac{du}{dx} = 4x^3$

$x = \sqrt{3} \rightarrow u = \left(\sqrt{3}\right)^4 - 9 = 9 - 9 = 0$

$x = \sqrt{5} \rightarrow u = \left(\sqrt{5}\right)^4 - 9 = 25 - 9 = 16$

$\int_{\sqrt{3}}^{\sqrt{5}} x^3\sqrt{x^4 - 9}\, dx = \frac{1}{4}\int_0^{16} \sqrt{u}\, du$

3 marks

b. $\frac{1}{4}\int_0^{16} \sqrt{u}\, du = \frac{1}{4} \times \frac{2}{3}\left(u^{\frac{3}{2}}\right)\begin{matrix}16\\0\end{matrix}$

$= \frac{1}{6}\left(16^{\frac{3}{2}} - 0\right)$

$= \frac{1}{6}(4^3)$

$= \frac{64}{6}$

$= \frac{32}{3}$

2 marks

Question 4

a. $I_1 = \int_1^e \ln(x)\, dx$

Let $u = \ln(x)$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$ and $v = x$

$\int_1^e \ln(x)\, dx = uv - \int v\frac{du}{dx}\, dx$

$\int_1^e \ln(x)\, dx = \left[x\ln(x) - \int x\frac{1}{x}\, dx\right]_1^e$

$\int_1^e \ln(x)\, dx = [x\ln(x) - \int 1\, dx]_1^e$

$\int_1^e \ln(x)\, dx = [x\ln(x) - x]_1^e$

$\int_1^e \ln(x)\, dx = e\ln(e) - e - \ln(1) + 1$

$\int_1^e \ln(x)\, dx = 1$

3 marks

b. $I_2 = \int_1^e (\ln(x))^2 \, dx$

$Let\ u = (\ln(x))^2$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2\ln(x)}{x}$ and $v = x$

$\int_1^e (\ln(x))^2 \, dx = x(\ln(x))^2 - \int_1^e x \frac{2\ln(x)}{x} \, dx$

$I_2 = [x(\ln(x))^2]_1^e - 2I_1$

$I_2 = e - 2I_1$

2 marks

c. $I_n = \int_1^e (\ln(x))^n \, dx$

$Let\ u = (\ln(x))^n$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = n\frac{(\ln(x))^{n-1}}{x}$ and $v = x$

$I_n = x(\ln(x))^n - \int_1^e xn \frac{(\ln(x))^{n-1}}{x} \, dx$

$I_n = [x(\ln(x))^n]_1^e - n\int_1^e (\ln(x))^{n-1} \, dx$

$I_n = e - nI_{n-1}$

2 marks

Question 5

$$\frac{1}{\sqrt{3}\sin(x)+3\cos(x)}=\frac{1}{2\sqrt{3}\left(\frac{1}{2}\sin(x)+\frac{\sqrt{3}}{2}\cos(x)\right)}=\frac{\frac{\sqrt{3}}{6}}{\sin(x)\cos\left(\frac{\pi}{3}\right)+\cos(x)\sin\left(\frac{\pi}{3}\right)}=\frac{\frac{\sqrt{3}}{6}}{\sin\left(x+\frac{\pi}{3}\right)}$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{3}\sin(x)+3\cos(x)}\,dx=\int_0^{\frac{\pi}{2}}\frac{\frac{\sqrt{3}}{6}}{\sin\left(x+\frac{\pi}{3}\right)}\,dx$$

$$=\frac{\sqrt{3}}{6}\int_0^{\frac{\pi}{2}}\frac{1}{\sin\left(x+\frac{\pi}{3}\right)}\,dx$$

$$=\frac{\sqrt{3}}{6}\int_0^{\frac{\pi}{2}}\frac{\sin\left(x+\frac{\pi}{3}\right)}{\sin^2\left(x+\frac{\pi}{3}\right)}\,dx$$

$$=\frac{\sqrt{3}}{6}\int_0^{\frac{\pi}{2}}\frac{\sin\left(x+\frac{\pi}{3}\right)}{1-\cos^2\left(x+\frac{\pi}{3}\right)}\,dx$$

$$Let\ \cos\left(x+\frac{\pi}{3}\right)=u$$

$$=\frac{\sqrt{3}}{6}\int_{\frac{1}{2}}^{-\frac{\sqrt{3}}{2}}-\frac{1}{1-u^2}\,du$$

$$=-\frac{\sqrt{3}}{6}\int_{\frac{1}{2}}^{-\frac{\sqrt{3}}{2}}\frac{1}{2}\left(\frac{1}{1-u}-\frac{1}{1+u}\right)\,du$$

$$=-\frac{\sqrt{3}}{12}\left(\ln\left|\frac{1-u}{1+u}\right|\right)_{\frac{1}{2}}^{-\frac{\sqrt{3}}{2}}$$

$$=-\frac{\sqrt{3}}{12}\left(\ln\left|\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right|-\ln\left|\frac{\frac{1}{2}}{\frac{3}{2}}\right|\right)$$

$$=-\frac{\sqrt{3}}{12}\ln\left|\frac{3(2+\sqrt{3})}{2-\sqrt{3}}\right|$$

$$=-\frac{\sqrt{3}}{12}\left(\ln\left|3(2+\sqrt{3})^2\right|\right)$$

$$=-\frac{\sqrt{3}}{12}\ln\left(21+12\sqrt{3}\right)$$

6 marks