Neap

Trial Examination 2023

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (2 marks)

$\mathrm{E}(X + 2X - 4Y) = \mathrm{E}(X) + 2\mathrm{E}(X) - 4\mathrm{E}(Y)$

$= 3 + 6 - 20$

$= -11$ A1

$\mathrm{Var}(X + 2X - 4Y) = \mathrm{Var}(X) + 4\mathrm{Var}(X) + 16\mathrm{Var}(Y)$

$= 2 + 8 + 64$

$= 74$ A1

Question 2 (3 marks)

a. If $p + q$ is not even, then $p^2 + q^2 + 1$ is not odd. A1

OR

If $p + q$ is odd, then $p^2 + q^2 + 1$ is even. A1

b. Letting $p + q$ be odd gives:

$p^2 + q^2 + 1 = (p+q)^2 - 2pq + 1$ M1

$= \text{odd} - \text{even} + 1$

$= (\text{odd} + 1) - \text{even}$

$= \text{even} - \text{even}$

$= \text{even}$ M1

Question 3 (3 marks)

$V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, where r is the radius of the spherical balloon.

$\frac{dV}{dt} = 24 \text{ cm}^3/\text{s}$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ M1

$24 = \left(4\pi \times 8^2\right) \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{24}{256\pi}$

$= \frac{3}{32\pi}$ A1

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= (8\pi \times 8) \times \frac{3}{32\pi}$

$= 6 \text{ cm}^2/\text{s}$ A1

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Question 4 (3 marks)

$z^3 = -64$

Letting $z = r\text{cis}(\theta)$ gives:

$r^3\text{cis}(3\theta) = 64\text{cis}(\pi)$ M1

When $r^3 = 64$ and $3\theta = \pi + 2k$, $k \in Z$:

$r = 4$ and $\theta = \dfrac{\pi + 2\pi k}{3}$, $k \in Z$ A1

Choosing $k = 0, 1, 2$ gives:

$z_1 = 4\text{cis}\left(\dfrac{\pi}{3}\right)$, $z_2 = 4\text{cis}(\pi)$, $z_3 = 4\text{cis}\left(-\dfrac{\pi}{3}\right)$ A1

Question 5 (3 marks)

Proving for $n = 1$:

LHS $= 1^3 = 1$

$\text{RHS} = \dfrac{1}{4} \times 1^2 \times 2^2 = 1$ M1

Assuming true for $n = k$:

$1^3 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{1}{4}k^2(k+1)^2$

Proving true for $n = k + 1$:

$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \dfrac{1}{4}(k+1)^2(k+2)^2$ M1

$$\begin{aligned}
\text{LHS} &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\
&= (k+1)^2\left(\frac{k^2}{4} + k + 1\right) \\
&= (k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right) \\
&= (k+1)^2\,\frac{(k+2)^2}{4} \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \text{RHS}
\end{aligned}$$

M1 (at the line $= \frac{1}{4}(k+1)^2(k+2)^2$)

Question 6 (6 marks)

a. $\begin{cases} x = 2\cos(t) \\ y = -\sin(t) \end{cases}$

$\begin{cases} \cos(t) = \frac{x}{2} \\ \sin(t) = -y \end{cases}$ M1

$\begin{cases} \cos^2(t) = \frac{x^2}{4} \\ \sin^2(t) = y^2 \end{cases}$

Using the identity $\cos^2(t) + \sin^2(t) = 1$ gives:

$\frac{x^2}{4} + y^2 = 1$ A1

b. $\dot{\underset{\sim}{r}}(t) = -2\sin(t)\underset{\sim}{i} - \cos(t)\underset{\sim}{j}$ M1

$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{4\sin^2(t) + \cos^2(t)}$

$\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right)\right| = \sqrt{4 \times \frac{1}{2} + \frac{1}{2}}$

$= \sqrt{\frac{5}{2}}$ A1

c. $\ddot{\underset{\sim}{r}}(t) = -2\cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$ M1

$\left|\ddot{\underset{\sim}{r}}(t)\right| = \sqrt{4\cos^2(t) + \sin^2(t)}$

$= \sqrt{3\cos^2(t) + 1}$

The particle's maximum acceleration occurs when $\cos^2(t) = 1$. Therefore:

$\left|\ddot{\underset{\sim}{r}}(t)\right|_{max} = \sqrt{(3 \times 1) + 1}$

$= 2$ A1

Question 7 (4 marks)

a. If A, B and C are collinear, $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ for some constant λ.

$$\overrightarrow{AB} = \begin{bmatrix} 3-(-2) \\ 4-1 \\ -2-0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

$$\overrightarrow{AC} = \begin{bmatrix} -5-(-2) \\ -2-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$ M1

$5 = -3\lambda$

$\lambda = -\frac{5}{3}$

$\lambda = -\frac{5}{3}$ does not satisfy $3 = -3\lambda$. M1

Therefore, the points are not collinear.

b. $\text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 5 & 3 & -2 \\ -3 & -3 & 1 \end{vmatrix}$$ M1

$$= -3\underset{\sim}{i} + \underset{\sim}{j} - 6\underset{\sim}{k}$$

$$\text{area} = \frac{1}{2}\left|-3\underset{\sim}{i} + \underset{\sim}{j} - 6\underset{\sim}{k}\right|$$

$$= \frac{\sqrt{46}}{2}$$ A1

*Note: Consequential on answer to **Question 7a.***

Question 8 (4 marks)

When $\sin(2x) = 0$:

$$2x = 0, \pi, 2\pi$$

$$x = 0, \frac{\pi}{2}, \pi$$

When $\cos(2x) = 0$:

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Therefore, the smallest non-negative root of $f(x)$ is $\frac{\pi}{4}$. A1

$$A = \int_0^{\frac{\pi}{4}} \sin^2(2x)\cos^3(2x)dx$$

$$= \int_0^{\frac{\pi}{4}} \sin^2(2x)\cos^2(2x)\cos(2x)dx$$

$u = \sin(2x)$ M1

$$du = 2\cos(2x)dx$$

$$\cos^2(2x) = 1 - \sin^2(2x)$$

$$= 1 - u^2$$

$A = \frac{1}{2}\int_0^1 u^2\left(1 - u^2\right)du$ A1

$$= \frac{1}{2}\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$$

$= \frac{1}{15}$ A1

Question 9 (4 marks)

$u = x^2 \Rightarrow u' = 2x$

$v' = \sin(x) \Rightarrow v = -\cos(x)$ M1

Using integration by parts gives:

$$I = \int x^2 \sin(x)dx$$

$$= uv - \int u'v\,dx$$

$= -x^2\cos(x) + 2\int x\cos(x)dx$ A1

$m = x \Rightarrow m' = 1$

$n' = \cos(x) \Rightarrow n = \sin(x)$ M1

$$I = -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)dx\right)$$

$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + c$ A1

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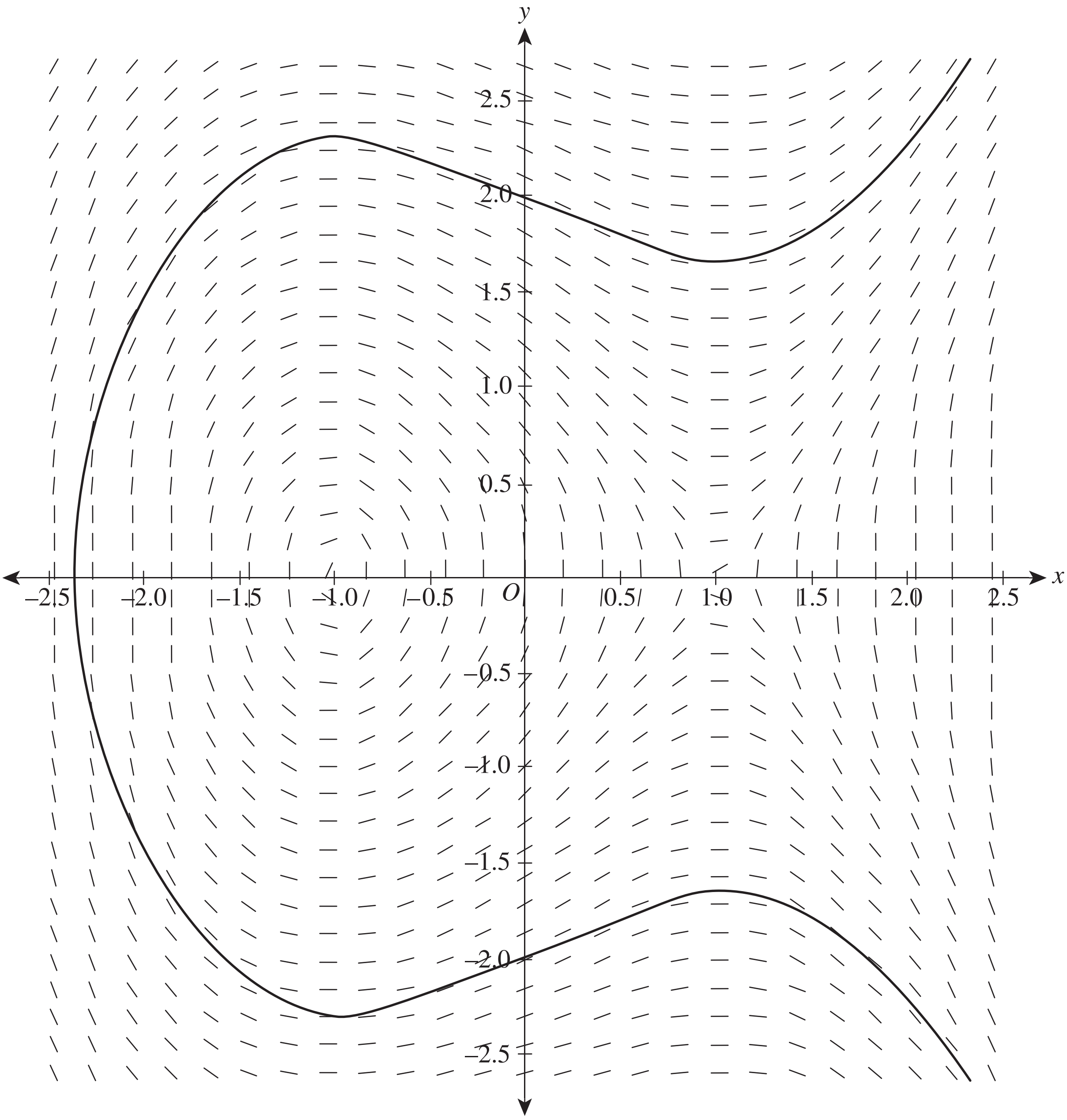
Question 10 (8 marks)

a.



correct shape A1

correct demonstration of zero gradient at $x = 1$ *and* $x = -1$ *and no gradient at* $y = 0$ A1

Note: The solution is obtained by using the initial condition $y(0) = 2$ *and drawing a solution curve that is tangential to the direction field.*

b. $\int y\,dy = \int (x^2 - 1)\,dx$ M1

$\frac{y^2}{2} = \frac{x^3}{3} - x + c$

$\begin{cases} x = 0 \\ y = 2 \end{cases} \Rightarrow c = 2$

$\frac{y^2}{2} = \frac{x^3}{3} - x + c$

$\frac{3y^2}{2} = x^3 - 3x + 6$

$3y^2 = 2x^3 - 6x + 12$ A1

c. $\frac{dy}{dx} = \frac{x^2 - 1}{y}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ M1

$= \frac{2xy - (x^2 - 1)\frac{dy}{dx}}{y^2}$

$= \frac{2xy - \frac{(x^2 - 1)^2}{y}}{y^2}$

$= \frac{2xy^2 - (x^2 - 1)^2}{y^3}$ A1

$\frac{d^2y}{dx^2} = 0 \Rightarrow 2xy^2 - (x^2 - 1)^2 = 0$ M1

$2xy^2 - (x^2 - 1)^2 = 0$

$2xy^2 = (x^2 - 1)^2$

$y^2 = \frac{(x^2 - 1)^2}{2x}$

$y = \frac{|x^2 - 1|}{\sqrt{2x}}$

$= \frac{1 - x^2}{\sqrt{2x}}$ M1