Neap

Trial Examination 2023

VCE Specialist Mathematics Units 1&2

Written Examination 1

Question and Answer Booklet

Reading time: 15 minutes
Writing time: 1 hour

Student's Name: ______________________________

Teacher's Name: ______________________________

Structure of booklet

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.

Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

No calculator is allowed in this examination.

Materials supplied

Question and answer booklet of 12 pages

Formula sheet

Working space is provided throughout the booklet

Instructions

Write your **name** and your **teacher's name** in the space provided above on this page.

Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.

All written responses must be in English.

At the end of the examination

You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, diagrams in this booklet are **not** drawn to scale.

Take the acceleration due to gravity to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (2 marks)

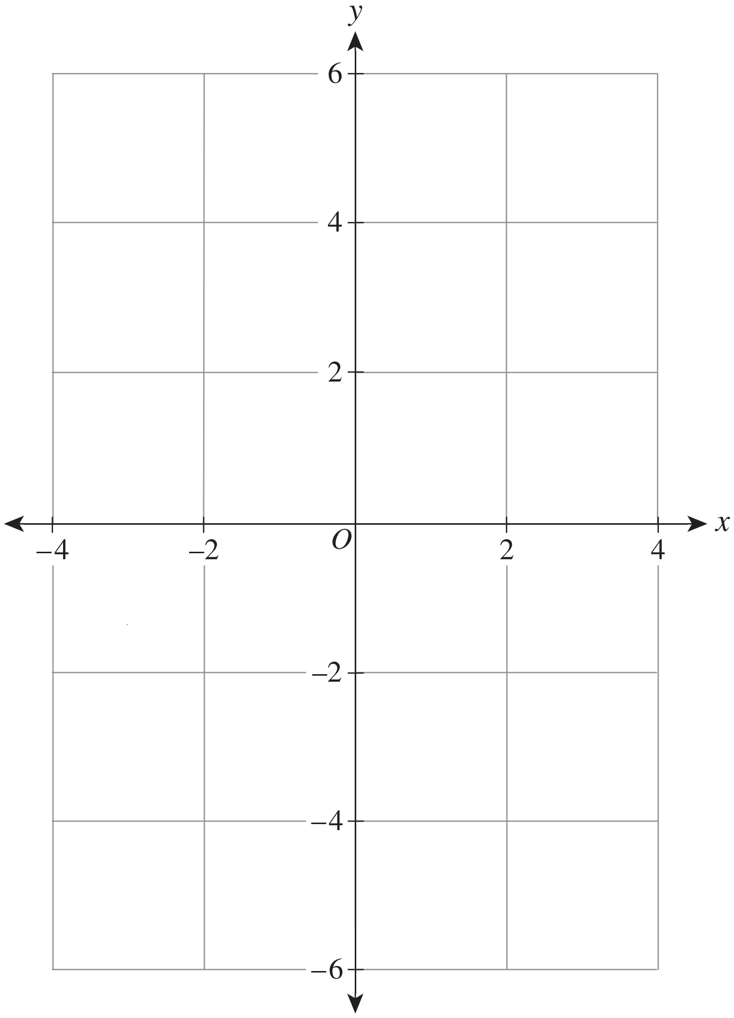
If $a \in Z$ and a^2 is divisible by 2, prove by contradiction that a must be even. 2 marks

Question 2 (6 marks)

Points A and B have the coordinates $(-a, 0)$ and $(a, 0)$ respectively. The locus of point P has the coordinates (x, y) and moves so that the product of the gradients of lines AP and BP equals $-k$, where $k > 0$ and $k \neq 1$.

a. Find the locus of point P and identify its shape. 3 marks

b. On the axes below, sketch the graph of the locus of point P when $a = 3$ and $k = 2$.
Label all axis intercepts. 3 marks



Question 3 (5 marks)

Prove the following using Boolean algebra.

a. $a \vee (a' \wedge b) = a \vee b$ 1 mark

b. $(a \vee b) \wedge (c \vee d) = (a \wedge c) \vee (b \wedge c) \vee (a \wedge d) \vee (b \wedge d)$ 1 mark

c. If $a \wedge b = a \wedge c$ and $a \vee c = a \vee b$, then $b = c$. 3 marks

Question 4 (6 marks)

It is known that sequence a_n satisfies $a_1 = 1$ and $na_{n+1} = 2(n+1)a_n$, and that sequence b_n satisfies $b_n = \frac{a_n}{n}$.

a. Find b_1, b_2 and b_3. 2 marks

b. Using direct proof, show that b_n is a geometric sequence. State its common ratio. 2 marks

c. Hence, express a_n in terms of n. 2 marks

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Question 5 (5 marks)

Three adults and five children are asked to stand in a line.

a. Find the number of ways that the line can be arranged if all the adults must stand together. 1 mark

b. Find the number of ways that the line can be arranged if all the adults must stand separately. 1 mark

c. Find the number of ways that the line can be arranged if an adult cannot stand at **either** end of the line. 1 mark

d. Find the number of ways that the line can be arranged if an adult cannot stand at **both** ends of the line. 2 marks

Question 6 (6 marks)

a. Expand $(1 - i)^9$. Express your answer in Cartesian form. 2 marks

b. Show that $-i$ is a root of the polynomial $P(z) = 2z^4 + 2iz^3 - 6z - 6i$. 1 mark

c. Hence, find the solutions to $P(z) = 2z^4 + 2iz^3 - 6z - 6i$. Express your answers in polar form. 3 marks

Question 7 (5 marks)

a. Find the angle θ between $\underset{\sim}{u} = \left(0, \sqrt{3}, 0\right)$ and $\underset{\sim}{v} = \left(0, 1, \sqrt{3}\right)$. 2 marks

b. Consider the points $M(1, 1, 1)$, $N(3, 3, -1)$ and $Q(3, -1, -1)$. Point P is closest to point Q and lies on the line L that passes through points M and N.

Find the coordinates of point P. 3 marks

Question 8 (5 marks)

a. It is known that $0 < \alpha < \frac{\pi}{2} < \beta \le \pi$, $\tan\left(\frac{\alpha}{2}\right) = \frac{1}{2}$ and $\cos(\alpha - \beta) = \frac{\sqrt{2}}{10}$.

i. Show that $\sin(\alpha) = \frac{4}{5}$. 1 mark

ii. Hence, find the value of β. 2 marks

b. If $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\dfrac{m}{n}$, express $\dfrac{\tan(\beta)}{\tan(\alpha)}$ in terms of *m* and *n*. 2 marks

END OF QUESTION AND ANSWER BOOKLET

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Trial Examination 2023

VCE Specialist Mathematics Units 1&2

Written Examinations 1&2

Formula Sheet

Instructions

This formula sheet is provided for your reference.
A question and answer booklet is provided with this formula sheet.

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Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure

Boolean algebra	$a \wedge b = b \wedge a$ $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ $a \wedge a = a$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $a \wedge 1 = a$	$a \wedge 0 = 0$ $\neg(\neg a) = a$ $\neg(a \wedge b) = \neg a \vee \neg b$ $a \wedge \neg a = 0$ $\neg 0 = 1$

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\text{cis}(\theta)$	$\lvert z \rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \text{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$	de Moivre's theorem	$z^n = r^n \text{cis}(n\theta)$

Data analysis, probability and statistics

for independent random variables $X_1, X_2 \ldots X_n$	$\text{E}(aX_1 + b) = a\,\text{E}(X_1) + b$ $\text{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1\text{E}(X_1) + a_2\text{E}(X_2) + \ldots + a_n\text{E}(X_n)$	
	$\text{Var}(aX_1 + b) = a^2\text{Var}(X_1)$ $\text{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + \ldots + a_n^2\text{Var}(X_n)$	
for independent identically distributed variables $X_1, X_2 \ldots X_n$	$\text{E}(X_1 + X_2 + \ldots + X_n) = n\mu$	
	$\text{Var}(X_1 + X_2 + \ldots + X_n) = n\sigma^2$	
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$	
distribution of sample mean $\bar{X}$	mean	$\text{E}(\bar{X}) = \mu$
	variance	$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e \lvert x \rvert + c$
$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$	$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$	$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = a\sec^2(ax)$	$\int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}(\cot(ax)) = -a\,\text{cosec}^2(ax)$	$\int \text{cosec}^2(ax)dx = -\frac{1}{a}\cot(ax) + c$
$\frac{d}{dx}(\sec(ax)) = -a\sec(ax)\tan(ax)$	$\int \sec(ax)\tan(ax)dx = \frac{1}{a}\sec(ax) + c$
$\frac{d}{dx}(\text{cosec}(ax)) = -a\,\text{cosec}(ax)\cot(ax)$	$\int \text{cosec}(ax)\cot(ax)dx = -\frac{1}{a}\text{cosec}(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(ax)\right) = \frac{a}{\sqrt{1-(ax)^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\cos^{-1}(ax)\right) = \frac{-a}{\sqrt{1-(ax)^2}}$	$\int \frac{-1}{\sqrt{a^2 - x^2}}dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\tan^{-1}(ax)\right) = \frac{a}{1+(ax)^2}$	$\int \frac{a}{a^2 + x^2}dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$
	$\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$

Calculus – continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
Euler's method	If $\frac{dy}{dx} = f(x, y)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$.
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about *x*-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about *y*-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about *x*-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about *y*-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u + v)t$

Vectors in two or three dimensions

$\underset{\sim}{r}(t) = x(i)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \qquad y(t) = y_1 + y_2t \qquad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s, t) = \mathrm{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2$ $= (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s, t) = x_0 + x_1s + x_2t, \quad y(s, t) = y_0 + y_1s + y_2t, \quad z(s, t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Functions, relations and graphs

The hyperbola with equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ has asymptotes given by $y - k = \pm\frac{b}{a}(x - h)$	
$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \mathrm{cosec}^2(x)$
$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

END OF FORMULA SHEET