Neap

Trial Examination 2023

VCE Specialist Mathematics Units 1&2

Written Examination 1

Suggested Solutions

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Question 1 (2 marks)

If a is an odd number, then $a = 2m + 1$, $m \in Z$.

$$a^2 = (2m+1)^2$$
$$= 4m^2 + 4m + 1$$
$$= 4m(m+1) + 1 \qquad \text{M1}$$

Since $4m(m + 1) + 1$ is not divisible by 2, it is a contradiction. Therefore, a is even. A1

Question 2 (6 marks)

a. Using the coordinates of point P, (x, y), gives

$$AP \text{ gradient} = \frac{y}{x+a}$$

$$BP \text{ gradient} = \frac{y}{x-a}$$

both gradients M1

Therefore:

$$\frac{y}{x+a} \times \frac{y}{x-a} = -k$$
$$\frac{y^2}{x^2-a^2} = -k$$
$$y^2 = -kx^2 + ka^2$$

Hence, $\frac{x^2}{a^2} + \frac{y^2}{ka^2} = 1$. A1

The shape of the locus is an ellipse. A1

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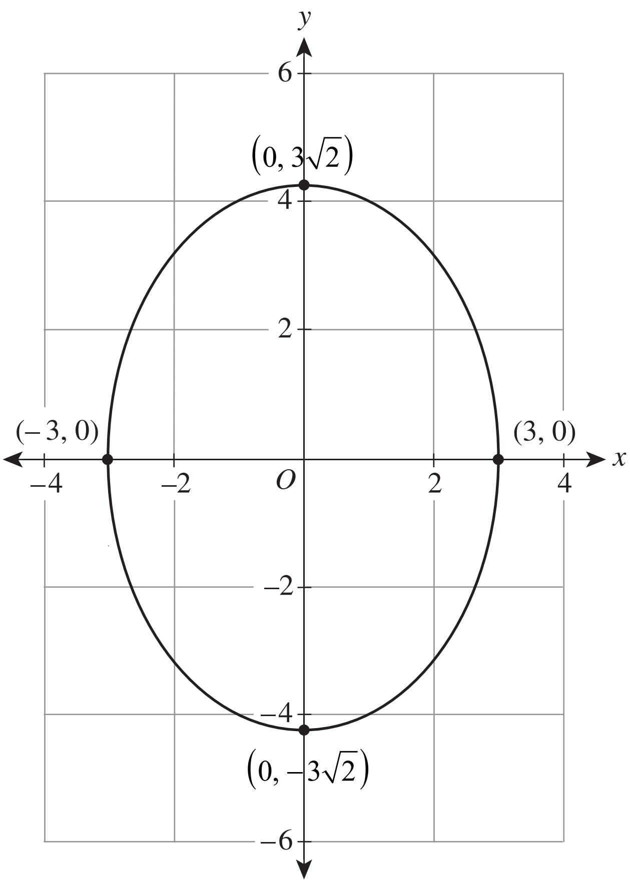
b. When $a = 3$ and $k = 2$, the shape of the ellipse is given by $\frac{x^2}{9} + \frac{y^2}{18} = 1$.

Substituting $y = 0$ to find the x-intercepts gives:

$$\frac{x^2}{9} + \frac{0^2}{18} = 1$$
$$x = \pm 3$$

Substituting $x = 0$ to find the y-intercepts gives:

$$\frac{0^2}{9} + \frac{y^2}{18} = 1$$
$$y = \pm 3\sqrt{2}$$



correct shape A1
correct x-intercepts A1
correct y-intercepts A1
Note: Consequential on answer to ***Question 2a.***

Question 3 (5 marks)

a. $a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b)$

$= 1 \wedge (a \vee b)$

$= a \vee b$ A1

b. $(a \vee b) \wedge (c \vee d) = \big((a \vee b) \wedge c\big) \vee \big((a \vee b) \wedge d\big)$

$= \big((a \wedge c) \vee (b \wedge c)\big) \vee \big((a \wedge d) \vee (b \wedge d)\big)$

$= (a \wedge c) \vee (b \wedge c) \vee (a \wedge d) \vee (b \wedge d)$ A1

c. Using the absorption property of Boolean algebra gives:

$b = b \vee (a \wedge b)$ M1

$= b \vee (a \wedge c)$

$= (b \vee a) \wedge (b \vee c)$

$= (a \vee c) \wedge (b \vee c)$ M1

$= (a \wedge b) \vee c$

$= (a \wedge c) \vee c$ M1

$= c$

Question 4 (6 marks)

a. Using $a_1 = 1$ to find a_2 gives:

$1 \times a_2 = 2 \times (1+1)a_1$

$a_2 = 4a_1$

$= 4$

Using a_2 to find a_3 gives:

$2 \times a_3 = 2 \times (2+1)a_2$

$a_3 = 3a_2$

$= 12$

finding a_2 and a_3 A1

Hence, $b_1 = \frac{a_1}{1} = 1$, $b_2 = \frac{a_2}{2} = 2$ and $b_3 = \frac{a_3}{3} = 4$. A1

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b. $na_{n+1} = 2(n+1)a_n$

$\frac{a_{n+1}}{n+1} = \frac{2a_n}{n}$

Since $b_n = \frac{a_n}{n}$ and $b_{n+1} = \frac{a_n + 1}{n+1}$: M1

$$\frac{b_{n+1}}{b_n} = \frac{\frac{a_{n+1}}{n+1}}{\frac{a_n}{n}}$$

$$= \frac{2 \times \frac{a_n}{n}}{\frac{a_n}{n}}$$

$$= 2$$

$b_1 = 1$

Therefore, b_n is a geometric sequence with a common ratio of 2 and starting term of 1. A1

Note: Consequential on answer to ***Question 4a.***

c. Since b_n is a geometric sequence with a common ratio of 2 and starting value of 1:

$b_n = 2^{n-1}$ M1

$b_n = \frac{a_n}{n}$

$a_n = n \times b_n$

$= n \times 2^{n-1}$ A1

Note: Consequential on answer to ***Question 4b.***

Question 5 (5 marks)

a. Since the three adults must be together, they can be grouped together as one object. Therefore, with the five children, there are six objects to arrange and thus 6! arrangements.

Within the group of three adults, there are 3! arrangements.

Hence, there are $3! \times 6! = 4320$ possible arrangements. A1

b. Since all adults must be separate, the children need to be arranged first. Therefore, there are 5! arrangements for the children.

Each adult must stand between two children. Each adult can stand in six possible places.

Therefore, there are $\frac{6!}{3!}$ ways to arrange the three adults.

Hence, there are $5! \times \frac{6!}{3!} = 14\ 400$ possible arrangements. A1

c. Since an adult cannot stand at either end of the line, a child must stand at either end.

That is two out of the five children; therefore, there are $\frac{5!}{3!}$ arrangements.

For the remaining six people (three adults and three children), there are 6! arrangements.

Hence, there are $\frac{5!}{3!} \times 6! = 14\ 400$ possible arrangements. A1

d. There are two possibilities that fulfil the requirements of the question.

The first possibility is that there is a child standing at both ends of the line. There are $\frac{5!}{3!} \times 6 = 14\ 400$ arrangements.

The second possibility is that there is an adult standing at one end of the line and a child standing at the other end. There are $2 \times \frac{3!}{2!} \times \frac{5!}{4!} \times 6! = 21\ 600$ arrangements.

both possibilities A1

Hence, there are 14 400 + 21 600 = 36 000 possible arrangements. A1

*Note: Consequential on answer to **Question 5c.***

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Question 6 (6 marks)

a. Converting the complex number to polar form gives:

$$r = \sqrt{1^2 + (-1)^2}$$
$$= \sqrt{2}$$

As $\tan(\theta) = -1$ and θ is in quadrant 4, $\theta = -\frac{\pi}{4}$.

$$1 - i = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$$ M1

$$(1-i)^9 = \left(\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)\right)^9$$
$$= 16\sqrt{2}\text{cis}\left(-\frac{9\pi}{4}\right)$$
$$= 16\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$$
$$= 16\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 16 - 16i$$ A1

b. $$P(-i) = 2\times(-i)^4 + 2i\times(-i)^3 - 6\times(-i) - 6i$$
$$= 0$$ A1

c. Since $-i$ is a root of the polynomial, $z + i$ is a factor.

Using long division gives:

$$\begin{array}{r} 2z^3 - 6 \\ z+i\,\overline{)\,2z^4 + 2iz^3 - 6z - 6i} \\ \underline{2z^4 + 2iz^3 } \\ -6z - 6i \\ \underline{-6z - 6i} \\ 0 \end{array}$$

Therefore, $P(z) = 2(z + i)(z^3 - 3)$. A1

As $z^3 - 3 = 0$:

$$z^3 = 3\text{cis}(2k\pi),\ k = 0, 1, 2$$ A1

$$z = \sqrt[3]{3}\text{cis}\left(\frac{2k\pi}{3}\right),\ k = 0, 1, 2$$

$$= \sqrt[3]{3},\ \text{cis}\left(-\frac{\pi}{2}\right), \sqrt[3]{3}\text{cis}\left(\frac{2\pi}{3}\right) \text{ and } \sqrt[3]{3}\text{cis}\left(-\frac{2\pi}{3}\right)$$ A1

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Question 7 (5 marks)

a. $\cos(\theta) = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\|\underset{\sim}{u}\|\|\underset{\sim}{v}\|}$

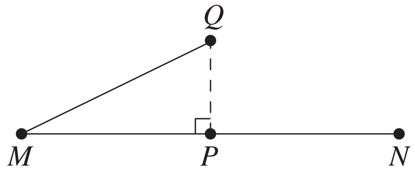
$= \dfrac{\left(0, \sqrt{3}, 0\right) \cdot \left(0, 1, \sqrt{3}\right)}{\left\|\left(0, \sqrt{3}, 0\right)\right\| \left\|\left(0, 1, \sqrt{3}\right)\right\|}$ M1

$= \dfrac{\sqrt{3}}{2\sqrt{3}}$

$= \dfrac{1}{2}$

Therefore, $\theta = \dfrac{\pi}{3}$. A1

b.



From the diagram above, it can be seen that *MP* is the parallel component of the vector projection of *MQ* onto *MN*.

$\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = (2, 2, -2)$ and $\overrightarrow{MQ} = \overrightarrow{MO} + \overrightarrow{OQ} = (2, -2, -2)$

Letting $\overrightarrow{MP} = k\overrightarrow{MN}$ gives:

$\overrightarrow{PQ} = \overrightarrow{MQ} - \overrightarrow{MP}$

$= (2 - 2k, -2 - 2k, -2 + 2k)$

Since $\overrightarrow{PQ} \perp \overrightarrow{MN}$:

$\overrightarrow{MN} \cdot \overrightarrow{PQ} = 0$

$(2, 2, -2) \cdot (2 - 2k, -2 - 2k, -2 + 2k) = 0$ M1

$k = \dfrac{1}{3}$ A1

$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

$= (1, 1, 1) + \dfrac{1}{3}(2, 2, -2)$

$= \left(\dfrac{5}{3}, \dfrac{5}{3}, \dfrac{1}{3}\right)$ A1

Note: A diagram is not required to obtain full marks.

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Question 8 (5 marks)

a. **i.** Applying the double angle formula gives:

$$\tan(\alpha) = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1 - \tan^2\left(\frac{\alpha}{2}\right)}$$

$$= \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{4}{3}$$

$\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ and $\sin^2(\alpha) + \cos^2(\alpha) = 1$; therefore, given that α is in quadrant 1,

$\sin(\alpha) = \dfrac{4}{5}$ and $\cos(\alpha) = \dfrac{3}{5}$. A1

ii. Given that $\cos(\alpha - \beta) = \dfrac{\sqrt{2}}{10}$, using trigonometric identity gives:

$$\sin(\alpha - \beta) = \pm\sqrt{1 - \left(\frac{\sqrt{2}}{10}\right)^2}$$

Since $\alpha < \beta$, $\alpha - \beta$ is in quadrant 4. Therefore:

$$\sin(\alpha - \beta) = -\sqrt{1 - \left(\frac{\sqrt{2}}{10}\right)^2}$$

$$= -\frac{7\sqrt{2}}{10}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

$$= \frac{\frac{4}{3} - \tan(\beta)}{1 + \frac{4}{3}\tan(\beta)}$$

$$= \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}$$

$$= -7$$ A1

$\tan(\beta) = -1$

Therefore, $\beta = \dfrac{3\pi}{4}$. A1

b. $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\dfrac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)}$

$=\dfrac{m}{n}$ M1

$n\times\sin(\alpha)\cos(\beta)+n\times\cos(\alpha)\sin(\beta)=m\times\sin(\alpha)\cos(\beta)-m\times\cos(\alpha)\sin(\beta)$

$(n+m)\cos(\alpha)\sin(\beta)=(m-n)\sin(\alpha)\cos(\beta)$

$\dfrac{\cos(\alpha)\sin(\beta)}{\sin(\alpha)\cos(\beta)}=\dfrac{\tan(\beta)}{\tan(\alpha)}$

$=\dfrac{m-n}{m+n}$ A1

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