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YEAR 12 *Trial Exam Paper*

2023
SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book includes:

- correct solutions, with full working
- explanatory notes
- mark allocations
- tips.

SECTION A – Multiple-choice questions

Question	Answer
1	A
2	B
3	D
4	B
5	B
6	D
7	E
8	B
9	E
10	C
11	B
12	C
13	A
14	D
15	A
16	D
17	E
18	C
19	D
20	A

Question 1

Answer: A

Explanatory notes

The graph has a vertical asymptote $x=0$. Away from the origin, it follows the curve $y=x^2$.

Hence we can expect that the equation of the graph will be of the form $y=x^2 \pm \dfrac{a}{x}=\dfrac{x^3 \pm a}{x}$.

If $a>0$, then the graph of $y=\dfrac{x^3-a}{x}$ has a positive *x*-intercept. Therefore, $y=\dfrac{x^3+a}{x}$.

Tip

- *Use CAS to sketch* $y=\dfrac{x^3+1}{x}$ *to show that the correct answer is*

  $y=\dfrac{x^3+a}{x},\ a>0$.

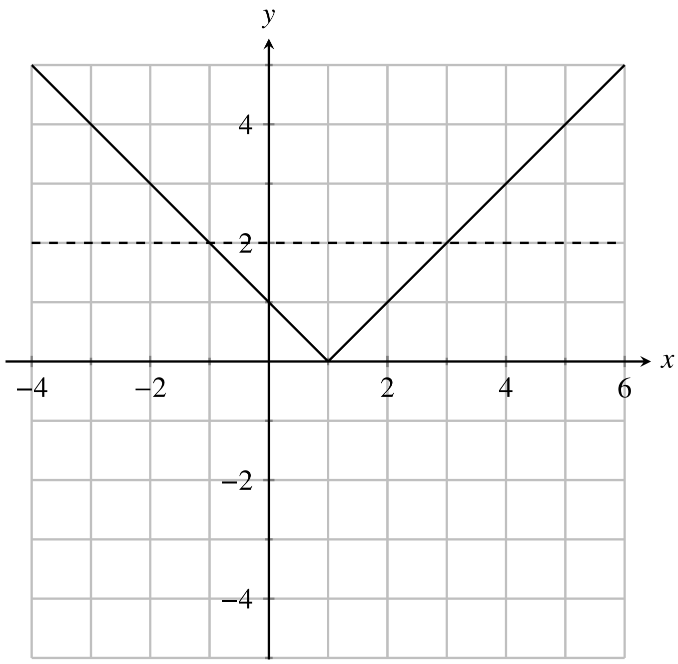
Question 2

Answer: B

Explanatory notes

We require $-1 \le \frac{|x-1|}{2} \le 1$, but since $|x-1| \ge 0$ it follows that $0 \le |x-1| \le 2$.

Consider the graph of $y = |x-1|$:



From the graph we see that $\text{dom } f = [-1, 3]$.

As $\frac{|x-1|}{2} \in [0, 1]$, $\text{ran } f = \left[0, \frac{\pi}{2}\right]$.

Tips

- *Draw the graph of* $y = |x-1|$ *to help find the domain of f.*
- *Be careful! The range of* f *is not the full range of* $y = \arccos(x)$.
- *Sketching the function on CAS can be useful.*

Question 3

Answer: D

Explanatory notes

Note that $z\,\bar{z} = (x+iy)(x-iy) = x^2 + y^2$.

So the annulus, which is the area between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$, is given by $4 \le z\,\bar{z} \le 16$.

Tips

- *The incorrect options all look reasonable but are wrong for various reasons.*
- *Ensure that you remember the standard forms of a circle of radius r centred at the origin in complex form, $|z| = r$, and in Cartesian form, $x^2 + y^2 = r^2$.*

Question 4

Answer: B

Explanatory notes

The coefficients of $p(z)$ are all real and so, by the conjugate root theorem, $(z-1-3i)$ is also a linear factor. Therefore, we can find a quadratic factor for $p(z)$:

$$\begin{aligned}(z-1+3i)(z-1-3i) &= (z-1)^2 + 9 \\ &= z^2 - 2z + 10\end{aligned}$$

The remaining linear factor of $p(z)$ is $(z+3)$. Therefore

$$\begin{aligned}p(z) &= (z^2 - 2z + 10)(z+3) \\ &= z^3 + z^2 + 4z + 30\end{aligned}$$

So $a = 1$, $b = 4$ and $\frac{a}{b} = \frac{1}{4}$.

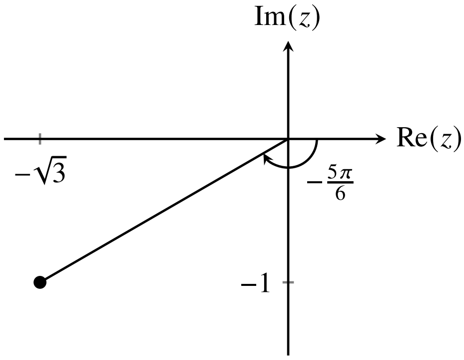
Tips

- *The conjugate root theorem applies when the coefficients of the polynomial are all real.*
- *The linear factor $(z+3)$ is identified by inspection. Use of long division (or similar) is not required here.*

Question 5

Answer: B

Explanatory notes



We can see that $a = -\sqrt{3}$.

Tip

- *Use of a diagram is recommended here.*

Question 6

Answer: D

Explanatory notes

Note that $\int_0^{\frac{\pi}{9}} \sin^3(3x)dx = \int_0^{\frac{\pi}{9}} \left(1-\cos^2(3x)\right)\sin(3x)dx$.

Let $u = \cos(3x)$, so $\frac{du}{dx} = -3\sin(3x) \Rightarrow -\frac{1}{3}du = \sin(3x)dx$.

When $x = 0$, $u = 1$; and when $x = \frac{\pi}{9}$, $u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Therefore

$$\begin{aligned}\int_0^{\frac{\pi}{9}} \sin^3(3x)dx &= -\frac{1}{3}\int_1^{\frac{1}{2}} \left(1-u^2\right)du \\ &= \frac{1}{3}\int_{\frac{1}{2}}^{1} \left(1-u^2\right)du\end{aligned}$$

Tips

- *Use the trigonometric identity* $\sin^2(A) + \cos^2(A) = 1$.
- *Note the reversal of the terminals when the sign of the integral is changed.*

Question 7

Answer: E

Explanatory notes

Water drains from the funnel at a rate of $15\,\text{cm}^3\ \text{min}^{-1}$ and so $\dfrac{dV}{dt} = -15$.

The radius of the cone is one-third of its height.

Therefore, $V = \dfrac{1}{3}\pi\left(\dfrac{1}{3}h\right)^2 h = \dfrac{1}{27}\pi h^3$ and so $\dfrac{dV}{dh} = \dfrac{1}{9}\pi h^2$.

Then

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV}\cdot\frac{dV}{dt}\\ &= \frac{9}{\pi h^2}\cdot -15\\ &= -\frac{135}{\pi h^2}\end{aligned}$$

So the rate of decrease is $\dfrac{135}{\pi h^2}$.

Tips

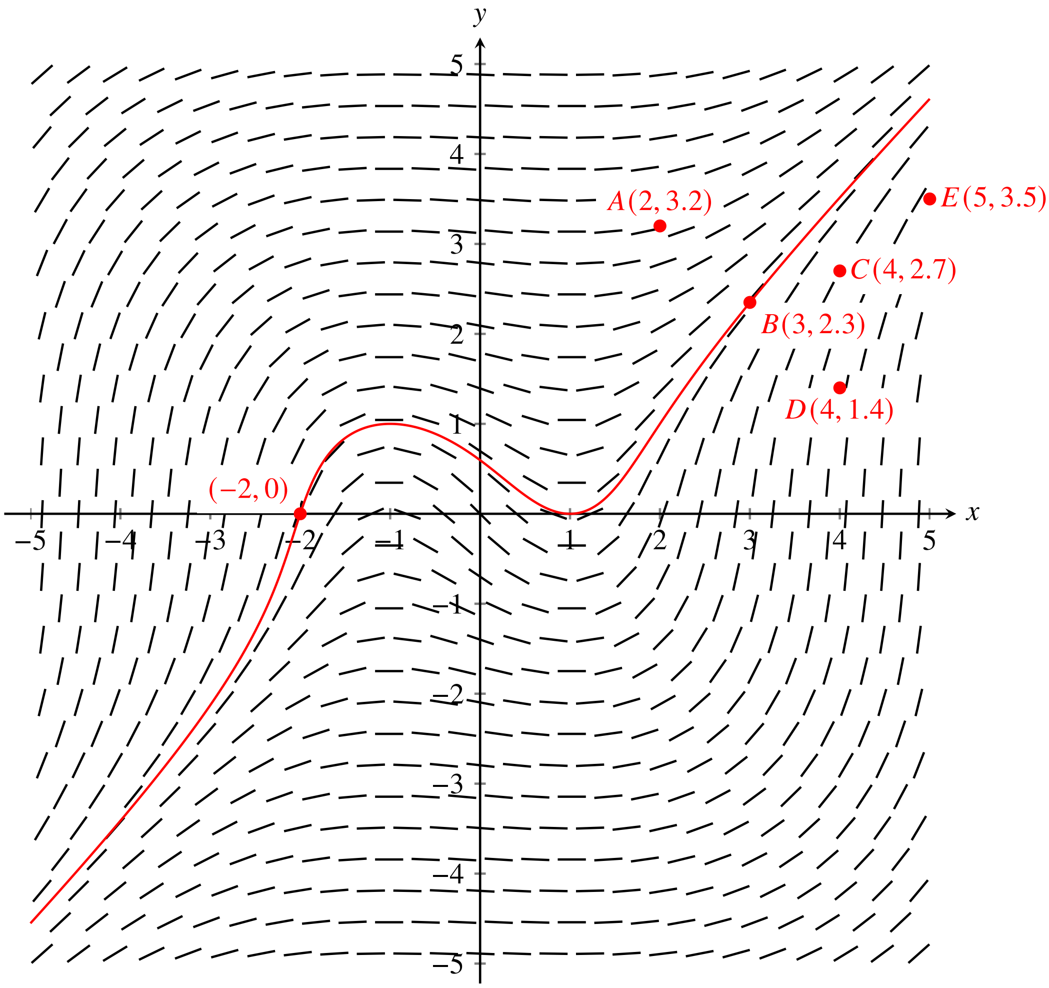
- *Begin by writing what we know:* $\dfrac{dV}{dt} = -15$ *and* $V = \dfrac{1}{3}\pi\left(\dfrac{1}{3}h\right)^2 h$ *(using the formula for the volume of a cone of base radius* r *and height* h*).*
- *Use the chain rule to find* $\dfrac{dh}{dt} = \dfrac{dh}{dV}\cdot\dfrac{dV}{dt}$. *(Note the cancelling of the* dV *terms.)*

Question 8

Answer: B

Explanatory notes

Identify point (–2, 0) on the axes and follow the direction field around.



Tips

- *Carefully follow the curve around.*
- *See which of the options fits best, remembering that your curve may not follow the field precisely.*

Question 9

Answer: E

Explanatory notes

Since $\frac{dx}{dt}=2$ and $\frac{dy}{dt}=t+1$, the area of the curved part of the surface is found by evaluating the integral $\int_2^4 2\pi(2t-1)\sqrt{2^2+(t+1)^2}\,dt=\int_2^4 2\pi(2t-1)\sqrt{t^2+2t+5}\,dt$.

Tips

- *From the formula sheet, the area of the curved part of the surface obtained when the curve described by the parametric equations $x=x(t)$ and $y=y(t)$ between $t=a$ and $t=b$ is rotated about the y-axis is* $\int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$.
- *The differentiation and expanding of the terms under the square root are best done, in this instance, by hand.*

Question 10

Answer: C

Explanatory notes

We have $\overrightarrow{AB}=-2\underset{\sim}{i}-3\underset{\sim}{j}+2\underset{\sim}{k}$ and $\overrightarrow{AC}=\underset{\sim}{i}-3\underset{\sim}{k}$.

Therefore, a vector perpendicular to the plane containing points *A*, *B* and *C* is $\overrightarrow{AB}\times\overrightarrow{AC}=9\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}$.

So the equation of the plane containing points *A*, *B* and *C* is $9x-4y+3z=26$.

Tips

- *Use the cross product to find a vector normal (i.e. perpendicular) to the plane.*
- *Use CAS to find the cross product of the vectors.*
- *The value of the constant (on the right-hand side of the equation) is found by substituting one of the points into the equation $9x-4y+3z=d$*

  Note: *A dot product could also be calculated using either of the three points. For example:*

  $\left(9\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}\right)\cdot\left(3\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)=27-4+3=26$

Question 11

Answer: B

Explanatory notes

The area of the triangle *ABC* is

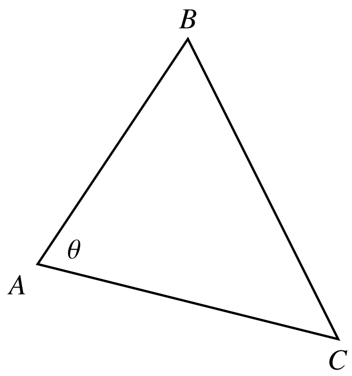
$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|9\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}\right|$$

$$= \frac{1}{2}\sqrt{81+16+9}$$

$$= \frac{\sqrt{106}}{2}$$

Tips

- *The cross product can be used to find the area of a triangle:*



Let θ *be the angle BAC. Then the area is* $\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$.

- *Use CAS to compute the length (i.e. the normal) of the vector* $9\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$ *(or use the result from Question 10).*

Question 12

Answer: C

Explanatory notes

Let $\underset{\sim}{n}_1$ be a vector normal (i.e. perpendicular) to plane Π_1 and let $\underset{\sim}{n}_2$ be a vector normal (i.e. perpendicular) to plane Π_2:

$\underset{\sim}{n}_1 = 5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{n}_2 = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

Then $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = -7\underset{\sim}{i} - \underset{\sim}{j} + 17\underset{\sim}{k}$.

Therefore, the line of intersection of planes Π_1 and Π_2 is parallel to $7\underset{\sim}{i} + \underset{\sim}{j} - 17\underset{\sim}{k}$.

Tips

- *The line of intersection of two planes is contained in both planes and is therefore perpendicular to the normal of both planes.*
- *A plane in Cartesian form $ax + by + cz = d$ has a normal vector $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$.*
- *Use the cross product to find the direction of the line of intersection because the cross product is a vector perpendicular to two given vectors, which are, in this case, the normal vectors to each of the two planes.*

Question 13

Answer: A

Explanatory notes

Show that $O(1) = 1$ and assume that $O(n) = n^2$, and from this deduce that $O(n+1) = (n+1)^2$.

Tip

- *A proof by induction follows a strict sequence of steps.*

Question 14

Answer: D

Explanatory notes

A trace table (or desk check) may be used here.

i	Output	Digits in output
0	0	1
1	1	1
4	4	1
13	13	2
40	40	2

A total of seven digits are printed.

Tip

- *The question asks how many digits are printed, not how many numbers are printed.*

Question 15

Answer: A

Explanatory notes

Use of a table is recommended when stepping through Euler's method:

n	x_n	y_n	y'_n
0	1	1	2
1	$1+h$	$1+2h$	$2(1+h)^2$
2	$1+2h$	$1+2h+2h(1+h)^2$	

Expanding the final result gives $1+2h+2h(1+h)^2 = 1+4h+4h^2+2h^3$.

Question 16

Answer: D

Explanatory notes

By the fundamental theorem of calculus, if $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

$$y(3) - y(1) = \int_1^3 \frac{dy}{dx}\,dx$$
$$\Rightarrow y(3) = \int_1^3 \sqrt{\cos(x)+3}\,dx + \sqrt{2}$$
$$\approx 4.657$$

Question 17

Answer: E

Explanatory notes

From the formula sheet:

$$a = v\cdot\frac{dv}{dx} = \frac{1}{\sqrt{2-x^2}}\cdot\frac{x}{\left(2-x^2\right)^{\frac{3}{2}}}$$
$$= \frac{x}{\left(2-x^2\right)^2}$$

CAS may be used to differentiate and simplify.

Question 18

Answer: C

Explanatory notes

By considering the area of a trapezium, in the first 10 seconds the particle travels $\frac{1}{2}(10+4)\times 4 = 28\text{ m}$ away from the starting point.

Between $t = 10$ and $t = a$, the particle travels $\frac{1}{2}(a-10)^2$ m.

Solving $\frac{1}{2}(a-10)^2 = 28$ for a gives $a \approx 17.48$.

Therefore, the particle passes the starting point when t is between 17 and 18 seconds.

Question 19

Answer: D

Explanatory notes

Use the results

$\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$

$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(A) + b^2\mathrm{Var}(Y)$

to find

$$\begin{aligned}\mathrm{E}(W) &= 4\times 5 - 3\times 4\\ &= 8\end{aligned}$$

$$\mathrm{Var}(W) = 16\sigma^2 + 9\sigma^2 = 25\sigma^2$$

$$\Rightarrow \mathrm{sd}(W) = 5\sigma$$

Now $\Pr(W < 10.5) = \Pr\left(Z < \frac{10.5 - 8}{5\sigma}\right)$ and so $\frac{2.5}{5\sigma} = 2 \Rightarrow \sigma = \frac{1}{4}$.

Question 20

Answer: A

Explanatory notes

The sampling distribution is $\bar{X}$, where $\mathrm{E}(\bar{X}) = 120$ and $\mathrm{sd}(\bar{X}) = \frac{6}{\sqrt{10}}$.

Therefore, $\Pr(\bar{X} < 118) \approx 0.1459$.

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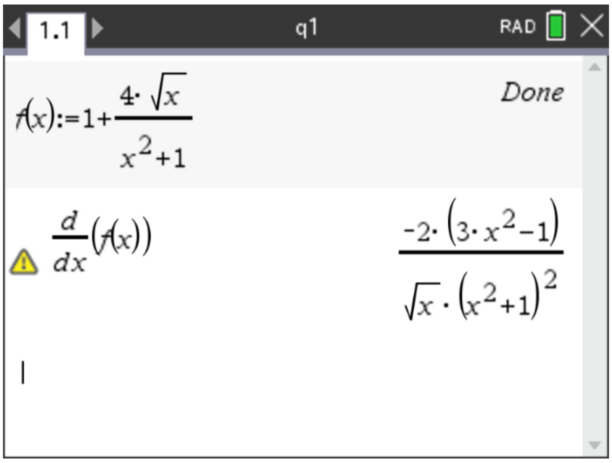
SECTION B

Question 1a.i.

Worked solution

Calculate the derivative by hand or use CAS to find $f'(x) = \dfrac{-6x^2+2}{\sqrt{x}\left(1+x^2\right)^2}$.



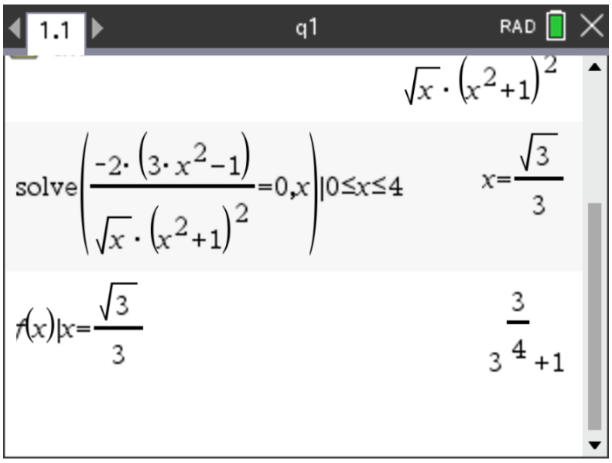
Mark allocation: 1 mark

- 1 mark for the correct answer, in the required form

Question 1a.ii.

Worked solution

The coordinates of the stationary point are $\left(\dfrac{\sqrt{3}}{3}, 3^{\frac{3}{4}}+1\right) = \left(\dfrac{1}{\sqrt{3}}, 3^{\frac{3}{4}}+1\right)$.



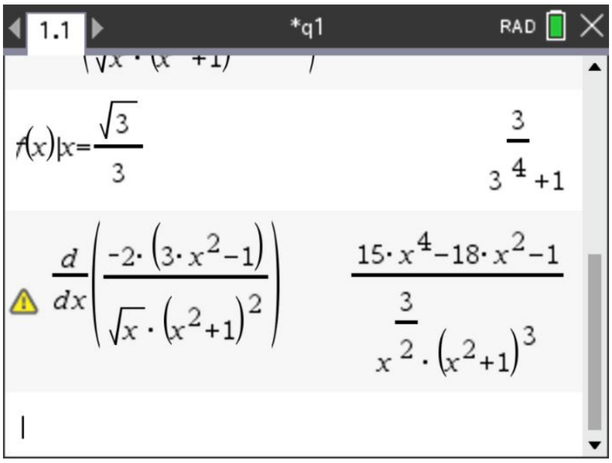
Mark allocation: 1 mark

- 1 mark for the correct coordinates (rationalisation not required)

Question 1b.i.

Worked solution

The point of inflection occurs when the second derivative is zero. Use CAS to find the second derivative.
The required polynomial equation is the numerator equated to zero:
$15x^4 - 18x^2 - 1 = 0$



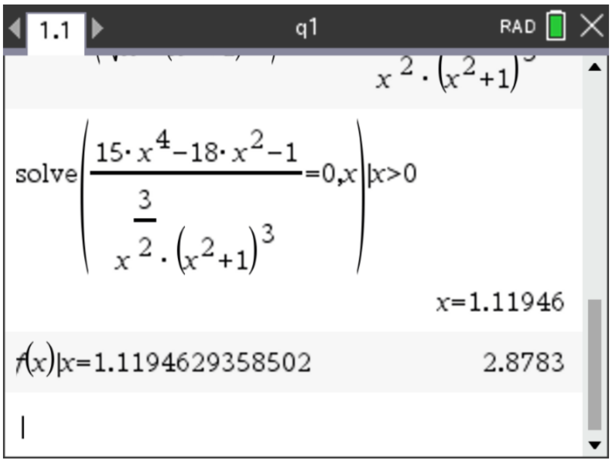
Mark allocation: 1 mark

- 1 mark for the correct polynomial equation
  Note: It is not sufficient to write down just the polynomial. You must equate it to zero.

Question 1b.ii.

Worked solution

Equate the second derivative to zero. Substitute the value found into the function $f(x)$.
The coordinates, correct to three decimal places, are (1.119, 2.878).



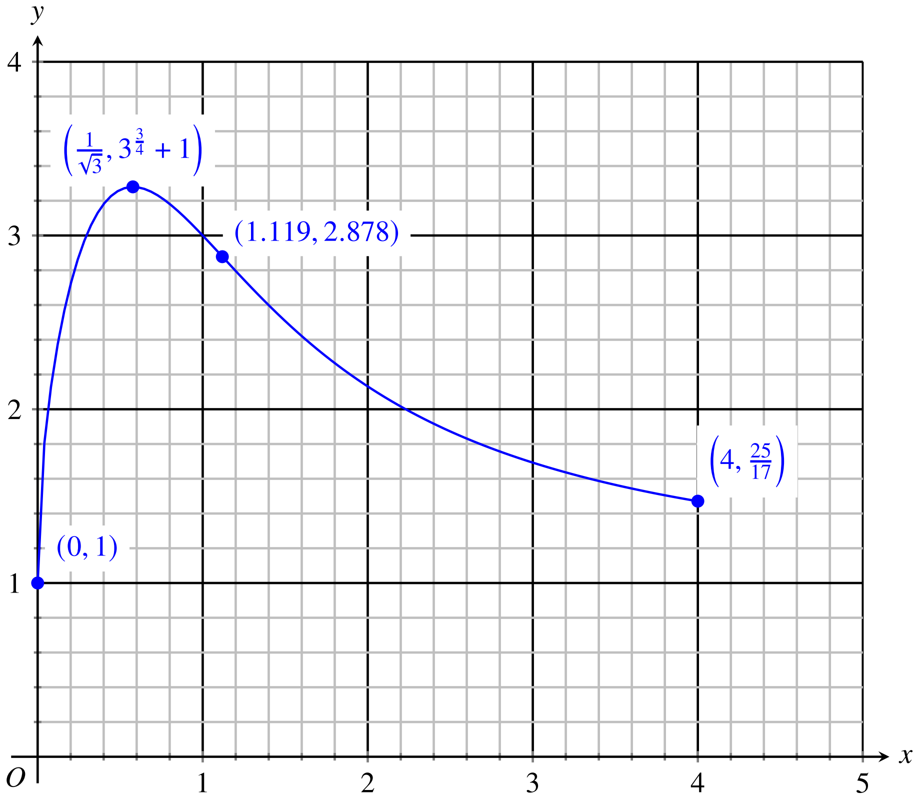
Mark allocation: 1 mark

- 1 mark for the correct coordinates to three decimal places

Question 1c.

Worked solution

The graph of f is shown below.



Mark allocation: 3 marks

- 1 mark for the correct shape
- 1 mark for labelling the end points with their correct coordinates
- 1 mark for the stationary point and point of inflection labelled with their correct coordinates

Question 1d.i.

Worked solution

An equation involving an integral is $\pi\int_0^h\left(1+\frac{4\sqrt{x}}{x^2+1}\right)^2 dx = 50$.

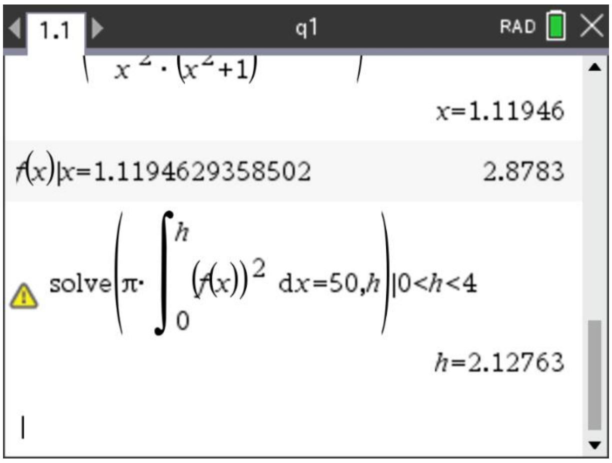
Mark allocation: 1 mark

- 1 mark for a correct integral

Question 1d. ii.

Worked solution

Solving for *h* gives $h = 2.13$ (correct to two decimal places).



Mark allocation: 1 mark

- 1 mark for the value of *h*, correct to two decimal places

Question 1e.

Worked solution

Note that

$$1+\left(f'(x)\right)^2 = \frac{4\left(3x^2-1\right)^2}{x\left(x^2+1\right)^4}+1$$

$$= \frac{4\left(3x^2-1\right)^2 + x\left(x^2+1\right)^4}{x\left(1+x^2\right)^4}$$

The values of m and n are 4 and 1, respectively.

Mark allocation: 2 marks

- 1 mark for simplifying the expression $1+\left(f'(x)\right)^2$
- 1 mark for finding the correct values of m and n

Question 2a.

Worked solution

Let $z = x + iy$. Hence

$$x^2 + (y-2)^2 = \left(x+\sqrt{3}\right)^2 + (y-3)^2$$

$$x^2 + y^2 - 4y + 4 = x^2 + 2\sqrt{3}x + 3 + y^2 - 6y + 9$$

$$2y = 2\sqrt{3}x + 8$$

$$y = \sqrt{3}x + 4$$

Mark allocation: 2 marks

- 1 mark for substituting $z = x + iy$ into the equation and squaring both sides to remove the square roots
- 1 mark for the result obtained correctly

Question 2b.

Worked solution

The set of points described form the points on the circumference of a circle of radius 2 centred at (0, 2).

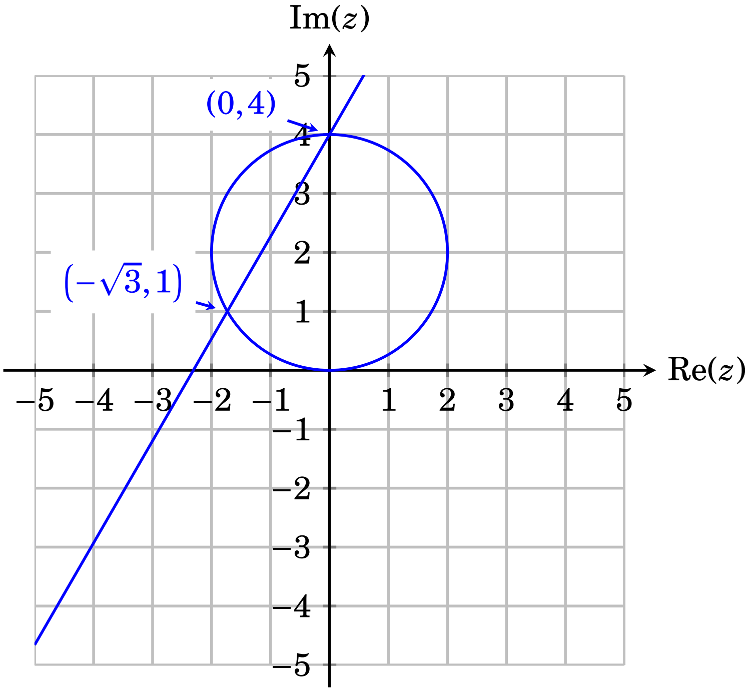
The Cartesian equation is $x^2 + \left(y-2\right)^2 = 4$.

Mark allocation: 1 mark

- 1 mark for the correct Cartesian equation of the circle

Question 2c.i. and ii.

Worked solution

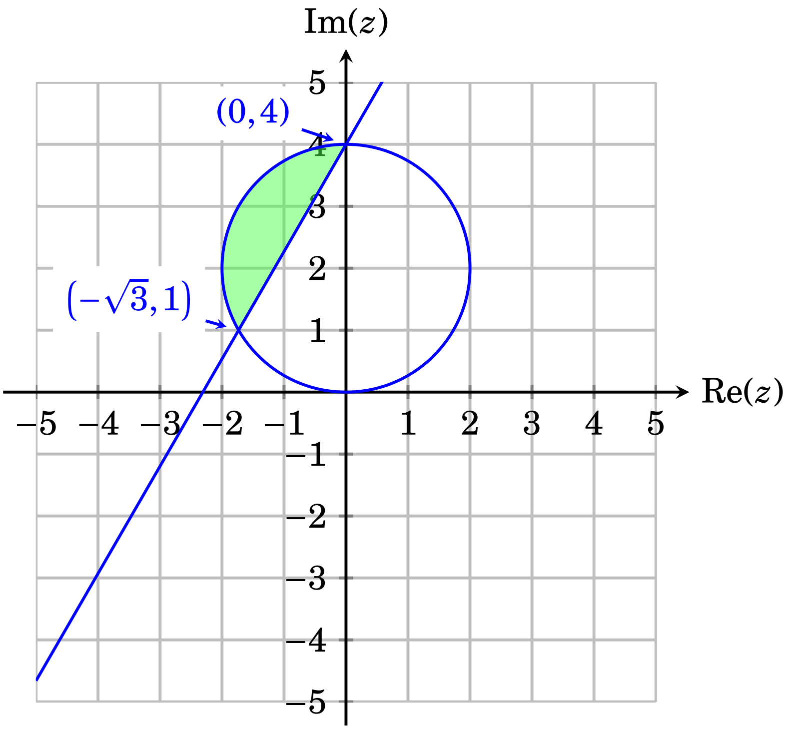


Mark allocation: 3 marks

- 1 mark for the correctly drawn straight line
- 1 mark for the correctly drawn circle
- 1 mark for the correct coordinates of each point of intersection

Question 2d.

Worked solution



Mark allocation: 1 mark

- 1 mark for shading the correct region

Question 2e.

Worked solution

The area A of the region is

$$A = \frac{1}{2} \times 2^2 \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3} \right) \right)$$

$$= 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right)$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

$$= \frac{4\pi - 3\sqrt{3}}{3}$$

Mark allocation: 2 marks

- 1 mark for use of the segment formula with angle $\frac{2\pi}{3}$
- 1 mark for the correct answer, in the required form

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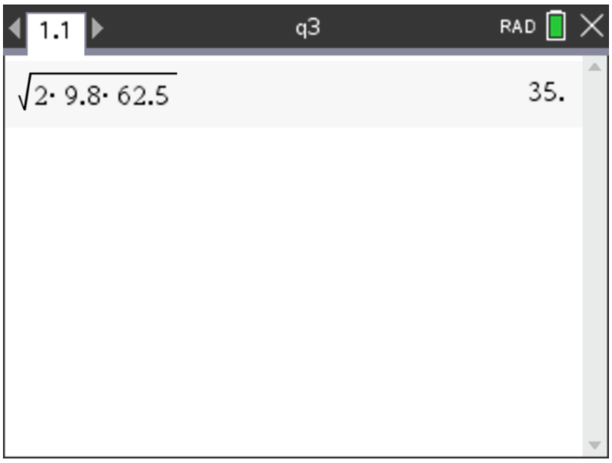
Question 3a.

Worked solution

$$v^2 = u^2 + 2as$$
$$= 0 + 2 \times 9.8 \times 62.5$$
$$\Rightarrow v = 35 \text{ m s}^{-1}$$



Mark allocation: 2 marks

- 1 mark for using the constant acceleration formula
- 1 mark for the correct answer

Question 3b.

Worked solution

The acceleration is

$$a = g - \frac{1}{16}v^2$$
$$= -\frac{1}{16}g\left(v^2 - 16\right)$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3c.

Worked solution

The terminal velocity occurs when the acceleration is zero; that is, when $v = 4 \text{ m s}^{-1}$.

Mark allocation: 1 mark

- 1 mark for the correct answer

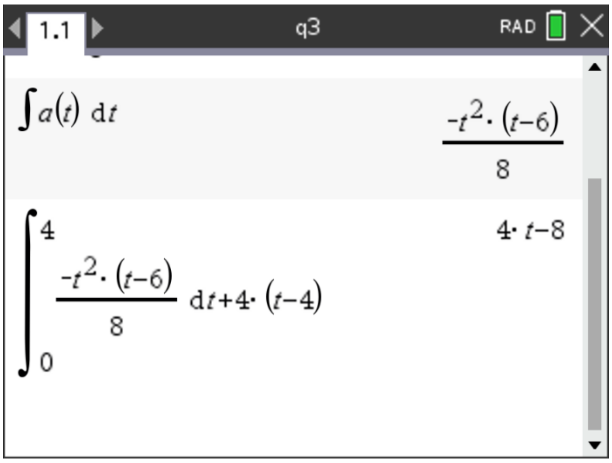
Question 3d.

Worked solution

The velocity is given by

$$v(t)=\begin{cases}\dfrac{-t^2(t-6)}{8} & 0\le t\le 4\\ 4 & t>4\end{cases}$$

So the distance travelled is $\int_0^4 \frac{-t^2(t-6)}{8}\,d\,t+4(T-4)=4\,T-8$.



Mark allocation: 3 marks

- 1 mark for stating the velocity function, where $0\le t\le 4$
- 1 mark for integrating the velocity function, including $4(T-4)$
- 1 mark for the correct answer

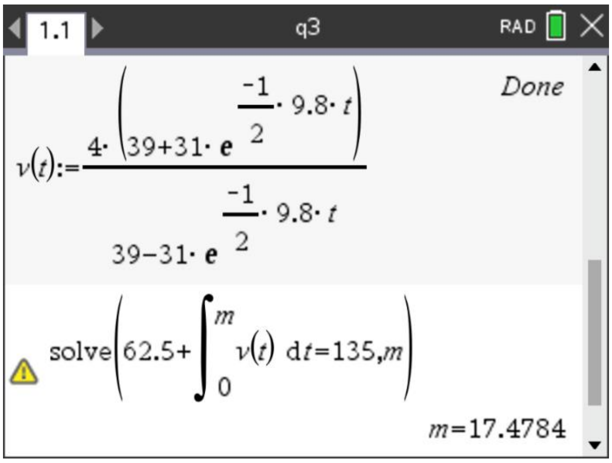
Question 3e.

Worked solution

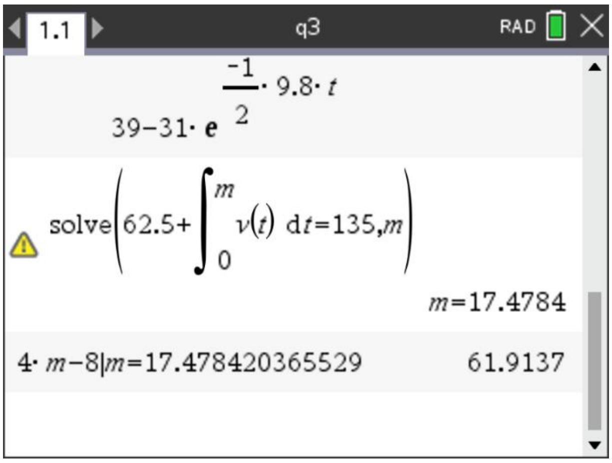
Find the time, m, the BASE jumper has fallen after opening their parachute:

$$62.5+\int_0^m v(t)\,d\,t=135$$

$$\Rightarrow m=17.48 \text{ seconds}$$



The distance the drone travels upwards in the same time is 4 × 17.48 – 8 = 61.91 m.



The height of the bridge is 135 + 61.91 = 197 m, correct to the nearest metre.

Mark allocation: 3 marks

- 1 mark for deriving 17.48 seconds, the time spent by the BASE jumper after their parachute was opened and they had travelled a total distance of 135 m
- 1 mark for the correct distance travelled upwards by the drone
- 1 mark for the correct height of the bridge, correct to the nearest metre

Question 4a.

Worked solution

Find two points on the line $l_1 : A(1,\ -1,\ 3)$ and $B(2,\ 5,\ 1)$, for example.

To do this, consider writing down the parametric equations for the line:

$x = 1 + t$

$y = -1 + 6t$

$z = 3 - 2t$

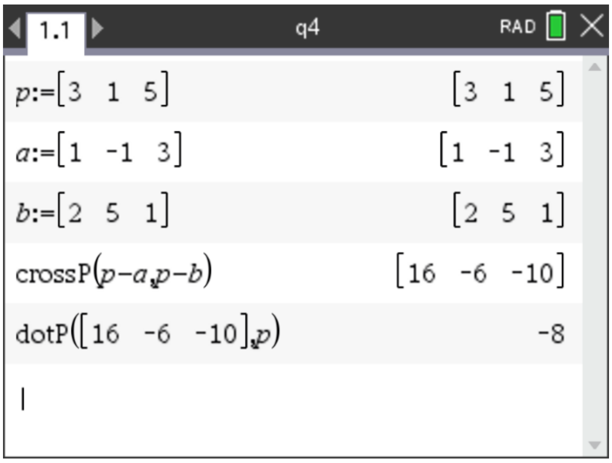
and letting $t = 0$ and $t = 1$ to find two points.

Then $\overrightarrow{PA} \times \overrightarrow{PB} = 16\underset{\sim}{i} - 6\underset{\sim}{j} - 10\underset{\sim}{k}$, the normal to plane Π_1.

Therefore the Cartesian equation of plane Π_1 is:

$16x - 6y - 10z = -8$

$\Rightarrow 8x - 3y - 5z = -4$



Mark allocation: 3 marks

- 1 mark for finding two points on the line
- 1 mark for the correct cross product
- 1 mark for the correct Cartesian equation of the plane

Question 4b.i.

Worked solution

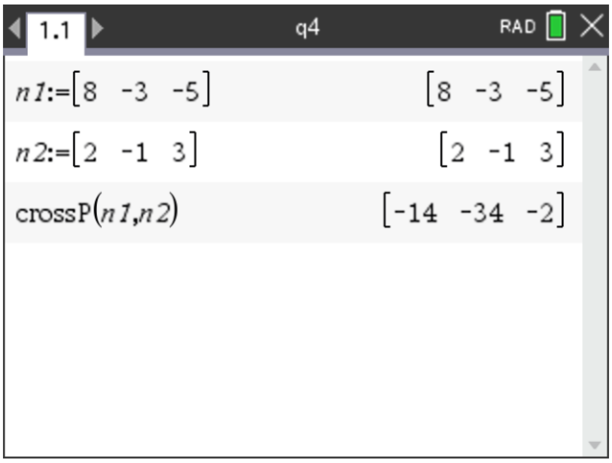
A vector normal (i.e. perpendicular) to plane Π_1 is $8\underset{\sim}{i}-3\underset{\sim}{j}-5\underset{\sim}{k}$.

A vector normal (i.e. perpendicular) to plane Π_2 is $2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}$.

The line of intersection is parallel to the cross product of the normal vectors to planes Π_1 and Π_2:

$$\left(8\underset{\sim}{i}-3\underset{\sim}{j}-5\underset{\sim}{k}\right)\times\left(2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)=-14\underset{\sim}{i}-34\underset{\sim}{j}-2\underset{\sim}{k}$$

and so is parallel to $7\underset{\sim}{i}+17\underset{\sim}{j}+\underset{\sim}{k}$.



Mark allocation: 1 mark

- 1 mark for writing down the normal vector of both planes and finding the cross product, from which the required result can be derived

Question 4b.ii.

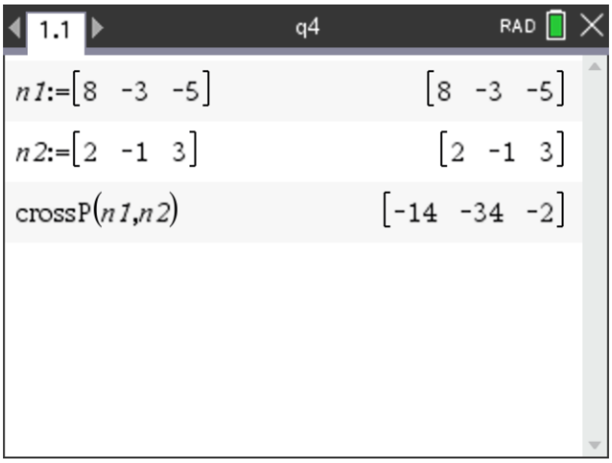
Worked solution

Find a point common to both planes. For instance, let $z = 0$. Then

$8x - 3y = -4$

$2x - y = 2$

This gives $(-5,\ -12,\ 0)$ as a point common to both planes.



Then $\underset{\sim}{r}(t) = -5\underset{\sim}{i} - 12\underset{\sim}{j} + \left(7\underset{\sim}{i} + 17\underset{\sim}{j} + \underset{\sim}{k}\right)t$.

Mark allocation: 2 marks

- 1 mark for finding a point common to both planes
- 1 mark for a correct vector equation

Question 4c.

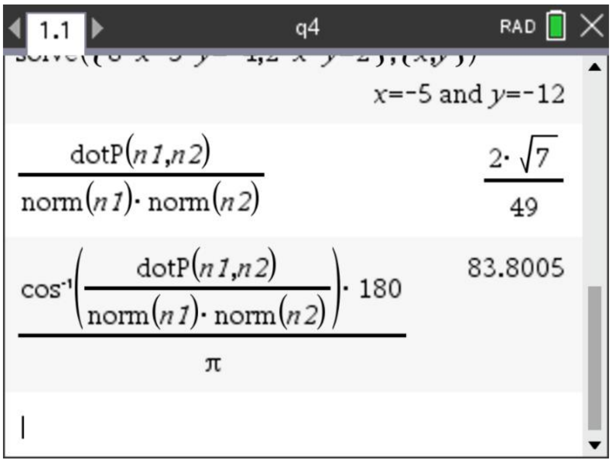
Worked solution

The angle θ between the planes Π_1 and Π_2 is the angle between their normal (i.e. perpendicular) vectors:

$$\cos(\theta) = \frac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|}$$

$$= \frac{2\sqrt{7}}{49}$$

So $\theta = 83.80°$.



Mark allocation: 2 marks

- 1 mark for using the dot (scalar) product of the two normal vectors to the planes
- 1 mark for the angle, correct to two decimal places

Question 4d.i.

Worked solution

The vector equation of the line is found by combining the given point and the appropriate direction, as follows:

$$\underset{\sim}{r}_3(t) = 4\underset{\sim}{i} + 9\underset{\sim}{j} + \underset{\sim}{k} + \left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)t$$

Mark allocation: 1 mark

- 1 mark for the correct equation of the line

Question 4d.ii.

Worked solution

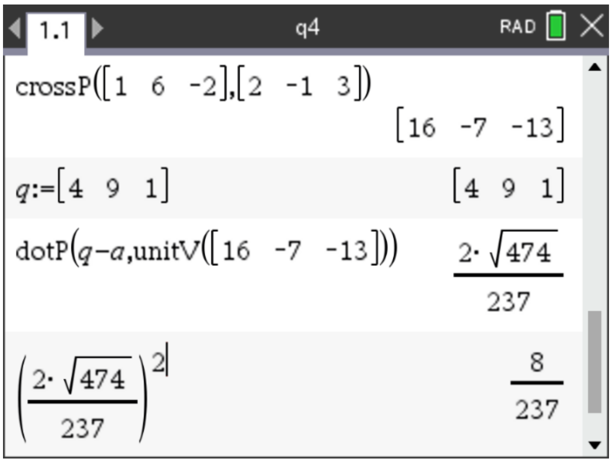
The distance between two lines is found by first finding a vector normal to the two planes that contain each of the lines:

$$\begin{aligned}\underset{\sim}{n} &= \left(\underset{\sim}{i}+6\underset{\sim}{j}-2\underset{\sim}{k}\right)\times\left(2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)\\ &=16\underset{\sim}{i}-7\underset{\sim}{j}-13\underset{\sim}{k}\end{aligned}$$

Let $A(1,\ -1,\ 3)$ be a point on line l_1 and $Q(4,\ 9,\ 1)$ be a point on l_3.

The distance between the lines is the scalar projection of $\overrightarrow{AQ}$ on $\underset{\sim}{n}$:

$$\overrightarrow{AQ}\cdot\hat{\underset{\sim}{n}}=\sqrt{\frac{8}{237}}$$



Mark allocation: 2 marks

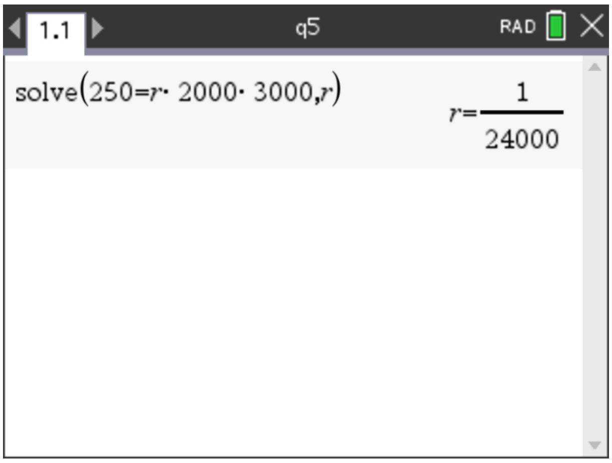
- 1 mark for using scalar projection with $\underset{\sim}{n}$ and $\overrightarrow{AQ}$
- 1 mark for the answer in correct form

Question 5a.

Worked solution

$250 = r \times 2000 \times 3000$

$\Rightarrow r = \dfrac{1}{24\,000}$



Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5b.

Worked solution

Rearrange the equation to obtain $\int \frac{dN}{N(5000-N)} = \int r dt$.

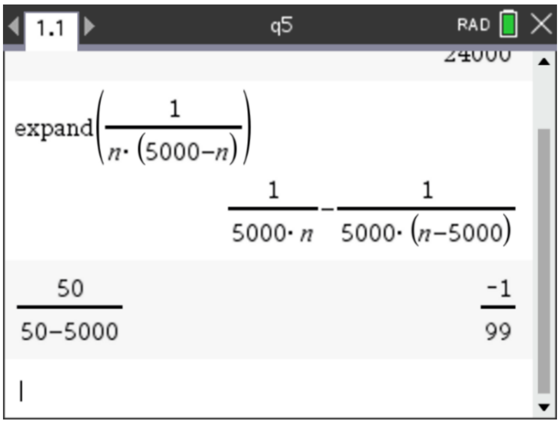
By partial fractions (use CAS to find this form), we get $\frac{1}{N(5000-N)} = \frac{1}{5000}\left(\frac{1}{N} - \frac{1}{N-5000}\right)$.

Therefore

$$\frac{1}{5000}\left(\log_e N - \log_e(N-5000)\right) = \frac{1}{24\ 000}t + c$$

$$\Rightarrow \frac{N}{N-5000} = Ae^{\frac{5}{24}t}$$

When $t = 0$, $N = 50$, so $A = -\frac{1}{99}$.



Therefore

$$N = -\frac{1}{99}e^{\frac{5}{24}t}(N-5000)$$

$$\Rightarrow N\left(1+\frac{1}{99}e^{\frac{5}{24}t}\right) = \frac{5000}{99}e^{\frac{5}{24}t}$$

$$\Rightarrow N = \frac{\frac{5000}{99}e^{\frac{5}{24}t}}{1+\frac{1}{99}e^{\frac{5}{24}t}}$$

$$= \frac{5000e^{\frac{5}{24}t}}{99+e^{\frac{5}{24}t}}$$

Mark allocation: 3 marks

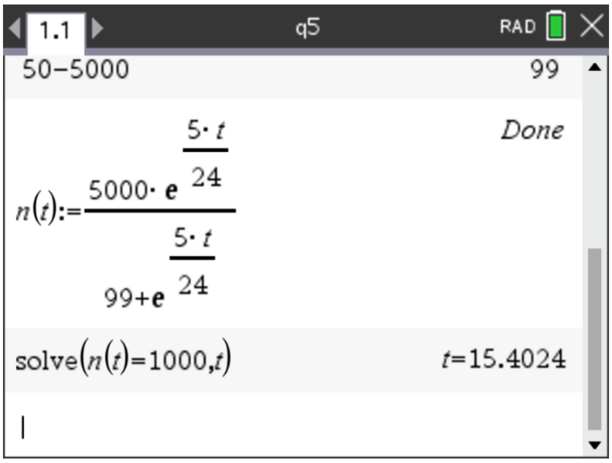
- 1 mark for rearranging the equation and using partial fractions
- 1 mark for applying the initial condition
- 1 mark for the correct answer for N, expressed in the required form

Question 5c.

Worked solution

Solve $N(t) = 1000$ for t.

15.40 days



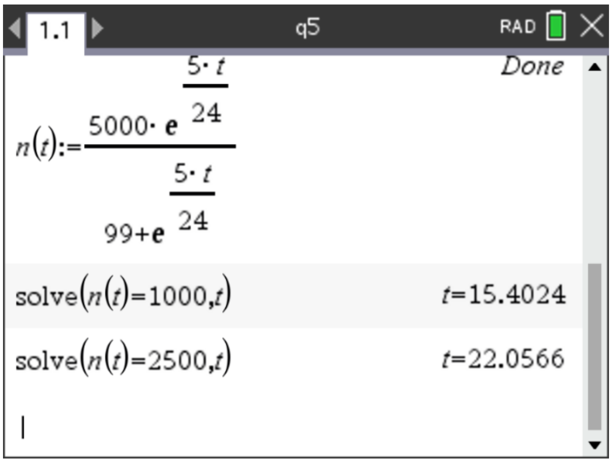
Mark allocation: 1 mark

- 1 mark for the correct answer, to two decimal places

Question 5d.i.

Worked solution

The rate of increase is greatest halfway between the equilibrium solutions, $N = 0$ and $N = 5000$; that is, when $N = 2500$ and $t = 22.06$ days.



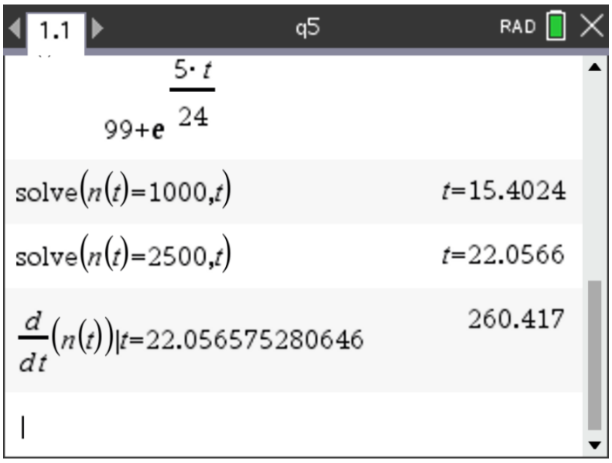
Mark allocation: 2 marks

- 1 mark for solving the equation $N(t) = 2500$ for t correctly
- 1 mark for the correct answer, to two decimal places

Question 5d.ii.

Worked solution

$\frac{dN}{dt} = 260$ (correct to the nearest person) when $t = 22.06$.

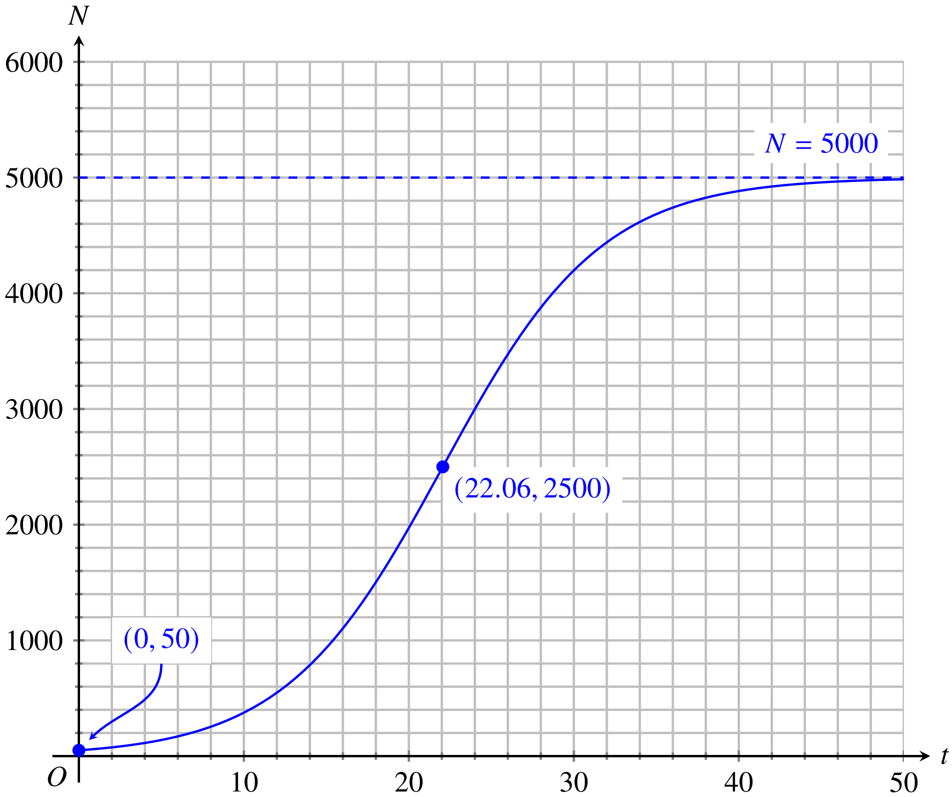


Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5e.

Worked solution



Mark allocation: 3 marks

- 1 mark for the correct shape of curve
- 1 mark for labelling the point of inflection (point of greatest rate of increase) correctly
- 1 mark for labelling the starting point and the asymptote correctly

Question 6a.

Worked solution

$H_0: \quad \mu = 400$

$H_1: \quad \mu < 400$

Mark allocation: 1 mark

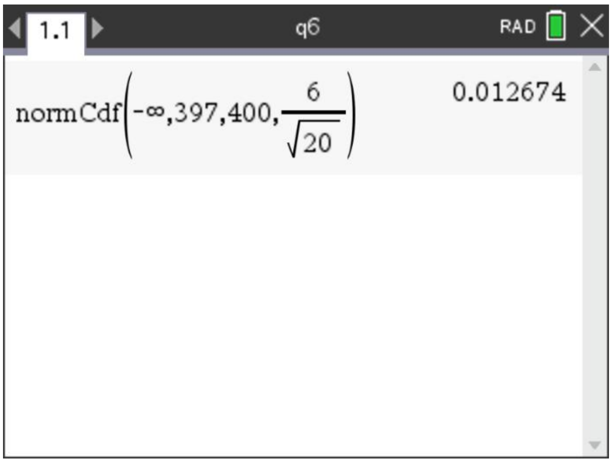
- 1 mark for correctly written hypotheses

Note: Hypotheses must be written exactly as presented in the solution to earn the mark.

Question 6b.

Worked solution

$\Pr\left(\bar{X} < 397 | \mu = 400\right) = 0.0127$



Mark allocation: 1 mark

- 1 mark for the correct answer, to four decimal places

Question 6c.

Worked solution

Yes, the null hypothesis should be rejected at the 5% level of significance because the p value is less than 0.05.

Mark allocation: 1 mark

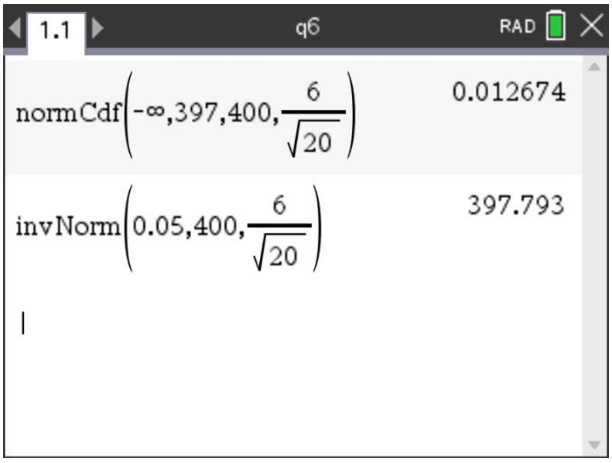
- 1 mark for the correct conclusion

Question 6d.

Worked solution

$$\Pr\left(\bar{X} < c \mid \mu = 400\right) = 0.05$$

$$\Rightarrow c = 397.79$$

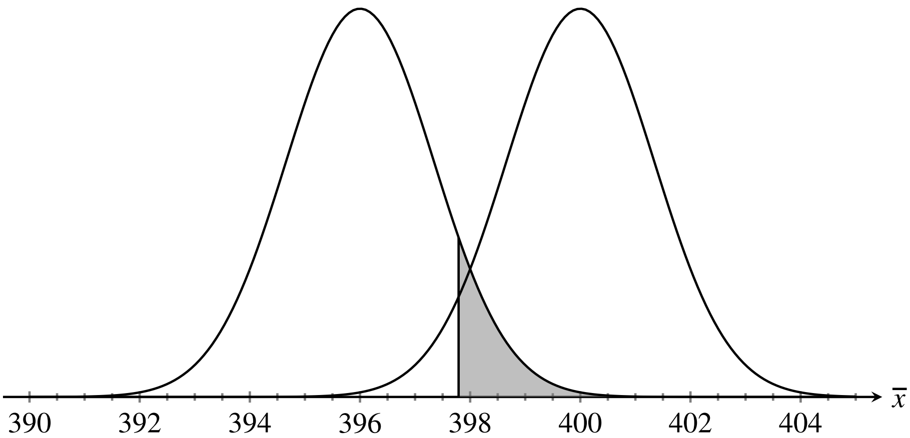


Mark allocation: 2 marks

- 1 mark for the correct probability equation
- 1 mark for the correct answer, to two decimal places

Question 6e.i. and ii.

Worked solution



Mark allocation: 1 mark (part i.)

- 1 mark for drawing a graph with the correct mean, height and shape

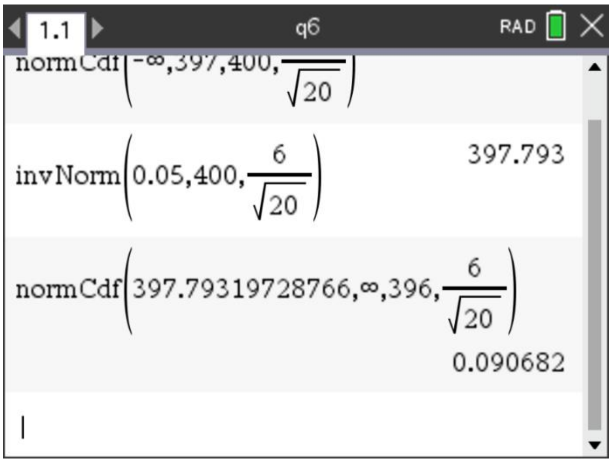
Mark allocation: 1 mark (part ii.)

- 1 mark for shading the region to the right of 397.79

Question 6e.iii

Worked solution

$\Pr\left(\bar{X} > 397.79 | \mu = 396\right) = 0.0907$



Mark allocation: 1 mark

- 1 mark for the correct answer, to four decimal places

END OF SOLUTIONS BOOKLET