2022 MAV Specialist Maths Trial Exam 2

VCE Further Maths (Monash University)

The Mathematical Association of Victoria

Trial Examination 2022

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 29 pages.
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A - Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

The curve given by $x = 2\operatorname{cosec}(t) + 1$ and $y = 3\cot(t)$ can be expressed in cartesian form as

A. $\dfrac{(x+1)^2}{4} - \dfrac{y^2}{9} = 1$

B. $\dfrac{(x-1)^2}{4} - \dfrac{y^2}{9} = 1$

C. $\dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1$

D. $\dfrac{(x+1)^2}{9} - \dfrac{y^2}{4} = 1$

E. $\dfrac{(x+1)^2}{9} + \dfrac{y^2}{4} = 1$

Question 2

The number of real solutions to $x^4 + \tan^2(x) - \sec^2(x) = 0$ is

A. 0

B. 1

C. 2

D. 3

E. 4

SECTION A – continued
TURN OVER

Question 3

Suppose $a, b > 0$. The domain and range of the function with rule $f(x) = \arcsin(|ax - b|) + \frac{\pi}{2}$ are respectively

A. $\left[\frac{-1-b}{a}, \frac{1+b}{a}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$

B. $\left[\frac{-1+b}{a}, \frac{1+b}{a}\right]$ and $[0, \pi]$

C. $\left[\frac{-1+b}{a}, \frac{1+b}{a}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$

D. $\left[\frac{-1-b}{a}, \frac{1-b}{a}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$

E. $\left[\frac{-1-b}{a}, \frac{1-b}{a}\right]$ and $[0, \pi]$

Question 4

The circle defined by $|z - a| = 2|z + 2i|$, where $a \in R$, has the cartesian equation

A. $\left(x + \frac{a}{3}\right)^2 + \left(y + \frac{8}{3}\right)^2 = \frac{16 + 4a^2}{9}$

B. $\left(x - \frac{a}{3}\right)^2 + \left(y - \frac{8}{3}\right)^2 = \frac{16 + 4a^2}{9}$

C. $\left(x - \frac{a}{3}\right)^2 + \left(y - \frac{8}{3}\right)^2 = \frac{9 + 2a^2}{16}$

D. $(x - a)^2 + (y + 2)^2 = 1$

E. $(x + a)^2 + (y - 2)^2 = 1$

Question 5

Let $z = \sqrt{3} + i$ and $w = 2 - 2i$.

The value of $\text{Arg}\left(\dfrac{z^3}{w^7}\right)$ is

A. $\dfrac{\pi}{2}$

B. $-\dfrac{3\pi}{4}$

C. $\dfrac{2\pi}{3}$

D. $-\dfrac{\pi}{3}$

E. $\dfrac{\pi}{4}$

Question 6

The coordinates of the point of intersection of the rays $\text{Arg}(z-1) = \dfrac{\pi}{3}$ and $\text{Arg}(z-i) = \dfrac{\pi}{4}$ are

A. $(1,1)$

B. $\left(2+\sqrt{3}, 3+\sqrt{3}\right)$

C. $\left(1+\sqrt{3}, 2+\sqrt{3}\right)$

D. $\left(-1-\sqrt{3}, -2-\sqrt{3}\right)$

E. $\left(-2-\sqrt{3}, -3-\sqrt{3}\right)$

Question 7

With a suitable substitution, $\int_{\sqrt{2}}^{\sqrt{5}} x^3 \sqrt{x^2+1}\,dx$ can be expressed as

A. $\frac{1}{2}\int_{\sqrt{2}}^{\sqrt{5}} (u+1)\sqrt{u}\,du$

B. $2\int_{\sqrt{2}}^{\sqrt{5}} (u-1)\sqrt{u}\,du$

C. $\int_{3}^{6} (u+1)\sqrt{u}\,du$

D. $\frac{1}{2}\int_{3}^{6} (u-1)\sqrt{u}\,du$

E. $2\int_{3}^{6} (u-1)\sqrt{u}\,du$

Question 8

If $y = e^{-2x}\sin(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + ky = 0$$

then k must equal

A. 4

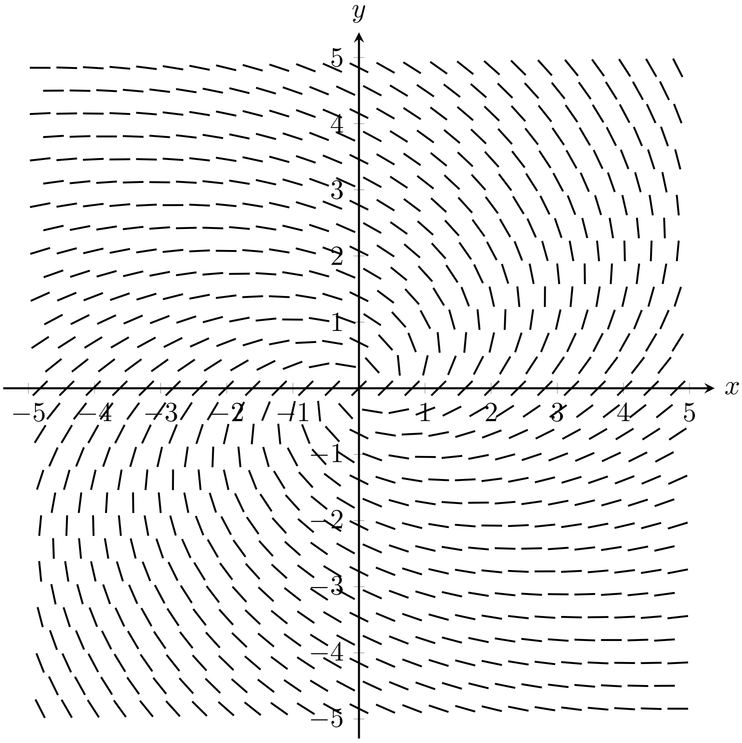
B. 5

C. 9

D. -13

E. 13

Question 9



The differential equation that has the diagram above as its direction field is

A. $\dfrac{dy}{dx} = \dfrac{x+y}{y-2x}$

B. $\dfrac{dy}{dx} = \dfrac{x+y}{2x-y}$

C. $\dfrac{dy}{dx} = \dfrac{x+y}{x-2y}$

D. $\dfrac{dy}{dx} = \dfrac{x-y}{x-2y}$

E. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

Question 10

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}(x+y)$, where $y(x_0) = y_0$.

Using Euler's method with a step-size of h, which one of the following is equal to y_2?

A. $y_0 + h(1 + x_0 + y_0) + \frac{1}{4}h^2(2 + x_0 + y_0)$

B. $y_0 + h(1 + x_0 + y_0) + \frac{1}{4}h^2(x_0 + y_0)$

C. $y_0 + h(x_0 + y_0) + \frac{1}{4}h^2(x_0 + y_0)$

D. $y_0 + h(x_0 + y_0) + \frac{1}{4}h^2(1 + x_0 + y_0)$

E. $y_0 + h(x_0 + y_0) + \frac{1}{4}h^2(2 + x_0 + y_0)$

Question 11

Let $\underset{\sim}{a} = 4\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$. If θ is the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ then $\tan(\theta)$ is equal to

A. $\frac{2\sqrt{6}}{5}$

B. $\frac{2\sqrt{6}}{7}$

C. $\frac{5\sqrt{6}}{12}$

D. $\frac{7}{5}$

E. $\frac{5}{7}$

Question 12

The vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 6\underset{\sim}{k}$ and $\underset{\sim}{c} = 5\underset{\sim}{i} + m\underset{\sim}{j} - 6\underset{\sim}{k}$ are linearly independent if

A. $m \in R$

B. $m \in R \setminus \{-5\}$

C. $m = -5$

D. $m = 5$

E. $m \in R \setminus \{5\}$

Question 13

The scalar resolute of $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ in the direction of the vector $\underset{\sim}{b}$ is 6.

The magnitude of $\underset{\sim}{d}$, where $\underset{\sim}{d}$ is the component of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is

A. $2\sqrt{3}$

B. $2\sqrt{11}$

C. $2\sqrt{14}$

D. $\sqrt{14}$

E. $5\sqrt{2}$

Question 14

The velocity of a particle of mass 2 kg is given by $\underset{\sim}{v}(t) = 4\cos(2t)\underset{\sim}{i} + 3\sin(2t)\underset{\sim}{j} + 4t\underset{\sim}{k}$.

The least magnitude of the force, in newtons, applied to the particle is

A. $2\sqrt{10}$

B. $2\sqrt{26}$

C. $4\sqrt{13}$

D. 40

E. 80

Question 15

A person is standing in an elevator that is accelerating upwards at 0.7ms^{-2}. The reaction force of the elevator floor on the person is 714 newtons.

The mass in kg of the person is

A. 78

B. 76

C. 72

D. 68

E. 64

SECTION A – continued
TURN OVER

Question 16

A particle travelling in a straight line has velocity $v\,\text{ms}^{-1}$ when its position relative to a fixed origin is x m. Its acceleration is given by $a=\sqrt{v^2+1}$.
The distance that the particle travels as its velocity increases from 1 ms^{-1} to 3 ms^{-1} is

A. $\int_1^3 \frac{1}{\sqrt{v^2+1}}dv$

B. $\int_1^3 \left(v^2+1\right)dv$

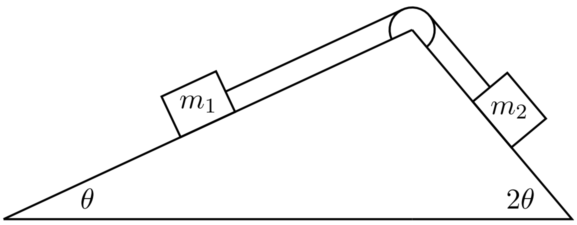
C. $\int_1^3 \frac{\sqrt{v^2+1}}{v}dv$

D. $\int_1^3 \frac{2v}{\sqrt{v^2+1}}dv$

E. $\int_1^3 \frac{v}{\sqrt{v^2+1}}dv$

Question 17

Two masses are placed on smooth inclined planes. The masses are connected by a light inextensible wire which passes over a smooth pulley and the planes and inclined at angles of θ and 2θ, $\theta>0$, as shown below.



Given that $m_2=2m_1$ and that the masses are in equilibrium then it must be the case that $\cos(\theta)$ is equal to

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{1}{\sqrt{2}}$

E. $\frac{1}{\sqrt{3}}$

Question 18

Newborn kittens have a mean mass of 120 g with a standard deviation of 15 g.
An $a\%$ confidence interval for the mean μ of a sample of 50 newborn kittens is $(114.90, 125.10)$.
The value of a correct to the nearest integer is

A. 99

B. 98

C. 97

D. 96

E. 95

Question 19

The scores of end of year physics and chemistry examinations at a particular school are normally distributed.
Physics scores have a mean of 65 marks with a standard deviation of 6 marks. Chemistry scores have a mean of 72 marks an a standard deviation of 5 marks.
Assume that physics and chemistry scores are independent of each other.
The probability that the scores of a randomly selected physics test and a randomly selected chemistry test differ by more than 10 marks is closest to

A. 0.851

B. 0.796

C. 0.365

D. 0.271

E. 0.165

Question 20

The mean mass of a packet of chips should be 100g with a standard deviation of 5g. It is believed that the packets are underweight and so a sample of 25 packets is taken and a one-sided statistical test is carried out at the 5% level of significance.
If in fact the mass of a packet of chips is 97.5g with a standard deviation of 5g, the probability that the null hypothesis would be incorrectly accepted is closest to

A. 0.8038

B. 0.8739

C. 0.3712

D. 0.1962

E. 0.1261

END OF SECTION A

TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (8 marks)

Let $f:[0,1] \to R$, $f(x) = \arcsin\left(\sqrt{x}\right)$.

a. **i.** Find an expression for $f''(x)$ in the form $\dfrac{ax+b}{2ax^{\frac{3}{2}}(1-x)^{\frac{3}{2}}}$ and state the values of a and b. 1 mark

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ii. Write down the coordinates of the point of inflection on the graph of $f(x) = \arcsin\left(\sqrt{x}\right)$. 1 mark

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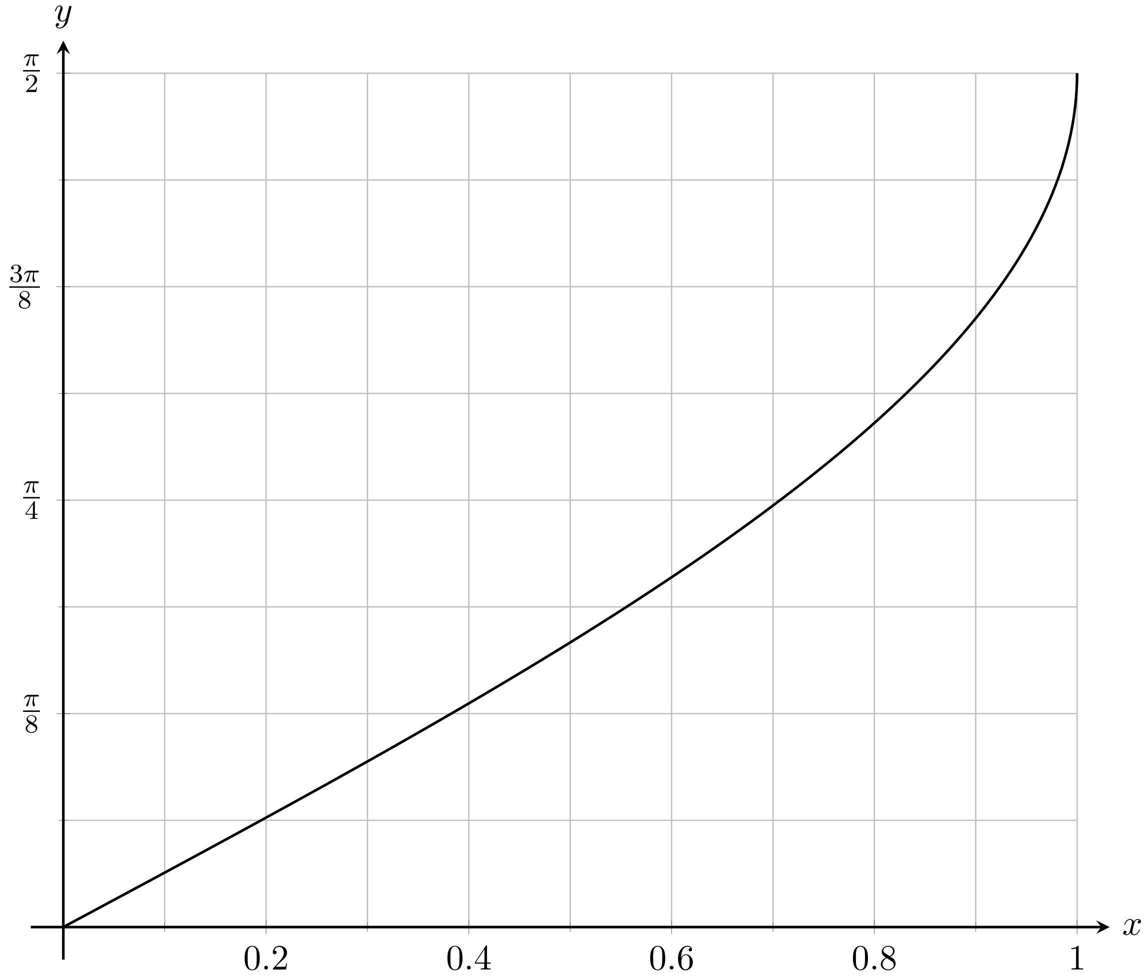
b. The graph of $y = \arcsin(x)$ is shown below.

On the same set of axes, sketch the graph of $f(x) = \arcsin\left(\sqrt{x}\right)$. Label the point of inflection with its coordinates.

2 marks



c. The region bounded by the graphs of $f(x) = \arcsin\left(\sqrt{x}\right)$ and $y = \arcsin(x)$ is rotated about the y-axis to form a solid of revolution. Write down an integral that gives the volume of the solid formed.

2 marks

d. Find the value of x for which the vertical distance between the graphs of $f(x) = \arcsin\left(\sqrt{x}\right)$ and $y = \arcsin(x)$ is maximised and find this vertical distance correct to three decimal places. 2 marks

Question 2 (9 marks)

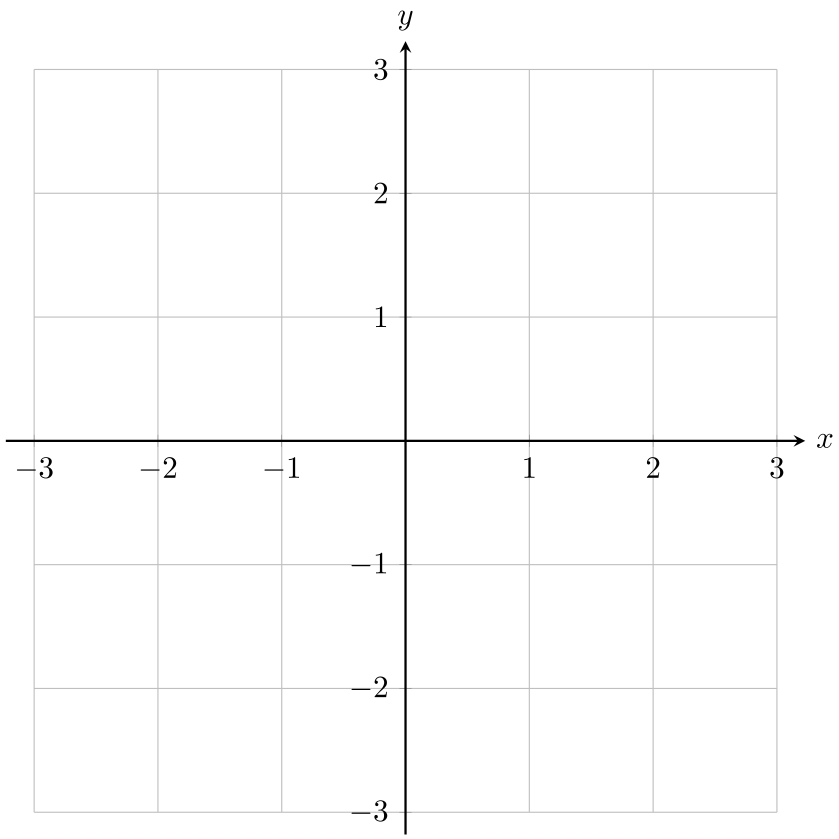
The paths of two particles A and B are described by the vector functions

$$\underset{\sim}{r}_A(t) = \left(1+\sin(t)\right)\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$$
$$\underset{\sim}{r}_B(t) = \frac{1}{2}t\underset{\sim}{i} + (t-1)\underset{\sim}{j}$$

respectively, where $t \geq 0$ is measured in seconds and distances are measured in metres.

a. Show that the cartesian equation of particle A is $(x-1)^2 + y^2 = 1$. 1 mark

b. Sketch the paths of the particles A and B on the axes below. Label the starting point of each particle with its coordinates and indicate the direction of motion with an arrow. Label the coordinates of the points of intersection of the paths with their coordinates. 3 marks



SECTION B – Question 2 – continued

TURN OVER

c. Point C is the point on the path of particle A at which the velocity vector of particle A is first parallel to the path of particle B.

i. Determine the coordinates of point C. 2 marks

ii. Determine the time in seconds, correct to three decimal places that particle A is first at point C. 1 mark

d. **i.** Find the value of t, correct to three decimal places, when the particles A and B are closest. 1 mark

ii. Find the minimum distance between particles A and B in metres, correct to two decimal places. 1 mark

SECTION B – continued

Question 3 (9 marks)

A 6 kg mass is attached to a 4 kg mass by a light inextensible wire. The 6 kg mass is placed at the end of a horizontal surface and the 4 kg mass hangs freely with the wire passing over a smooth pulley.

The system is initially at rest. When released, the 6 kg mass experiences resistance of $2v^2$ newtons, where v ms^{-1} is the speed of the mass.

a. On the diagram below, show all forces acting on the 4 kg mass. 1 mark

b. Show that the acceleration a ms^{-2} of the system is given by $a = \dfrac{2g - v^2}{5}$. 1 mark

SECTION B – Question 3 - continued
TURN OVER

c. **i.** Find an expression for the distance travelled by the 6 kg mass as a function of v. Give your answer in the form $a\log_e\left(\frac{b}{b-v^2}\right)$ where a and b are constants. 2 marks

ii. Determine the speed of the 6 kg mass when it has travelled 4m. Give your answer in ms^{-1}, correct to three decimal places. 1 mark

When the 6 kg mass has travelled 4 m, the wire breaks. The 6 kg mass begins to decelerate and reaches the end of the horizontal surface with a speed of 1 ms^{-1}. The mass continues to experience resistance of $2v^2$ newtons.

d. How long does it take for the 6 kg mass to reach the end of the horizontal surface? Give your answer in seconds, correct to two decimal places. 2 marks

e. How long is the horizontal platform? Give your answer in metres, correct to one decimal place. 2 marks

Question 4 (11 marks)

The mould *Vacraa Spuriosa* was observed to be spreading over the walls of a bathroom shower at a rate modelled by the differential equation

$$\frac{dP}{dt} = P(19-9P)-10\sqrt{P}\,,\quad t \geq 0 \text{ and } 0 \leq P \leq 1$$

where P is the **proportion** of the walls covered by mould after t weeks.

Let the function $P = g(t)$ be the solution to this differential equation when the initial proportion of shower walls covered in mould is 0.64.

a. **i.** Show that $P = g(t)$ is an increasing function. 1 mark

ii. Find $\lim\limits_{t\to\infty} g(t)$ and interpret this value as a feature on the graph of $P = g(t)$. 1 mark

SECTION B – Question 4 - continued

b. Show that $P = g(t)$ has a point of inflection and find, correct to four decimal places, the coordinates of this point. 4 marks

SECTION B – Question 4- continued

TURN OVER

c. Find, correct to four decimal places, the arc length of $P = g(t)$ over the interval $P \in [0.64,\ 0.9]$. 2 marks

SECTION B – Question 4 - continued

d. If the initial proportion of shower walls covered in mould is 0.64, use Euler's method with a step size of **1 day** to find an estimate of the proportion of shower walls covered in mould after:

i. **1 day**. Give your answer in exact form. 2 marks

ii. **5 days**. Give your answer correct to three decimal places. 1 mark

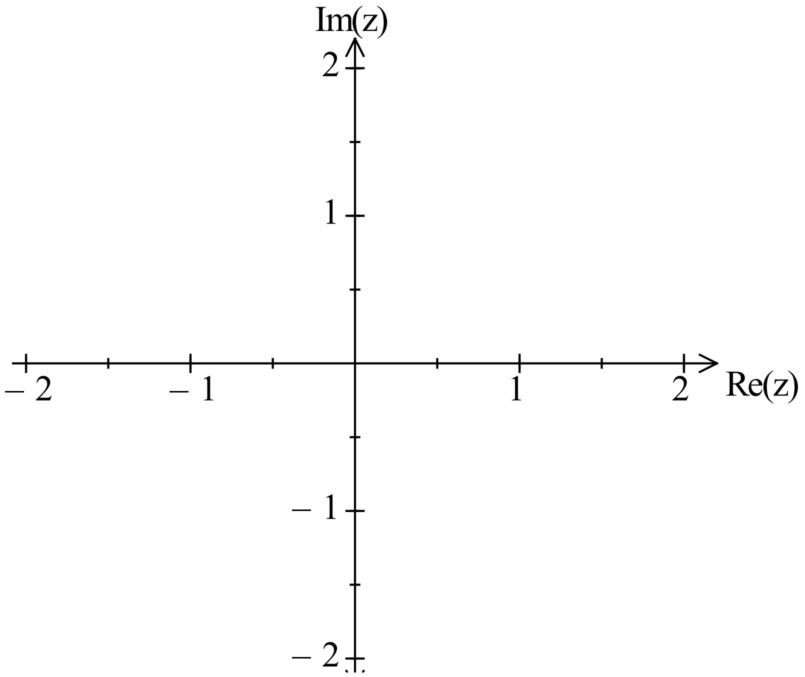
e. Find, correct to three decimal places, the proportion of shower walls covered in mould after **3 days** if the initial proportion of shower walls covered in mould is 0.64. 2 marks

Question 5 (12 marks)

a. **i.** Write $4z\bar{z}+(2-i)z+(2+i)\bar{z}=3$, $z \in C$, in the form $|z+z_1|=\rho$ where $z_1, \rho \in C$. 2 marks

ii. Let T be the circle defined by $4z\bar{z}+(2-i)z+(2+i)\bar{z}=3$.

Sketch a graph of T on the Argand diagram below, labelling the centre with its value of z. 1 mark



SECTION B – Question 5 - continued

iii. Let z_1 be the value of z that satisfies $4z\overline{z}+(2-i)z+(2+i)\overline{z}=3$ and has the minimum modulus. Find the principal argument of z_1. 1 mark

b. Sketch a graph of $\text{Arg}(2z+1-i)=-\frac{\pi}{4}$ on the Argand diagram above. 3 marks

c. Find in Cartesian form the value(s) of z that lie on T and satisfy $\text{Arg}(2z+1-i)=-\frac{\pi}{4}$. 2 marks

SECTION B – Question 5 - continued

TURN OVER

d. Find the values of β for which there is a value of z that satisfies both $\text{Arg}(2z+1-i)=\beta$ and $|z+1|=|z-3+2i|$. 2 marks

SECTION B – continued

Question 6 (11 marks)

The Gratuitous Citrus Farm grows oranges and lemons. The oranges have a mean mass of 200 grams with a standard deviation of 5 grams and the lemons have a mean mass of 70 grams with a standard deviation of 4 grams. The masses for each type of fruit are normally distributed.

a. What is the probability, correct to four decimal places, that a randomly selected orange will have at least three times the mass of a randomly selected lemon? 2 marks

b. What is the probability, correct to four decimal places, that two randomly selected lemons will have a combined mass of at least two-thirds the mass of a randomly selected orange? 2 marks

SECTION B – Question 6 - continued

TURN OVER

The Gratuitous Citrus Farm decides to test a new type of fertiliser that is claimed to increase the mass of oranges. The fertiliser was used for six months and then a random sample of 40 oranges was collected. A one-sided statistical test at the 2% level of significance was performed.

c. **i.** Write down suitable null and alternative hypotheses for this test. 1 mark

ii. Let $\bar{x}_c$ be the largest sample mean that can be observed and the null hypothesis **not** be rejected. Find, in grams and correct to two decimal places, the value of $\bar{x}_c$. 1 mark

iii. Giving a reason, state whether there is any evidence that the new type of fertiliser increases the mass of oranges if the total mass of oranges in the random sample is 8.095 kg. 1 mark

Another one-sided statistical test is performed using a new random sample of 40 oranges. A type 2 error is made when the null hypothesis is not true and is not rejected.

d. If the true mean mass of oranges grown on the Gratuitous Citrus Farm after using the new type of fertiliser is 202 grams and $\Pr(\text{type 2 error}) = 3\Pr(\text{type 1 error})$, find the level of significance of the test. Give your answer as a percentage, correct to two decimal places. 4 marks

END OF QUESTION AND ANSWER BOOK