Neap

Trial Examination 2021

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (4 marks)

a. $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$ A1

b.   **Method 1:**

$\left|\overrightarrow{OA}\right| = \sqrt{2},\ \left|\overrightarrow{OB}\right| = 5,\ \left|\overrightarrow{BC}\right| = \sqrt{2},\ \left|\overrightarrow{AC}\right| = 5$ A1

Take the scalar product between either $\overrightarrow{OA}$ and $\overrightarrow{OB}$, $\overrightarrow{OA}$ and $\overrightarrow{AC}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$
or $\overrightarrow{OB}$ and $\overrightarrow{BC}$.

For example:

$\overrightarrow{OA} \cdot \overrightarrow{OB} = \left(\underset{\sim}{i} + \underset{\sim}{j}\right) \cdot \left(-5\underset{\sim}{k}\right)$

$\qquad = 0$ M1

As $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0,$ OA is perpendicular to OB.   A1

Hence, $OACB$ is a rectangle.

Method 2:

Find $\left|\overrightarrow{OC}\right|$ and $\left|\overrightarrow{AB}\right|$.

$\left|\overrightarrow{OC}\right| = \sqrt{1^2 + 1^2 + (-5)^2}$ $\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-1)^2 + (-5)^2}$ M1

$\qquad = \sqrt{27}$ $\qquad = \sqrt{27}$ A1

As $\left|\overrightarrow{OC}\right| = \left|\overrightarrow{AB}\right| = \sqrt{27},$ then the magnitude of the diagonals are equal.   A1
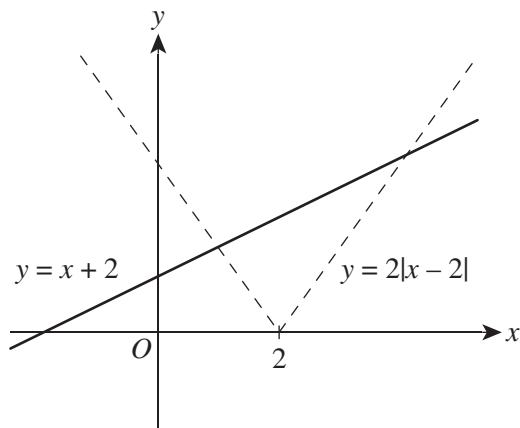
Hence, $OACB$ is a rectangle.

Question 2 (3 marks)

Method 1:

The values of x for which $|x + 2| > 2|x - 2|$ are the values of x for which the graph of $y = x + 2$
is above the graphs of $y = 2(x - 2)$ and $y = -2(x - 2)$.

Solve $x + 2 = 2(x - 2)$ and $x + 2 = -2(x - 2)$ for x.

$x + 2 = 2(x - 2)$ $x + 2 = -2(x - 2)$ M1

$\Rightarrow x = 6$ $\Rightarrow x = \dfrac{2}{3}$ A1

$\dfrac{2}{3} < x < 6$ A1

Method 2:

Square both sides of $|x+2| > 2|x-2|$ to obtain $(x+2)^2 > 4(x-2)^2$.   A1

Solve $(x+2)^2 > 4(x-2)^2$ for x.

$$(x+2)^2 > 4(x-2)^2$$
$$x^2 + 4x + 4 > 4\left(x^2 - 4x + 4\right)$$ M1

$$3x^2 - 20x + 12 < 0$$
$$(3x-2)(x-6) < 0$$
$$\frac{2}{3} < x < 6$$ A1

Question 3 (3 marks)

Find $\dot{\underset{\sim}{r}}(t)$.

$$\dot{\underset{\sim}{r}}(t) = -\pi \sin(\pi t)\underset{\sim}{i} + 2\pi \cos(2\pi t)\underset{\sim}{j}$$ M1

Find $\dot{\underset{\sim}{r}}(1)$ and $\dot{\underset{\sim}{r}}\left(\dfrac{1}{2}\right)$.

$$\dot{\underset{\sim}{r}}(1) = -\pi \sin(\pi)\underset{\sim}{i} + 2\pi \cos(2\pi)\underset{\sim}{j} \qquad \dot{\underset{\sim}{r}}\left(\frac{1}{2}\right) = -\pi \sin\left(\frac{\pi}{2}\right)\underset{\sim}{i} + 2\pi \cos(\pi)\underset{\sim}{j}$$

$$= 2\pi\underset{\sim}{j} \qquad\qquad\qquad\qquad\qquad = -\pi\underset{\sim}{i} - 2\pi\underset{\sim}{j}$$ A1

Let the change in momentum be $\Delta\underset{\sim}{p}$ and $\Delta\underset{\sim}{p} = m\Delta\underset{\sim}{v}$.

$$\Delta\underset{\sim}{p} = 3\left(\dot{\underset{\sim}{r}}(1) - \dot{\underset{\sim}{r}}\left(\frac{1}{2}\right)\right)$$

$$= 3\left(2\pi\underset{\sim}{j} - \left(-\pi\underset{\sim}{i} - 2\pi\underset{\sim}{j}\right)\right)$$

$$\Delta\underset{\sim}{p} = 3\pi\underset{\sim}{i} + 12\pi\underset{\sim}{j} \ (\text{kg ms}^{-1})$$ A1

Question 4 (4 marks)

Using $E(aX + bY) = aE(X) + bE(Y)$ gives $-2a - 2b = 2,$ and

using $\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y)$ gives $a^2 + 2b^2 = 9.$ **M1**

Method 1:

Substitute $a = -1 - b$ into $a^2 + 2b^2 = 9.$

$(-1 - b)^2 + 2b^2 = 9$

$\quad 3b^2 + 2b - 8 = 0$ **A1**

Solve for b.

$(b + 2)(3b - 4) = 0$

$$\Rightarrow b = -2, \frac{4}{3}$$ **M1**

$b \neq \dfrac{4}{3}$ as $b \in Z$ so $b = -2.$

$a = -1 - b$

$\;= -1 - (-2)$

$\;= 1$

$a = 1$ and $b = -2 \; (a, b \in Z)$ **A1**

Method 2:

Substitute $b = -1 - a$ into $a^2 + 2b^2 = 9.$

$a^2 + 2(-1 - a)^2 = 9$ **A1**

$\quad 3a^2 + 4a - 7 = 0$

Solve for a.

$(3a + 7)(a - 1) = 0$

$$\Rightarrow a = -\frac{7}{3}, 1$$ **M1**

$a \neq -\dfrac{7}{3}$ as $a \in Z$ so $a = 1.$

$b = -a - 1$

$\;= -(1) - 1$

$\;= -2$

$a = 1$ and $b = -2 \; (a, b \in Z)$ **A1**

Question 5 (4 marks)

a.
$$e^{-x}\frac{dy}{dx} - ye^{-x} = 1 + 2y\frac{dy}{dx}$$
M1

$$\frac{dy}{dx}\left(e^{-x} - 2y\right) = 1 + ye^{-x}$$

$$\frac{dy}{dx} = \frac{1 + ye^{-x}}{e^{-x} - 2y}\left(= \frac{e^x + y}{1 - 2ye^x}\right)$$
A1

b. At $P(0,1)$:

$$\frac{dy}{dx} = \frac{1 + e^0}{e^0 - 2}$$

$$= -2$$

Let the gradient of the normal be m_N and $m_N = \dfrac{-1}{\dfrac{dy}{dx}}$.

$$m_N = \frac{1}{2}$$
M1

$$y - 1 = \frac{1}{2}(x - 0)$$

$$y = \frac{x}{2} + 1$$
A1

Question 6 (4 marks)

a. Substitute $\operatorname{cosec}(x) = \dfrac{1}{\sin(x)}$ and $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$ into the equation.

$$\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} = 2\sin(x)$$

$$1 + \cos(x) = 2\sin^2(x)$$
M1

Use $\sin^2(x) = 1 - \cos^2(x)$ to obtain $1 + \cos(x) = 2\left(1 - \cos^2(x)\right)$.

$$1 + \cos(x) = 2\left(1 - \cos^2(x)\right)$$

$$\cos(x) + 1 = 2 - 2\cos^2(x)$$

$$2\cos^2(x) + \cos(x) - 1 = 0$$
A1

b. **Method 1:**
Factorise and solve $2\cos^2(x) + \cos(x) - 1 = 0$.
$$\left(2\cos(x) - 1\right)\left(\cos(x) + 1\right) = 0$$
M1

$$\cos(x) = \frac{1}{2}, -1$$

$$\Rightarrow x = \frac{\pi}{3} \text{ as } x \neq \pi$$
A1

Method 2:

Use the quadratic formula on $2\cos^2(x) + \cos(x) - 1 = 0$.

$$\cos(x) = \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{4}$$

M1

$$\cos(x) = \frac{1}{2}, -1$$

$$\Rightarrow x = \frac{\pi}{3} \text{ as } x \neq \pi$$

A1

Question 7 (4 marks)

a.

$$\frac{3}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$= \frac{A(x-2) + B(x+1)}{(x+1)(x-2)}$$

So, $3 = A(x-2) + B(x+1)$.

M1

Method 1:

When $x = 2$:   When $x = -1$:

$3 = A(2-2) + B(2+1)$ $3 = A(-1-2) + B(-1+1)$

$3 = B(3)$ $3 = A(-3)$

$B = 1$ $A = -1$

$$\frac{3}{(x+1)(x-2)} = \frac{1}{x-2} - \frac{1}{x+1}$$

A1

Method 2:

$3 = Ax - 2A + Bx + B$

Equate coefficients to obtain $A + B = 0$ and $-2A + B = 3$.

Solving simultaneously gives $A = -1$ and $B = 1$.

$$\frac{3}{(x+1)(x-2)} = \frac{1}{x-2} - \frac{1}{x+1}$$

A1

Method 3:

Use the cover-up method to obtain $\dfrac{3}{(x+1)(x-2)} = \dfrac{\frac{3}{-1-2}}{x+1} + \dfrac{\frac{3}{2+1}}{x-2}$.

$$\frac{3}{(x+1)(x-2)} = \frac{1}{x-2} - \frac{1}{x+1}$$

A1

b. Separate variables and antidifferentiate both sides.

$$\int \frac{3}{(x+1)(x-2)}dx = \int 3t^2 dt$$

$$\log_e(x-2) - \log_e(x+1) = t^3 + C \qquad \text{M1}$$

$$\log_e\left(\frac{x-2}{x+1}\right) = t^3 + C$$

$$\frac{x-2}{x+1} = e^{t^3+C}$$

$$= e^C e^{t^3}$$

Writing $k = e^C$ gives $\dfrac{x-2}{x+1} = ke^{t^3}$. $\qquad$ A1

Question 8 (3 marks)

Let the length of C be L, where $L = \displaystyle\int_0^{\log_e(k)} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{3x} - e^{-3x}\right)$$

Express L as a definite integral.

$$L = \int_0^{\log_e(k)} \sqrt{1+\frac{1}{4}\left(e^{3x}-e^{-3x}\right)^2}\,dx$$

$$= \int_0^{\log_e(k)} \sqrt{\frac{1}{4}\left(e^{3x}+e^{-3x}\right)^2}\,dx \qquad \text{M1}$$

So, $L = \displaystyle\int_0^{\log_e(k)} \frac{1}{2}\left(e^{3x}+e^{-3x}\right)dx$. $\qquad$ A1

$$L = \frac{1}{6}\left[e^{3x}-e^{-3x}\right]_0^{\log_e(k)}$$

$$= \frac{1}{6}\left(e^{3\log_e(k)} - e^{-3\log_e(k)}\right)$$

$$= \frac{1}{6}\left(e^{\log_e(k^3)} - e^{\log_e(k^{-3})}\right)$$

$$= \frac{1}{6}\left(k^3 - \frac{1}{k^3}\right) \qquad \text{A1}$$

$$L = \frac{1}{6}\left(k^3 - \frac{1}{k^3}\right)$$

Question 9 (5 marks)

a. From the formula sheet, $g'(x) = \dfrac{1}{1+x^2}$.

Use chain rule differentiation on $f(x) = \arctan\left(\dfrac{x+1}{1-x}\right)$.

Let $u = \dfrac{x+1}{1-x}$ so that $y = \arctan(u)$.

Find $\dfrac{du}{dx}\left(= \dfrac{d}{dx}\left(\dfrac{x+1}{1-x}\right)\right)$.

$$\frac{du}{dx} = \frac{(1-x)(1) - (x+1)(-1)}{(1-x)^2}$$ M1

$$= \frac{2}{(1-x)^2}$$ A1

$\dfrac{du}{dx} = \dfrac{2}{(1-x)^2}$ and $\dfrac{dy}{du} = \dfrac{1}{1+u^2}$.

$$f'(x) = \frac{1}{\left(1 + \left(\dfrac{x+1}{1-x}\right)^2\right)}\left(\frac{2}{(1-x)^2}\right)$$

$$= \left(\frac{(1-x)^2}{(1-x)^2 + (x+1)^2}\right)\left(\frac{2}{(1-x)^2}\right)$$

$$= \frac{2}{2 + 2x^2}$$

$$= \frac{1}{1+x^2}$$

$f'(x) = \dfrac{1}{1+x^2}$ and $f'(x) = g'(x)$. A1

b. Use $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A+B)$ with $A = \arctan(x)$ and $B = \arctan(1)$.

$$\text{LHS} = \frac{\tan(\arctan(x)) + \tan(\arctan(1))}{1 - \tan(\arctan(x))\tan(\arctan(1))}$$ M1

$$= \frac{x+1}{1-x}$$

$$\frac{x+1}{1-x} = \tan(\arctan(x) + \arctan(1))$$

$$\Rightarrow \arctan\left(\frac{x+1}{1-x}\right) = \arctan(x) + \arctan(1)$$
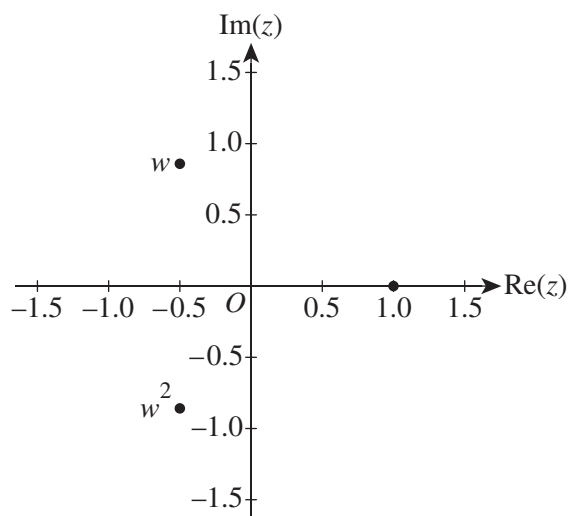
So, $f(x) = \arctan(x) + \arctan(1)$ (for $x < 1$) and

$$\frac{d}{dx}(\arctan(x) + \arctan(1)) = \frac{1}{1+x^2} = g'(x).$$ A1

Question 10 (6 marks)

a.

A1

$$w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and } w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

b.   **Method 1:**

$$w^3 = 1 \Rightarrow (w - 1)(w^2 + w + 1) = 0$$ A1

$$\Rightarrow 1 + w + w^2 = 0 \text{ for } w \neq 1$$ A1

Method 2:

$$1 + w + w^2 = 1 + \text{cis}\left(\frac{2\pi}{3}\right) + \text{cis}\left(\frac{4\pi}{3}\right)$$ A1

$$1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0 \text{ and so } 1 + w + w^2 = 0.$$ A1

Method 3:
Form the three vectors $\underset{\sim}{i}, -\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}$ and $-\frac{1}{2}\underset{\sim}{i} - \frac{\sqrt{3}}{2}\underset{\sim}{j}$.   A1

$$\underset{\sim}{i} + \left(-\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}\right) + \left(-\frac{1}{2}\underset{\sim}{i} - \frac{\sqrt{3}}{2}\underset{\sim}{j}\right) = \underset{\sim}{0} \text{ (can be shown geometrically)}$$

and so $1 + w + w^2 = 0.$ A1

Method 4:

$$1 + w + w^2 = \frac{1 - w^3}{1 - w}$$ A1

$$= \frac{0}{1 - w} = 0 \Rightarrow 1 + w + w^2 = 0$$ A1

Method 5:

Consider that, for $z^3 - 1 = 0$, the sum of the roots is zero.   A1
The sum of the roots (1, w and w^2) and so $1 + w + w^2 = 0.$ A1

c. **Method 1:**

$1+w = -w^2$ and $1+w^2 = -w$ (both obtained from $1+w+w^2 = 0$)

Expand $(z-1)(z+w^2)(z+w)$.

$(z-1)(z+w^2)(z+w) = z^3 - (1-w-w^2)z^2 + (-w-w^2+w^3)z - w^3$ M1, A1

$$= z^3 - (1-(w+w^2))z^2 + (-(w+w^2)+w^3)z - w^3$$

$$= z^3 - 2z^2 + 2z - 1$$

$z^3 - 2z^2 + 2z - 1 = 0$ A1

Note: Attempts to expand $(z-1)(z-(1+w))(z-(1+w)^2)$
should be awarded marks as above.

Method 2:

The required cubic equation is of the form $z^3 + bz^2 + cz + d = 0$, where $b, c, d \in Z$.

Find the values of b, c and d.

Sum of roots:

$1+(1+w)+(1+w^2) = 3+w+w^2$

$$= 3+(-1)$$

$$= 2$$

Product of roots:

$(1)(1+w)(1+w^2) = 1+w+w^2+w^3$

$$= 0+1$$

$$= 1$$

Sum of the product of the roots taken two at a time:

$(1)(1+w)+(1)(1+w^2)+(1+w)(1+w^2) = 1+(1+w+w^2)+(1+w+w^2)+w^3$

$$= 1+0+0+1$$

$$= 2$$

attempts to find sum of roots,
product of roots and sum of product of roots taken two at a time M1
correct sum of roots,
product of roots and sum of product of roots taken two at a time A1

$b = -2$, $c = 2$ and $d = -1$

Hence, $z^3 - 2z^2 + 2z - 1 = 0$.   A1