Neap

Trial Examination 2021

VCE Specialist Mathematics Units 1&2

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	**B**	C	D	E
2	A	B	**C**	D	E
3	A	B	C	**D**	E
4	A	B	C	**D**	E
5	A	B	C	**D**	E
6	**A**	B	C	D	E
7	**A**	B	C	D	E
8	A	B	**C**	D	E
9	A	B	C	**D**	E
10	A	**B**	C	D	E
11	A	B	C	D	**E**
12	A	**B**	C	D	E
13	A	**B**	C	D	E
14	A	B	**C**	D	E
15	A	B	C	**D**	E
16	A	B	C	**D**	E
17	A	B	**C**	D	E
18	A	B	C	**D**	E
19	A	B	C	**D**	E
20	A	B	**C**	D	E

Question 1 B

$\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 3\underset{\sim}{j}$.

$$\begin{aligned} 2\underset{\sim}{b} - \underset{\sim}{a} &= 2(\underset{\sim}{i} - 3\underset{\sim}{j}) - (2\underset{\sim}{i} - \underset{\sim}{j}) \\ &= 2\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{i} + \underset{\sim}{j} \\ &= -5\underset{\sim}{j} \end{aligned}$$

Question 2 C

t_4 is the geometric mean of t_2 and t_6.

$$\begin{aligned} t_4 &= \sqrt{6 \times 24} \\ &= 12 \end{aligned}$$

Question 3 D

Area of ΔABD:

$$\frac{1}{2} \times b \times h = 24$$

$$\frac{1}{2} \times 4 \times h = 24$$

$$h = 12$$

Area of trapezium $ABCD$:

$$\begin{aligned} \frac{1}{2}(a+b)h &= \frac{1}{2}(4+6) \times 12 \\ &= 60 \end{aligned}$$

Question 4 D

As the largest angle is opposite the longest side, use the cosine rule.

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{6^2 + 12^2 - 13^2}{2 \times 6 \times 12}\right) \\ &= 85.61897126 \end{aligned}$$

Sum of the two smallest angles:

$$\begin{aligned} 180 - 85.61897126 &= 94.38102874 \\ &\approx 94° \end{aligned}$$

Question 5 D

$$\begin{aligned} y &= \frac{x}{x^2 + 3x + 2} \\ &= \frac{x}{(x+2)(x+1)} \end{aligned}$$

$x \in R \setminus \{-2, -1\}$

Question 6 A

$$\underset{\sim}{a}+\underset{\sim}{b}=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\\m\end{bmatrix}$$

$$=\begin{bmatrix}3\\2+m\end{bmatrix}$$

If $\underset{\sim}{a}+\underset{\sim}{b}$ is parallel to $6\underset{\sim}{i}-\underset{\sim}{j}$, then $\begin{bmatrix}3\\2+m\end{bmatrix}=k\begin{bmatrix}6\\-1\end{bmatrix}$.

$$\Rightarrow 3=6k$$

$$k=\frac{1}{2}$$

$$2+m=\frac{1}{2}\times -1$$

$$m=-\frac{5}{2}$$

Question 7 A

Solving on a CAS calculator gives:

$(5+12\cdot i)^2$	$-119+120\cdot i$
$\text{real}\left((5+12\cdot i)^2\right)$	-119

Question 8 C

$$\left|2(m\underset{\sim}{i}+m)\underset{\sim}{j}\right|=\sqrt{(2m)^2+(2m)^2}$$

$$=\sqrt{8m^2}$$

Let $\left|2\left(m\underset{\sim}{i}+m\underset{\sim}{j}\right)\right|=1$.

$$\sqrt{8m^2}=1$$

$$8m^2=1$$

$$m=\frac{\sqrt{2}}{4}\quad(\text{as } m>0)$$

Question 9 D

$$\overrightarrow{CA}+\overrightarrow{AB}=\underset{\sim}{q}$$

$$2\underset{\sim}{r}+\overrightarrow{AB}=\underset{\sim}{q}$$

$$\overrightarrow{AB}=\underset{\sim}{q}-2\underset{\sim}{r}$$

ΔABC is right angled (inscribed in semi-circle).

$\Rightarrow \overrightarrow{AB}.\underset{\sim}{q}=0$ (as $\overrightarrow{AB}$ is perpendicular to $\underset{\sim}{q}$)

$$(\underset{\sim}{q}-2\underset{\sim}{r}).\underset{\sim}{q}=0$$

$$\underset{\sim}{q}.\underset{\sim}{q}-2\underset{\sim}{r}.\underset{\sim}{q}=0$$

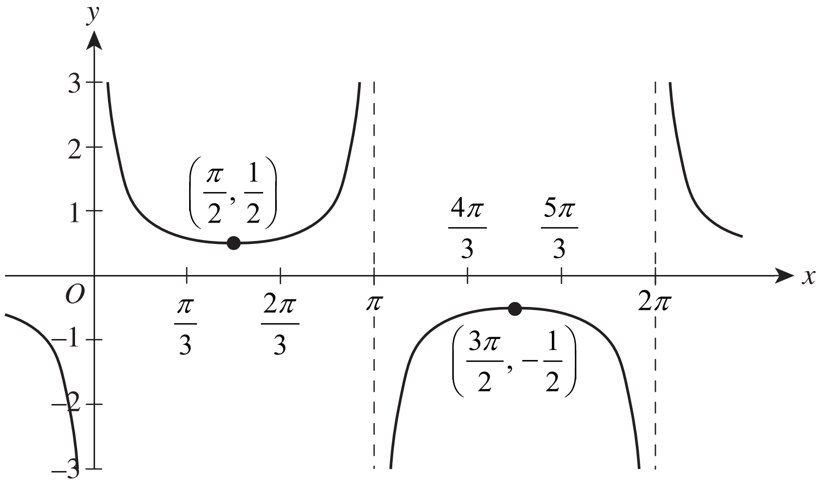
$$\underset{\sim}{q}.\underset{\sim}{q}=2\underset{\sim}{r}.\underset{\sim}{q}$$

Question 10 B

$$\frac{x+1}{x}=\frac{5}{4}$$
$$4x+4=5x$$
$$x=4\text{ cm}$$

Question 11 E

E is correct. The maximum and minimum of $y=\frac{1}{2\sin(x)}$ occur at $\pm\frac{1}{2}$ and therefore a vertical translation of $k\in\left(-\frac{1}{2},\frac{1}{2}\right)$ will take the graph to the *x*-axis without creating an *x*-intercept.



A, **B**, **C** and **D** are incorrect. These graphs will have *x*-intercepts.

Question 12 B

B is correct. Sketching each option on a CAS calculator gives **B** as the only match. **A**, **C**, **D** and **E** are incorrect. These equations will not result in the graph shown.

Question 13 B

$$S_n=a\left(\frac{1-r^n}{1-r}\right)$$
$$S_8=2\left(\frac{1-1.1^8}{1-1.1}\right)$$
$$=20\left(1.1^8-1\right)$$

Question 14 C

$$z=3+2i$$
$$\text{Im}(z)=2$$

Question 15 D

$$b^2 - 4ac = 0$$
$$(-2ki)^2 - 4(-3k) = 0$$
$$-4k^2 + 12k = 0$$
$$-4k(k-3) = 0$$
$$k = 0 \text{ or } 3$$

However, $k = 0$ gives a real solution. Therefore, $k = 3$ is the only solution.

$\text{cSolve}(z^2 - 2 \cdot k \cdot z \cdot i - 3 \cdot k = 0, z) \mid k=0$	$z=0$
$\text{cSolve}(z^2 - 2 \cdot k \cdot z \cdot i - 3 \cdot k = 0, z) \mid k=3$	$z=3 \cdot i$

Question 16 D

Triangles ΔABC and ΔDEC are similar.

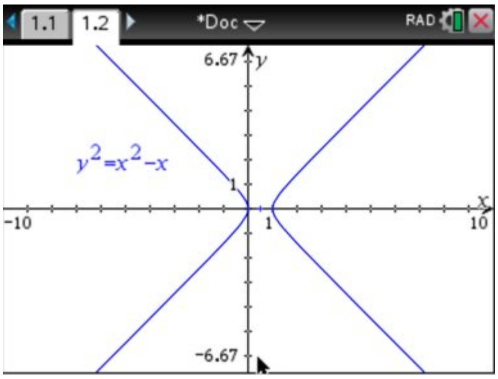
$$\text{area ratio} = \left(\frac{BC}{EC}\right)^2$$
$$= \left(\frac{5}{3}\right)^2$$
$$= \frac{25}{9}$$

Question 17 C

$$\underset{\sim}{a} . \underset{\sim}{b} = 0$$
$$m^2 + n - n^2 = 0$$
$$m^2 = n^2 - n$$

As $m^2 \geq 0$, $0 < n < 1$ is not possible.

Graphing the relation $y^2 = x^2 - x$ may be useful to visualise solutions to $m^2 = n^2 - n$.



Question 18 D

$$\sqrt{(x-1)^2+(y-2)^2}=\sqrt{(x-0)^2+(y-(-3))^2}$$
$$x^2-2x+1+y^2-4y+4=x^2+y^2+6y+9$$
$$-2x-4=10y$$
$$y=-\frac{1}{5}x-\frac{2}{5}$$

Question 19 D

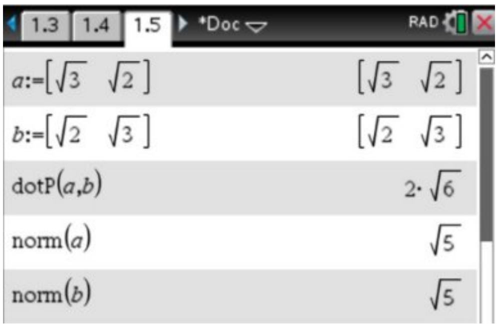
$$2\left(x^2+2ax\right)+y^2-2by+c=0$$
$$2\left((x+a)^2-a^2\right)+(y-b)^2-b^2+c=0$$
$$2(x+a)^2+(y-b)^2=2a^2+b^2-c$$

As the centre of the ellipse is (2, 1), $a=-2,\ b=-1.$

$$2a^2+b^2-c=1$$
$$2(-2)^2+(-1)^2-c=1$$
$$c=8$$

Question 20 C

$$\cos(\theta)=\frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$



$$\cos(\theta)=\frac{2\sqrt{6}}{\sqrt{5}\times\sqrt{5}}$$
$$=\frac{2\sqrt{6}}{5}$$

$$\sin^2(\theta)+\cos^2(\theta)=1$$
$$\Rightarrow \tan^2(\theta)=\sec^2(\theta)-1$$
$$\tan^2(\theta)=\left(\frac{5}{2\sqrt{6}}\right)^2-1$$
$$=\frac{1}{24}$$
$$\tan(\theta)=\frac{1}{\sqrt{24}}$$
$$=\frac{\sqrt{6}}{12}$$

SECTION B

Question 1 (5 marks)

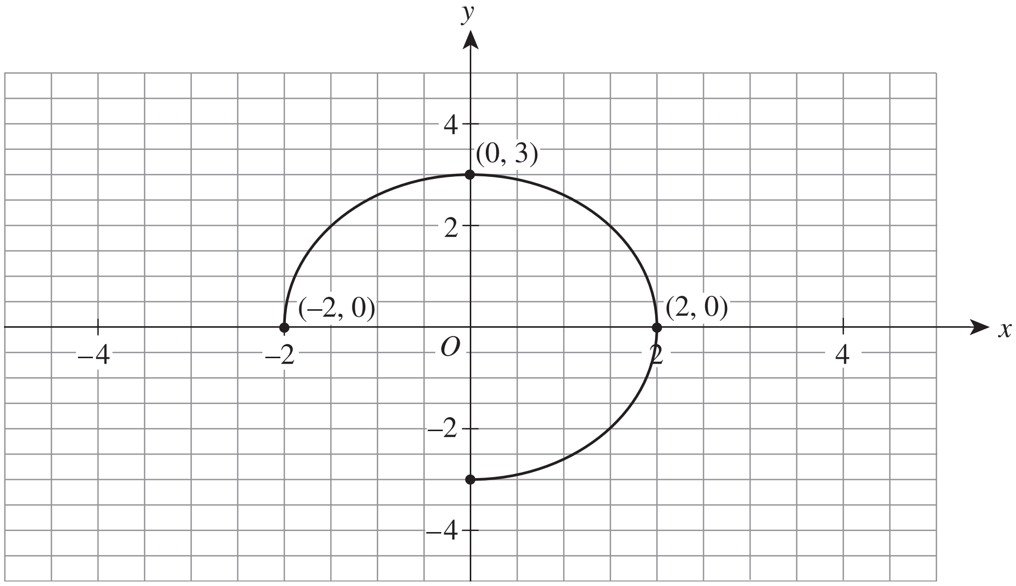
a. $-\frac{x}{2} = \cos(t) \quad \frac{y}{3} = \sin(t)$ M1

$\sin^2(t) + \cos^2(t) = 1$

$\left(-\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ A1

b.



correct intercepts A1
correct endpoints A1
correct shape A1
*Note: Consequential on answer to **Question 1a.***

Question 2 (13 marks)

a. i. $\overrightarrow{AB} = -30\underset{\sim}{i} + 50\underset{\sim}{j}$ A1

ii. $\left|\overrightarrow{AB}\right| = \sqrt{(-30)^2 + (50)^2}$

$= 10\sqrt{34}$ m A1

b. $\overrightarrow{AB}.\overrightarrow{OA} = 80 \times -30 + 40 \times 50$

$= -400$ A1

$\cos(\theta) = \dfrac{\overrightarrow{AB}.\overrightarrow{OA}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{OA}\right|}$

$\left|\overrightarrow{AB}\right| = 10\sqrt{34}$ $\quad \left|\overrightarrow{OA}\right| = 40\sqrt{5}$

$\cos(\theta) = \dfrac{-400}{10\sqrt{34} \times 40\sqrt{5}}$

$= -\dfrac{1}{\sqrt{170}}$ M1

$\theta = \cos^{-1}\left(-\dfrac{1}{\sqrt{170}}\right)$

$\approx 94°$

$\angle OAB = 180° - 94°$

$= 86°$ A1

c. $\overrightarrow{OC} = 60\underset{\sim}{i} + 30\underset{\sim}{j}$ $\quad \overrightarrow{CA} = 20\underset{\sim}{i} + 10\underset{\sim}{j}$ M1

As $\overrightarrow{OC} = 3\overrightarrow{CA}$, O, C and A are all collinear. Hence, the student is standing on the footpath. A1

d. $\overrightarrow{BP} = (2p - 50)\underset{\sim}{i} + (p - 90)\underset{\sim}{j}$ A1

e. **i.** Minimum distance occurs when $\overrightarrow{BP}$ is perpendicular to $\overrightarrow{OA}$.

Let $\overrightarrow{BP}.\overrightarrow{OA} = 0$. M1

$(2p - 50) \times 80 + (p - 90) \times 40 = 0$

$p = 38$ A1

The coordinates are $P(76, 38)$. A1

ii. $\overrightarrow{BP} = (2 \times 38 - 50)\underset{\sim}{i} + (38 - 90)\underset{\sim}{j}$ M1

$= 26\underset{\sim}{i} - 52\underset{\sim}{j}$

$\left|\overrightarrow{BP}\right| + \left|\overrightarrow{OP}\right| = \sqrt{(26)^2 + (-52)^2} + \sqrt{(76)^2 + (38)^2}$

$= 64\sqrt{5}$ m A1

Question 3 (9 marks)

a. $L_{2021} = 1200$

$L_{2022} = 0.85 \times L_{2021} + 400$

$= 0.85 \times 1200 + 400$

$= 1420$

area = 1420 km^2 A1

b. $K_{2022} = 1.5 \times K_{2021}$

$= 1.5 \times 4000$

$= 6000\ \text{km}^2$ A1

Overpopulation ratio:

$\dfrac{K_{2022}}{L_{2022}} = \dfrac{6000}{1420}$

$= 4.23\ \text{kangaroos km}^{-2}$

$4.23 > 4$ A1

Thus, the kangaroo habitat will be overpopulated in 2022.

c. The maximum land available occurs when $L_{n+1} \to L_n$.

$L = 0.85L + 400$

$L_{\max} = 2666\frac{2}{3}$ M1

$\text{maximum number of kangaroos} = 2666\frac{2}{3} \times 4$

$= 10\,666$ A1

d. $4000 = 1.5 \times 4000 - k$

$= 2000$ A1

e. $L_{2023} = 0.85 \times L_{2022} + 400$

$= 0.85 \times 1420 + 400$

$= 1607\ \text{km}^2$

$\text{maximum kangaroos} = 4 \times 1607$

$= 6428\ \text{kangaroos}$ A1

$P_{2021} = 4000$

$P_{2022} = 1.5 \times P_{2021} - k$

$P_{2023} = 1.5 \times P_{2022} - k$

$6428 = 1.5(1.5 \times 4000 - k) - k$ M1

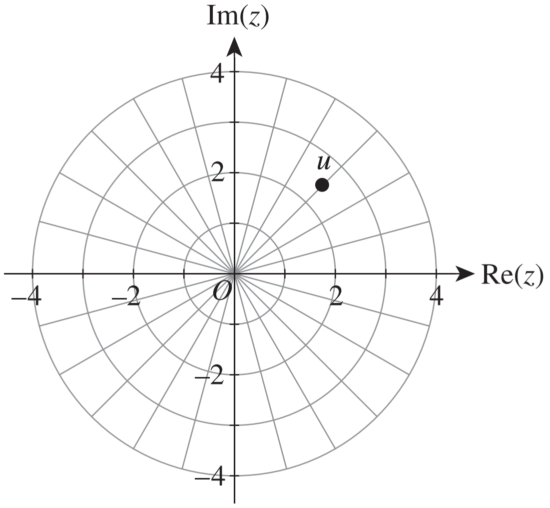
$k = 1028.8$

≈ 1029

Therefore, the minimum number of kangaroos that must be moved to other wildlife sanctuaries at the end of each year is 1029. A1

Question 4 (18 marks)

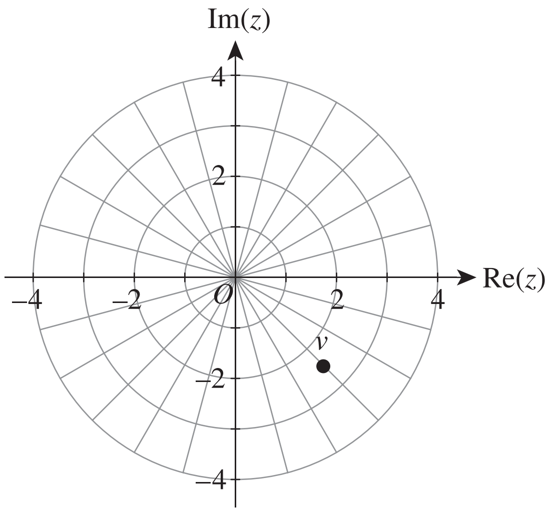
a.



correctly plotted point A1

b. **i.** $v = 2 - 2i$ A1

ii.



correctly plotted point A1

iii. $(z - (2 + 2i))(z - (2 - 2i)) = 0$ M1

$$z^2 - 4z + 8 = 0$$

expand((z−(2+2·i))·(z−(2−2·i)))

$z^2-4\cdot z+8$

$b = -4$ and $c = 8$ A1

c. **i.**

$$\begin{aligned} \text{RHS} &= iv \\ &= i(2-2i) \\ &= 2i - 2i^2 \\ &= 2 + 2i \\ &= u \end{aligned}$$

M1

ii. $n = 1 + 4k,\ k \in Z,$ as $i^1 = i^5 = i^n.$ A1

d. $u = 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ A1

$v = 2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$ A1

$u:=2+2\cdot i$	$2+2\cdot i$
$v:=2-2\cdot i$	$2-2\cdot i$
u▸Polar	$e^{\frac{i\cdot\pi}{4}}\cdot 2\cdot\sqrt{2}$
v▸Polar	$e^{\frac{-i\cdot\pi}{4}}\cdot 2\cdot\sqrt{2}$

e. **i.** $u - v = 4i$

$|u - v| = 4$ A1

ii. length of the diagonal of the square A1

iii. $w = 4 + 0i$ A1

$w = 4\text{cis}(0)$ A1

f. $z(z-4)\left(z^2 - 4z + 8\right) = 0$ M1

$z^4 - 8z^3 + 24z^2 - 32z = 0$

$q = -8,\ r = 24,\ s = -32$ A1

$\text{expand}\left(z\cdot(z-4)\cdot\left(z^2-4\cdot z+8\right)\right)$

$z^4-8\cdot z^3+24\cdot z^2-32\cdot z$

g. The solutions are $z = 0$, $z = k$, $z = \frac{k}{2} + \frac{k}{2}i$ and $z = \frac{k}{2} - \frac{k}{2}i$. M1

$z(z-k)\left(z - \left(\frac{k}{2} + \frac{k}{2}i\right)\right)\left(z - \left(\frac{k}{2} - \frac{k}{2}i\right)\right) = 0$ M1

$\left(z^2 - kz\right)\left(z^2 - kz + \frac{k^2}{2}\right) = 0$ A1

$z^4 - 2kz^3 + \frac{3k^2}{2}z^2 - \frac{k^3}{2}z = 0$

Question 5 (7 marks)

a. $\angle SPQ = \angle SQP$ (isosceles triangle) M1

$\angle OPQ = \angle OQP$ (isosceles triangle)

$\angle SQO = \angle ORS$ (isosceles triangle)

$\angle OPS = \angle SPQ - \angle OPQ$ M1

$\angle ORS = \angle SQP - \angle OQP$ M1

As $\angle OPS = \angle ORS$, $PRSO$ is a cyclic quadrilateral with $\angle OPS$ and $\angle ORS$ and equal angles subtended on the same arc of the circle with the points $PRSO$ on the circumference. A1

b. Area of ΔOSP:

$$A = \frac{1}{2}ab\sin(\theta)$$
$$= \frac{1}{2}OS \times OP \times \sin(\angle OSP)$$

$OP = r$

$$OS = \frac{1}{2}OP$$
$$= \frac{r}{2}$$

$\angle OSP = 180° - \angle SRP$ M1

$= 180° - 60°$

$= 120°$

$$A = \frac{1}{2} \times r \times \frac{r}{2} \times \sin(120°)$$
$$= \frac{\sqrt{3}r^2}{8}$$ A1

area of triangle : area of circle

$$\frac{\sqrt{3}r^2}{8} : \pi r^2$$

$$1 : \frac{8\sqrt{3}}{3}\pi$$ A1

Question 6 (8 marks)

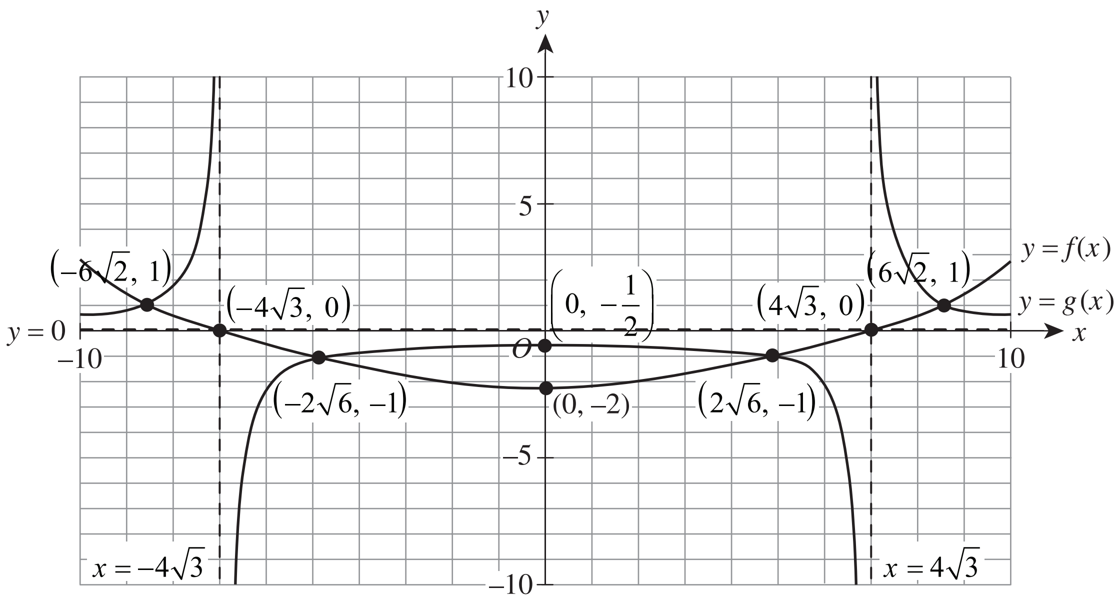
a. $\sqrt{x^2 + (y-4)^2} = \sqrt{0^2 + (y-(-8))^2}$ M1

$x^2 + y^2 - 8y + 16 = y^2 + 16y + 64$

$24y = x^2 - 48$ M1

$$f(x) = \frac{x^2}{24} - 2$$

b.



graph of $y = f(x)$ *with correct shape and intercepts* A1
graph of $y = g(x)$ *with correct shape and intercept* A1
correct asymptotes with equations A1
four correct points of intersection A1

c.

$$\sqrt{x^2 + (y-k)^2} = \sqrt{0^2 + (y-(-2k))^2}$$

$$x^2 + y^2 - 2ky + k^2 = y^2 + 4ky + 4k^2$$

$$6ky = x^2 - 3k^2$$

$$h(x) = \frac{x^2}{6k} - \frac{k}{2}$$ M1

Let $h(x) = \dfrac{1}{h(x)}$.

As $x^2 \geq 0$ for four solutions, $3k^2 + 6k \geq 0$ and $3k^2 - 6k \geq 0$.

$\Rightarrow k \in (-\infty, -2] \cup [2, \infty)$ for four solutions

$$\frac{x^2 - 3k^2}{6k} = \frac{6k}{x^2 - 3k^2}$$

$$(x^2 - 3k^2)^2 - 36k^2 = 0$$

$$(x^2 - 3k^2 - 6k)(x^2 - 3k^2 + 6k) = 0$$

$$x^2 = 3k^2 + 6k \text{ or } 3k^2 - 6k$$

Therefore, two solutions occur when $k \in (-2, 2) / \{0\}$. A1