VCAA 2020 MATHEMATICAL METHODS EXAMINATION 1 SOLUTIONS

By TWM Publications

Question 1a (1 mark)

By the product rule,

$$\frac{dy}{dx} = 2x\sin(x) + x^2\cos(x).$$

Question 1b (2 marks)

By the chain rule,

$$f'(x) = (2x-1)e^{x^2-x+3}.$$

$$\Rightarrow f'(1) = e^3.$$

Question 2a (1 mark)

Denote O for oilchange and A for air filter change.

$$\Pr(A \cap O') = \Pr(A) - \Pr(A \cap O)$$

$$= \frac{3}{20} - \frac{1}{20} = \frac{1}{10}.$$

Question 2b (2marks)

Now, $\Pr(A) = \frac{n}{m+n}$, $\Pr(O) = \frac{m}{m+n}$, $\Pr(A \cap O) = \frac{1}{m+n}$.

$$\Pr(A \cap O') = \frac{1}{20} = \frac{n}{m+n} - \frac{1}{m+n}.$$

$$\Rightarrow \frac{m+n}{20} = n-1 \Rightarrow m+n = 20n-20$$

$$\Rightarrow m = 19n-20$$

Question 3 (3 marks)

We have that

$$\tan(a+b) = \sqrt{3} \quad \text{and} \quad \tan(b-a) = -1.$$

$$\Rightarrow a+b = \frac{\pi}{3} \text{ and } b-a = \frac{-\pi}{4} \qquad (a>0 \text{ and } 0<b<1)$$

$$\Rightarrow a = \frac{7\pi}{24} \text{ and } b = \frac{\pi}{24}.$$

Question 4 (3 marks)

$$2\log_2(x+5) - \log_2(x+9) = 1 \qquad (x>-5)$$

$$\Rightarrow \log_2\left(\frac{(x+5)^2}{x+9}\right) = 1$$

$$\Rightarrow (x+5)^2 = 2(x+9)$$

$$\Rightarrow x^2+8x+7 = 0$$

$$\Rightarrow (x+1)(x+7) = 0$$

$$\Rightarrow x = -1 \quad (\text{as } x>-5).$$

Question 5a (4 marks)

Let $X \overset{d}{=} Bi(4, \frac{3}{5})$. Then

$$\begin{aligned}
Pr(X \geq 3) &= Pr(X=3) + Pr(X=4) \\
&= \binom{4}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^4 \\
&= \frac{27 \times 8}{625} + \frac{81}{625} \\
&= \frac{297}{625}.
\end{aligned}$$

Question 5b (2 marks)

$$\Pr(X=2 \mid X\geq 1) = \frac{\Pr(X=2)}{1-\Pr(X=0)}$$

$$= \frac{\binom{4}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2}{1-\left(\frac{2}{5}\right)^4}$$

$$= \frac{6\times 6^2}{5^4-2^4}$$

$$= \frac{6^3}{5^4-2^4}.$$

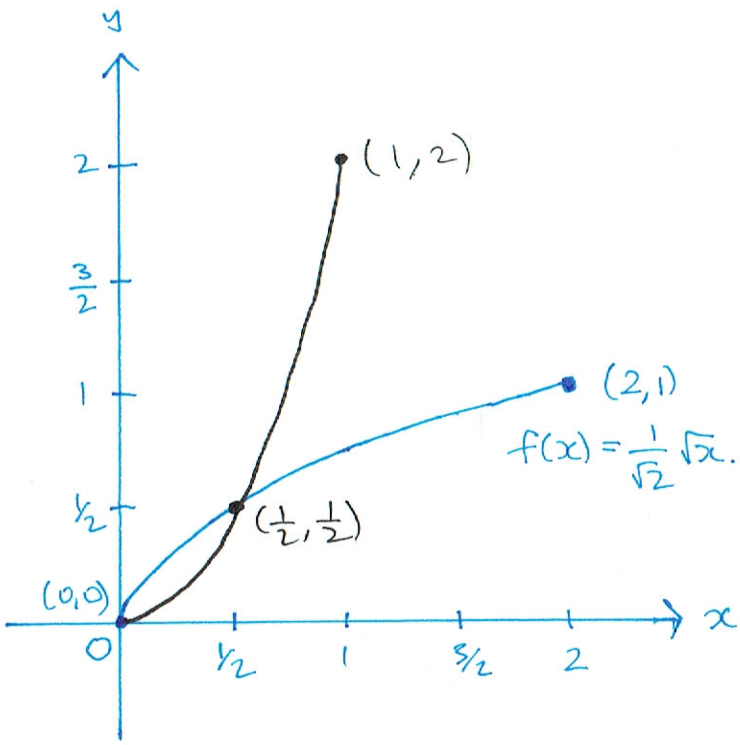
Question 6a (2marks)

We have

$$x = \sqrt{\frac{f^{-1}(x)}{2}} \Rightarrow f^{-1}(x) = 2x^2.$$

Then, $\operatorname{dom}(f^{-1}) = \operatorname{ran}(f) = [0,1]$.

Question 6b (2 marks)



Question 6c (4 marks)

$$A = \int_0^{1/2} \left(\frac{1}{\sqrt{2}}\sqrt{x} - 2x^2 \right) dx + \int_{1/2}^{1} \left(2x^2 - \frac{1}{\sqrt{2}}\sqrt{x} \right) dx$$

$$= \left[\frac{\sqrt{2}}{3} x^{3/2} - \frac{2}{3}x^3 \right]_0^{1/2} + \left[\frac{2}{3}x^3 - \frac{\sqrt{2}}{3} x^{3/2} \right]_{1/2}^{1}$$

$$= \left(\frac{1}{3} \times \frac{1}{2} - \frac{2}{3} \times \frac{1}{8} \right) + \left(\frac{2}{3} - \frac{\sqrt{2}}{3} \right) - \left(\frac{2}{3} \times \frac{1}{8} - \frac{1}{3} \times \frac{1}{2} \right)$$

$$= \frac{1}{6} - \frac{1}{12} + \frac{2}{3} - \frac{\sqrt{2}}{3} - \frac{1}{12} + \frac{1}{6}$$

$$= \frac{5}{6} - \frac{\sqrt{2}}{3}$$

$$= \frac{5 - 2\sqrt{2}}{6} \text{ units}^2.$$

Question 7a (1 marks)

$f(1) = 1^2 + 3 \times 1 + 5 = 1 + 3 + 5 = 9 \neq 0$,

so $(1, 0)$ is not on the graph of f.

Question 7b.i (1 mark)

$$m = \frac{f(a) - 0}{a - 1} = \frac{a^2 + 3a + 5}{a - 1}$$

Question 7b.ii (1 mark)

$f'(a) = 2a + 3$

Question 7b.iii (2 mark)

The tangent at Q is $y = (2a+3)(x-a) + a^2 + 3a + 5$.

$\Rightarrow 0 = (2a+3)(1-a) + a^2 + 3a + 5$

$\Rightarrow 0 = a^2 - 2a - 8$

$\Rightarrow (a+2)(a-4) = 0 \Rightarrow a = -2$ or $a = 4$.

Question 7b.iv (1 mark)

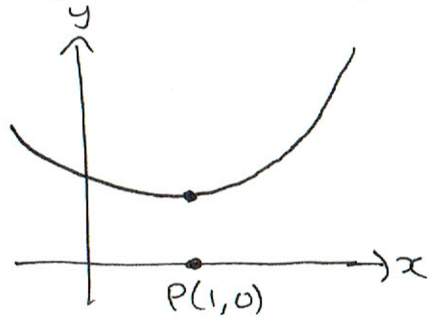
For $a=-2$: $y=-x+1$.

For $a=4$: $y=11x-11$.

(either one)

Question 7c (2 marks)

The shortest possible distance occurs when the turning point of $y=f(x-k)$ is directly above $P(1,0)$.



$$\Rightarrow \left.\frac{d}{dx}\left[f(x-k)\right]\right|_{x=1}=0$$

$$\Rightarrow 2(1-k)+3=0$$

$$\Rightarrow k=5/2.$$

Question 8a (2 marks)

By the product rule,

$$f'(x)=\log_e(x)+1.$$

So, $f'(a)=0 \Rightarrow \log_e(a)=-1 \Rightarrow a=\frac{1}{e}$.

$\Rightarrow Q=\left(\frac{1}{e},-\frac{1}{e}\right)$.

Question 8b (1 mark)

We have $\int(2x\log_e(x)+x)\,dx = x^2\log_e(x)$ up to an additive constant.

$$\Rightarrow 2\int x\log_e(x)\,dx = x^2\log_e(x)-\int x\,dx$$

$$= x^2\log_e(x)-\frac{x^2}{2}$$

$$\Rightarrow \int x\log_e(x)\,dx = \frac{x^2\log_e(x)}{2}-\frac{x^2}{4}\ (+c).$$

Question 8c (2 marks)

Here, $b=1$, so

$$A = -\int_{1/e}^{1} x\log_e(x)\,dx$$

$$= \left[\frac{x^2}{4} - \frac{x^2\log_e(x)}{2}\right]_{1/e}^{1}$$

$$= \left(\frac{1}{4} - 0\right) - \left(\frac{1}{4e^2} + \frac{1}{2e^2}\right)$$

$$= \frac{1}{4} - \frac{3}{4e^2} \text{ units}^2.$$

Question 8d.i (1 mark)

We wish to find x such that $f'(x) = 2$.

$\Rightarrow \log_e(x) = 1 \Rightarrow x = e$

Then, $g(e) = e + k$. But we require $g(e) = 2e$

$\Rightarrow e + k = 2e \Rightarrow k = e$.

Question 8d.ii (2 marks)

The graphs of g and g^{-1} will be tangential to eachother if $y = x$ is tangent to the graph of g.

$f'(x) = 1 \Rightarrow \log_e(x) = 0 \Rightarrow x = 1$.

Thus, g and g^{-1} are tangential when $g(1) = 1 \Rightarrow k = 1$.

So, the graphs do not intersect for $k \in (1, \infty)$.