The Mathematical Association of Victoria

Trial Examination 2020

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 32 pages.
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet
- Unless otherwise indicated, the diagrams are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Note: This examination was written for the Adjusted 2020 VCE Mathematics Study Design and accordingly does not include the Specialist Mathematics Area of Study 6 (Probability and Statistics).

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SECTION A - Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

The asymptotes of the graph of $f(x) = \dfrac{x^3 + 2x^2 + x - 1}{(x-1)(x+2)}$ have equations

A. $x = 1$, $x = -2$

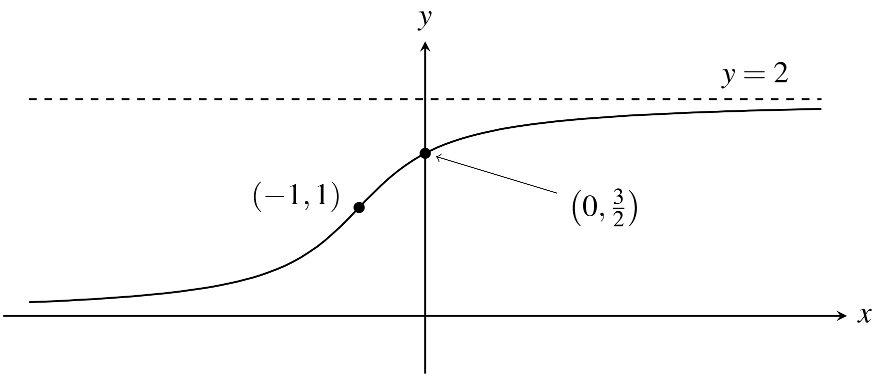
B. $x = -1$, $x = 2$

C. $y = 1 + x$, $x = -1$, $x = 2$

D. $y = 1 + x$, $x = 1$, $x = -2$

E. $y = 1 + x$, $x = -2$

Question 2



A rule for the function whose graph is shown above could be

A. $f(x) = \dfrac{2}{\pi}\arctan(x-1) + 1$

B. $f(x) = \dfrac{2}{\pi}\arctan(x+1) + 1$

C. $f(x) = \dfrac{\pi}{2}\arctan(x+1) + 1$

D. $f(x) = \dfrac{1}{\pi}\arctan(x+1) + 1$

E. $f(x) = \dfrac{1}{\pi}\arctan(x-1) + 1$

SECTION A – continued
TURN OVER

Question 3

The maximal domain of the function with rule $f(x) = 2\arcsin\left(\frac{x^2}{2}\right) - 1$ is

A. $[-1, 1]$

B. $[-1, 2]$

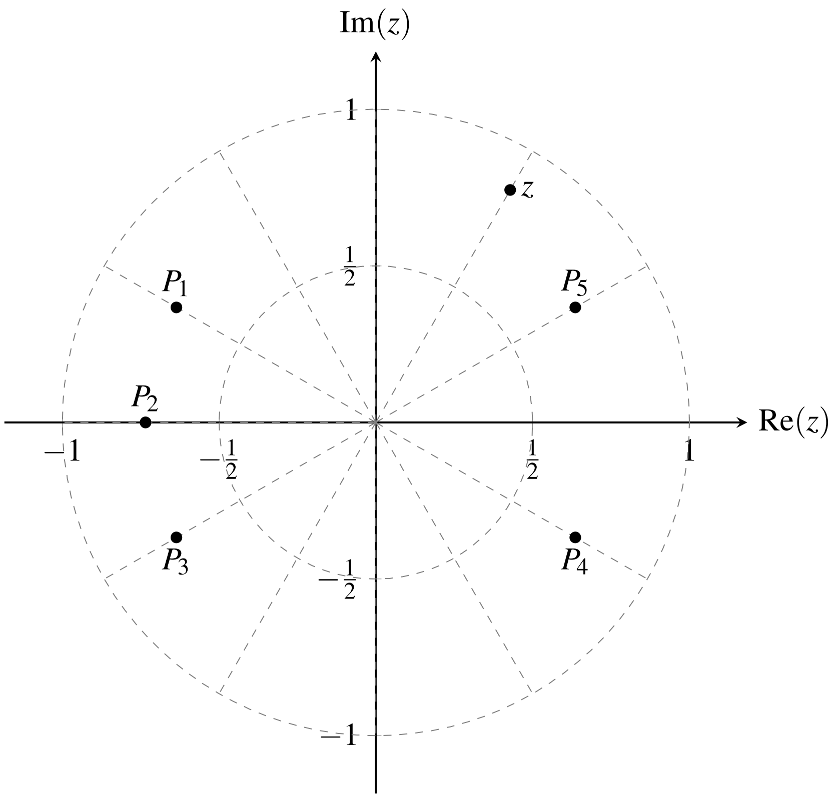
C. $[0, \sqrt{2}]$

D. $(-\sqrt{2}, \sqrt{2})$

E. $[-\sqrt{2}, \sqrt{2}]$

Question 4

A complex number z and points P_1, P_2, P_3, P_4, P_5 are plotted on the Argand diagram below:



The complex number iz^2 is best represented by the point

A. P_1

B. P_2

C. P_3

D. P_4

E. P_5

Question 5

The modulus and principal argument respectively of the complex number $z = \dfrac{(1+i)^2}{\left(1-\sqrt{3}i\right)^3}$ are

A. $\dfrac{1}{4}$ and $-\dfrac{\pi}{2}$

B. $\dfrac{1}{4}$ and $\dfrac{\pi}{2}$

C. $\dfrac{1}{2}$ and $-\dfrac{\pi}{2}$

D. $\dfrac{1}{2}$ and $\dfrac{\pi}{2}$

E. $\dfrac{1}{4}$ and $-\pi$

Question 6

The sum of all of the solutions of the equation $z^4 + z^3 - 2z^2 + 4z - 24 = 0$ is

A. $1-2i$

B. $1+2i$

C. 1

D. -1

E. 5

Question 7

The equation of the tangent to the curve $3x^2 - xy + y^2 = 5$ at the point $(1,2)$ is

A. $y = -\dfrac{4}{5}x + \dfrac{10}{5}$

B. $y = -\dfrac{4}{5}x + 1$

C. $y = -\dfrac{4}{3}x + 1$

D. $y = -\dfrac{4}{3}x - \dfrac{2}{3}$

E. $y = -\dfrac{4}{3}x + \dfrac{10}{3}$

SECTION A – continued
TURN OVER

Question 8

The length of the curve defined by the parametric equations $x(t) = \sin(4t)$, $y(t) = \cos(2t)$ between $t = 0$ and $t = \frac{\pi}{4}$ is given by

A. $\int_0^{\frac{\pi}{4}} \sqrt{10 - 2\cos(4t) + 8\cos(8t)}\, dt$

B. $\int_0^{\frac{\pi}{4}} \sqrt{8\cos^2(t)\left(3 - \cos(2t)\right)}\, dt$

C. $\int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{2}\left(2 + \cos(4t) - \cos(8t)\right)}\, dt$

D. $\int_0^{\frac{\pi}{4}} \sqrt{10 + 2\cos(4t) + 8\cos(8t)}\, dt$

E. $\int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{2}\left(2 - \cos(4t) + \cos(8t)\right)}\, dt$

Question 9

If $\frac{dy}{dx} = \sin\left(\frac{1}{\sqrt{x}}\right)$ and $y = 5$ when $x = 1$ then the value of y when $x = 2$ is closest to

A. –4.266

B. 3.559

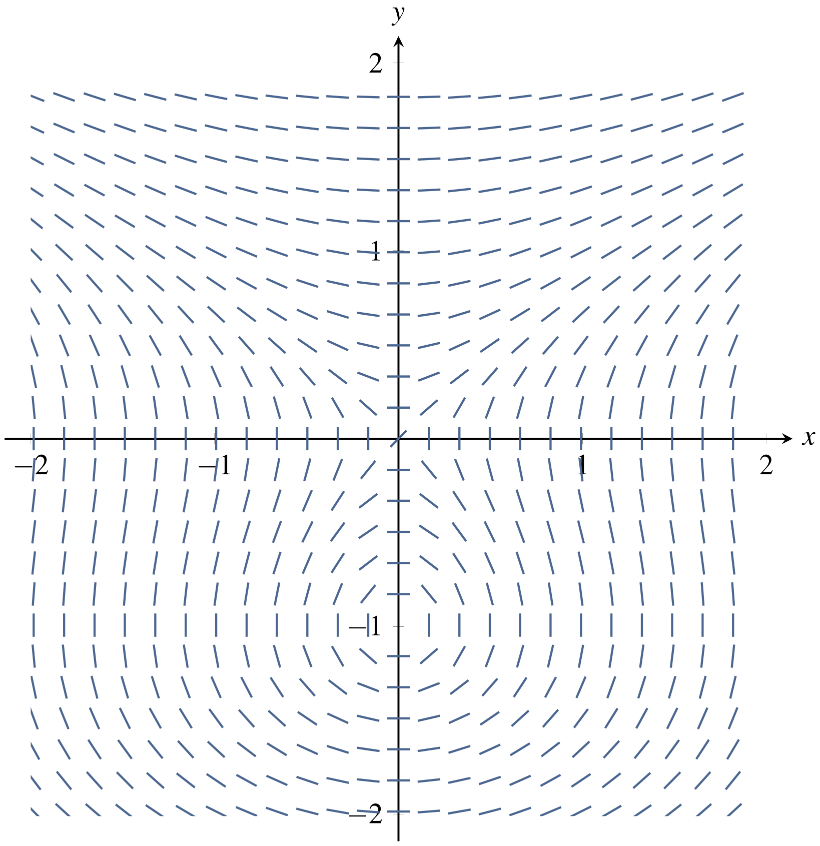
C. 4.293

D. 5.734

E. 6.559

SECTION A – continued

Question 10



The differential equation which best represents direction field shown above is

A. $\dfrac{dy}{dx} = \dfrac{xy}{1+y^2}$

B. $\dfrac{dy}{dx} = \dfrac{x}{y\left(1+y^2\right)}$

C. $\dfrac{dy}{dx} = \dfrac{y}{x\left(1+y^2\right)}$

D. $\dfrac{dy}{dx} = \dfrac{x}{y\left(1+y\right)}$

E. $\dfrac{dy}{dx} = \dfrac{y}{x\left(1+y\right)}$

Question 11

Using a suitable substitution, the definite integral $\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1+\log_e x^2}}\,dx$ is equivalent to

A. $\frac{1}{2}\int_1^2 \frac{1}{\sqrt{1+u}}\,du$

B. $2\int_1^2 \frac{1}{\sqrt{u}}\,du$

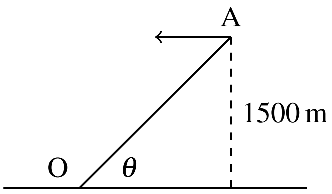
C. $\int_1^2 \frac{1}{\sqrt{u}}\,du$

D. $\frac{1}{2}\int_1^2 \frac{1}{\sqrt{u}}\,du$

E. $\frac{1}{2}\int_1^{\sqrt{e}} \frac{1}{\sqrt{u}}\,du$

Question 12

An aircraft A is flying at constant height of 1500m with a speed of 90ms^{-1} towards an observer O on the ground. The observer measures the angle of elevation θ, as shown in the diagram below.



What is the rate of change of θ in radians per second when $\theta = \frac{\pi}{3}$?

A. $\frac{1}{1500}$

B. $\frac{\sqrt{3}}{1500}$

C. $\frac{1}{200}$

D. $\frac{3}{200}$

E. $\frac{9}{200}$

SECTION A – continued

Question 13

In the parallelogram $OABC$, $\overrightarrow{OA} = -2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$. The acute angle between the diagonals of the parallelogram is closest to

A. 56.31°

B. 23.69°

C. 33.69°

D. 87.59°

E. 24.31°

Question 14

The vectors $\underset{\sim}{i} + p^2\underset{\sim}{j} + 3\underset{\sim}{k}$, $-2\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}$ and $2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, where p is a real constant, are linearly independent if

A. $p = 5$

B. $p \in R \setminus \{-5, 5\}$

C. $p \in \{-5, 5\}$

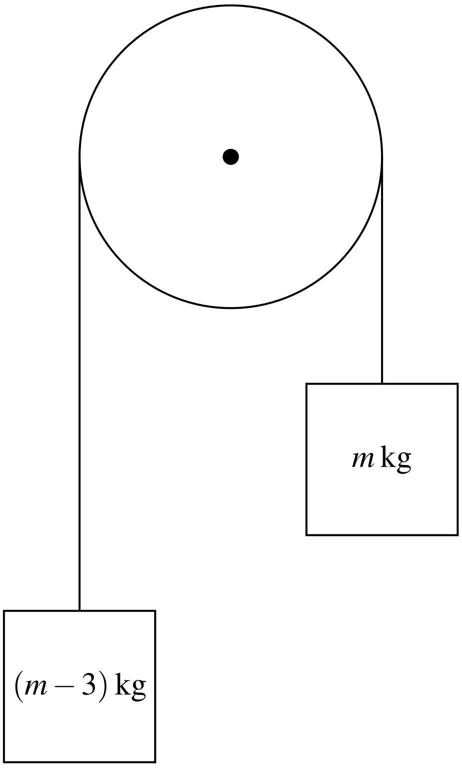
D. $p = -5$

E. $p \in R$

SECTION A – continued
TURN OVER

Question 15

Two masses of m kg and $(m-3)$ kg are connected by a light inextensible string which passes over a smooth pulley as shown below.



If the acceleration of the system is 2 ms^{-2} then the value of m is

A. $\dfrac{3g-6}{2}$

B. $\dfrac{3g+6}{2}$

C. $\dfrac{3g+6}{4}$

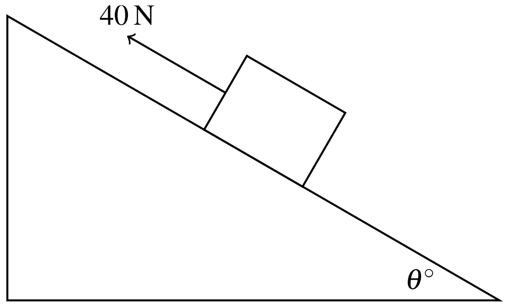
D. $\dfrac{3g-6}{4}$

E. $\dfrac{3g-3}{4}$

SECTION A – continued

Question 16

A mass of 5 kg is pulled up a smooth plane inclined at an angle of θ° to the horizontal by a force of 40 Newtons, as shown in the diagram below.



The mass accelerates at 3.5ms^{-2} up the plane. The value of θ in degrees is closest to:

A. 27.33°

B. 62.67°

C. 20.14°

D. 35.78°

E. 21.54°

Question 17

A particle of mass 2 kg is initially travelling with velocity $5\underset{\sim}{i}\text{ ms}^{-1}$. A force of $6\underset{\sim}{j}$ Newtons is applied to the particle for a period of 4 seconds.
The magnitude of the momentum of the particle in kg ms^{-1} after the force is removed is

A. 13

B. 26

C. $2\sqrt{41}$

D. $\sqrt{581}$

E. $2\sqrt{581}$

SECTION A – continued
TURN OVER

Question 18

The component of $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$ perpendicular to $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ is

A. $\frac{1}{7}\left(-5\underset{\sim}{i} + 10\underset{\sim}{j} - \underset{\sim}{k}\right)$

B. $\frac{1}{7}\left(12\underset{\sim}{i} + 4\underset{\sim}{j} - 20\underset{\sim}{k}\right)$

C. $-2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

D. $\frac{1}{\sqrt{7}}\left(11\underset{\sim}{i} - 13\underset{\sim}{j} - 5\underset{\sim}{k}\right)$

E. $\frac{1}{7}\left(11\underset{\sim}{i} - 13\underset{\sim}{j} - 5\underset{\sim}{k}\right)$

Question 19

A body is travelling in a straight line. Its velocity $v\,\text{ms}^{-1}$ is given by $v = 3 - x$ when it is x m from the origin at time t seconds. Given $x = 2$ when $t = 1$, the rule relating x to t is given by

A. $x = \log_e(2 - t) + 2$

B. $x = 3 - e^{1-t}$

C. $x = 3 - e^{t-1}$

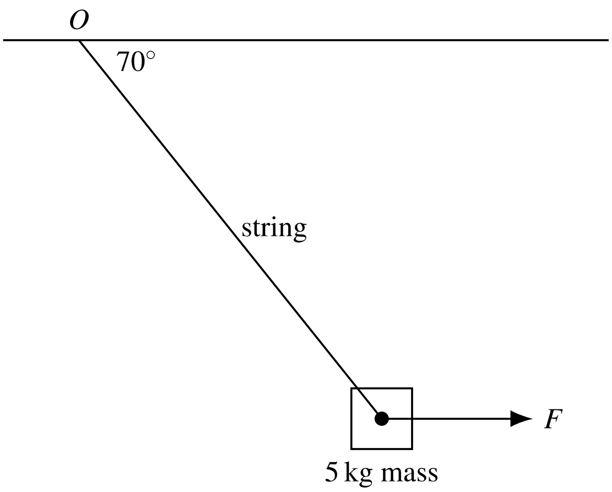
D. $x = \frac{3}{t}$

E. $x = 3 + e^{1-t}$

SECTION A – continued

Question 20

A mass of 5 kg is suspended on a light inextensible string from a point *O* on a horizontal ceiling.
When a horizontal force of F newtons is applied to the mass the string makes an angle of 70° with the horizontal ceiling as shown in the diagram below.



If the tension in the string is T newtons then the values of T and F respectively in newtons and correct to three decimal places are

A. 52.145 and 17.835

B. 143.266 and 134.626

C. 46.045 and 15.748

D. 16.759 and 15.748

E. 46.045 and 134.626

END OF SECTION A

TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (10 marks)

Let $f : D \to R$, $f(x) = \frac{\sqrt{x}}{1+x^2}$ where D is the maximal domain of f.

a. Find D. 1 mark

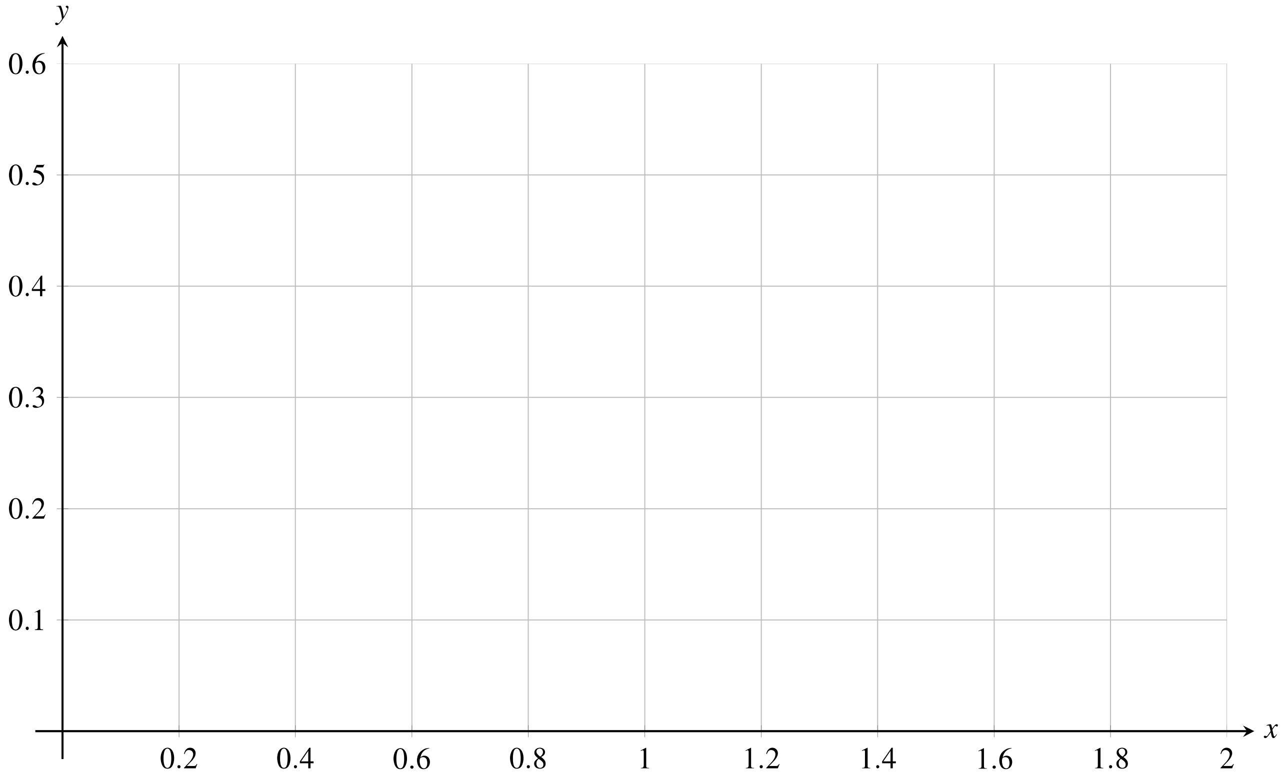
b. Find $f'(x)$ and hence write down the coordinates of the stationary point. 2 marks

SECTION B – Question 1 - continued

c. **i.** Write down a quartic equation which can be solved to give the x-coordinate of the point of inflection of f. 1 mark

ii. Hence find the coordinates of the point of inflection. Give your answer correct to three decimal places. 1 mark

d. Sketch the graph of $y = f(x)$ on the axes below. Include the coordinates of the stationary point and the point of inflection. 3 marks



e. The region bounded by the graph of f, the x-axis and the line $x = a$, where $a > 0$, is rotated about the x-axis to form a solid of revolution of volume $\frac{\pi}{12}$.

i. Write down an equation involving a definite integral that can be used to calculate the value of a. 1 mark

ii. Hence find the value of a. 1 mark

SECTION B – continued

Question 2 (10 marks)

a. Find the cartesian equation of the circle $C_1 = \{z : |z-2| = 2\}$. 1 mark

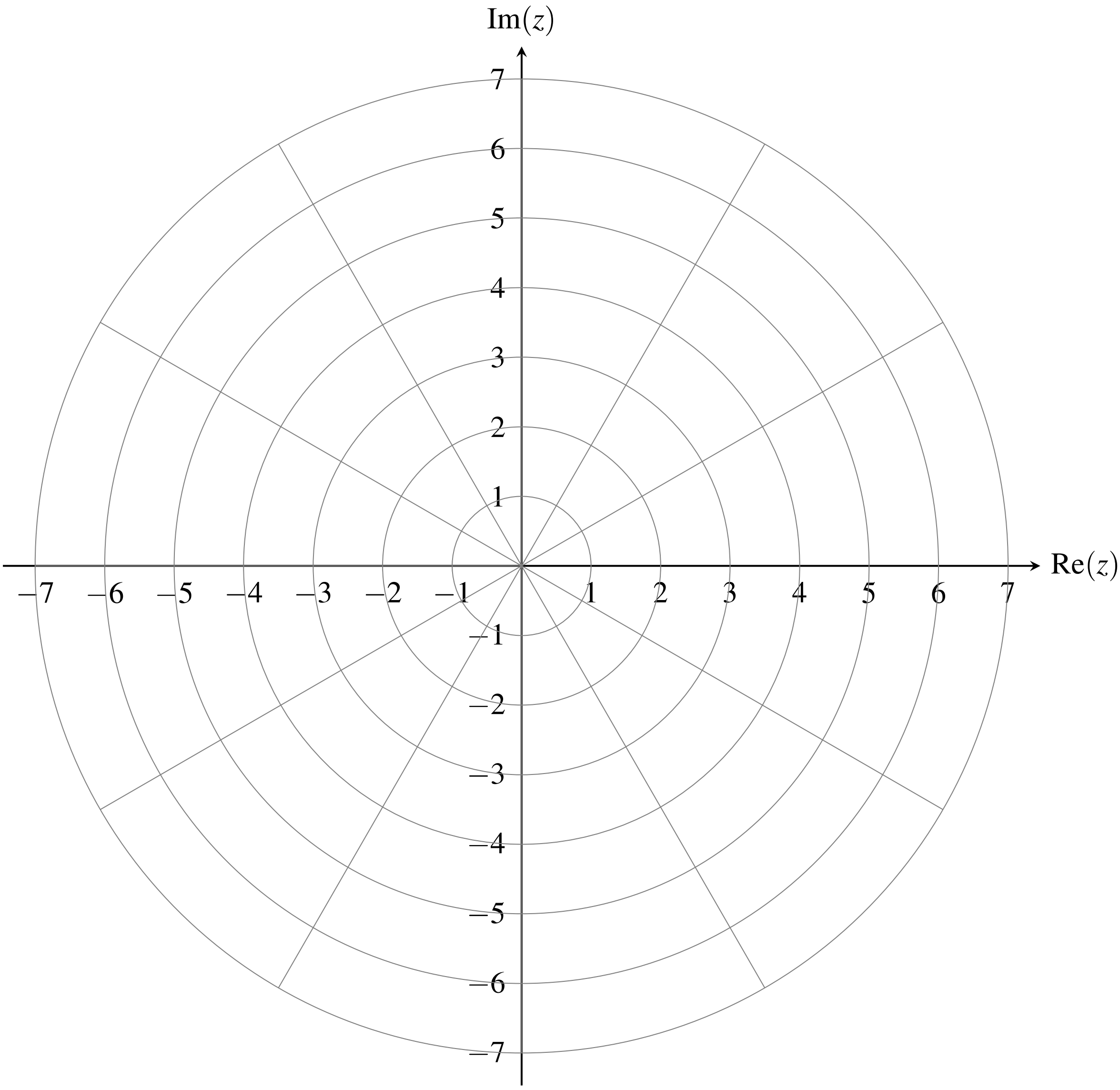
b. **i.** Show that the solutions of $z^2 - 6z + 12 = 0$ are $z = 3 \pm \sqrt{3}i$. 1 mark

ii. Show that $z_1 = 3 + \sqrt{3}i$ lies on the circle C_1. 1 mark

c. On the Argand diagram below:

i. Sketch the circle C_1. 1 mark

ii. Plot the solutions of $z^2 - 6z + 12 = 0$. 1 mark



SECTION B – Question 2 - continued

The circle C_2 given by the relation $\left|z-2\sqrt{3}i\right|=2\sqrt{3}$ intersects the circle C_1 at the origin and at z_1.

d. Sketch the circle C_2 and the line l which passes through the points of intersection of the circles C_1 and C_2 on the Argand diagram given in **part c.** 1 mark

e. The line l which passes through the points of intersection of the circles C_1 and C_2 can be written in the form $|z-2|=|z-\alpha|$ where $\alpha \in C$. Find α. 1 mark

f. Find the area of the region common to both of the circles C_1 and C_2. Give your answer correct to three decimal places. 2 marks

SECTION B– continued
TURN OVER

Question 3 (10 marks)

The temperature $x(t)$ in a room satisfies the differential equation

$$\frac{dx}{dt} = -k\left(x-20\right)$$

where x is measured in degrees and $t \geq 0$ is measured in minutes. The temperature in the room is initially 12°C.

a. If $x(20)=18$, find the value of k. 1 mark

In a second room a different heating system is used. The temperature in this room satisfies the differential equation

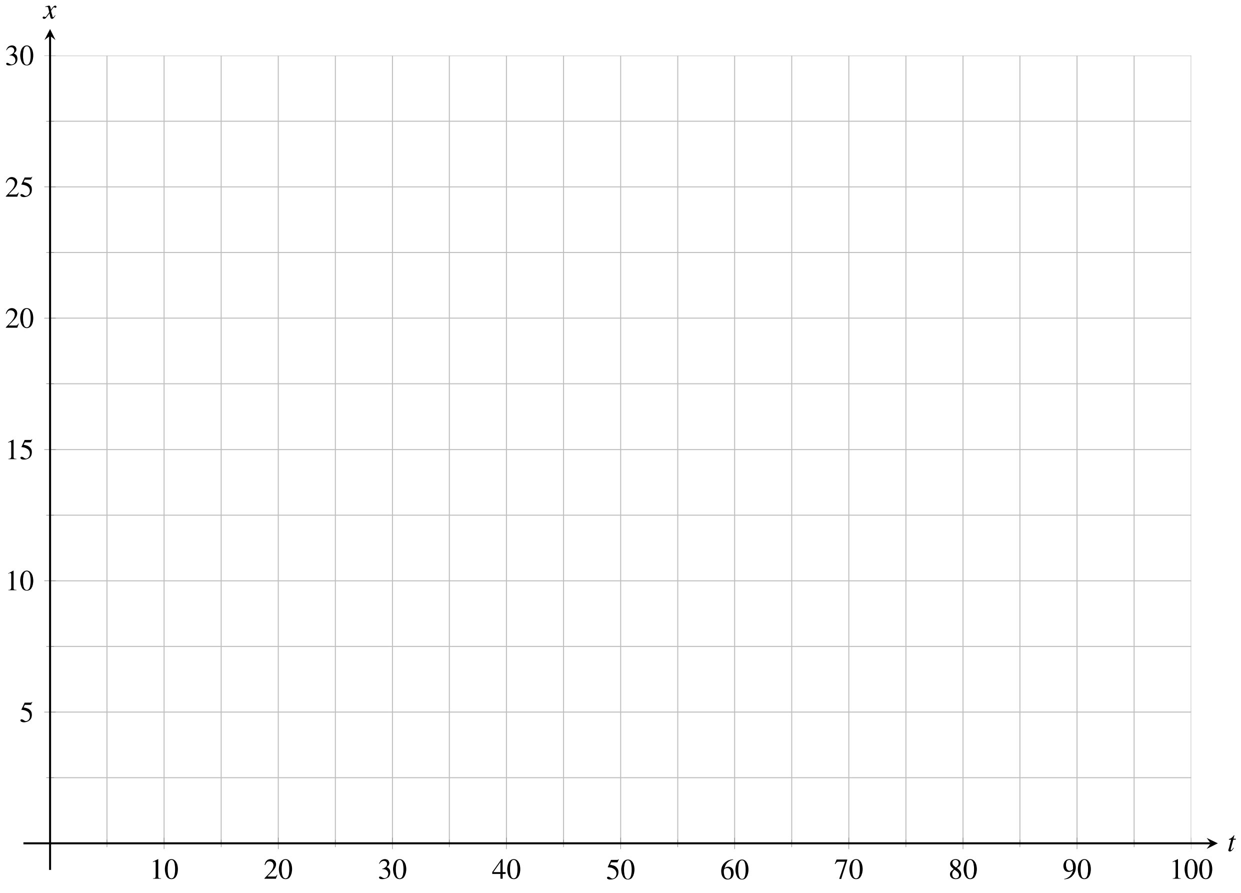
$$\frac{dx}{dt} = -\frac{1}{10}(x-20)+2e^{-\frac{1}{10}t},\ t \geq 0.$$

b. Verify by differentiation that $x(t) = 20+2te^{-\frac{1}{10}t} - 8e^{-\frac{1}{10}t}$ satisfies the differential equation, subject to the initial condition $x(0)=12$. 2 marks

SECTION B – Question 3 - continued

c. Find, correct to three decimal places, the maximum temperature in the second room and the time in seconds after $t = 0$ that this maximum temperature occurs. 2 marks

d. Sketch the graph of x versus t for the second room on the axes below. Label the stationary point and any asymptotes. 3 marks



SECTION B – Question 3 - continued
TURN OVER

In a third room the temperature satisfies the differential equation

$$\frac{dx}{dt} = -\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}},\ t \ge 0.$$

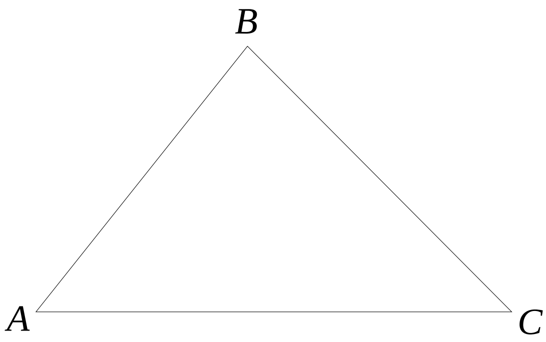
Assume that $x(0) = 12$.

e. **i.** Write down an integral that can be used to determine the time in minutes after $t = 0$ that the temperature in the third room is $r^{\circ}\text{C}$. 1 mark

ii. Hence determine how long after $t = 0$ it takes for the temperature in the third room to reach 18°C. Give your answer in minutes, correct to three decimal places. 1 mark

Question 4 (8 marks)

Relative to a fixed origin O, the vertices A, B and C of the triangle shown below have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively.



a. Show that the position vector of the midpoint M of side BC of this triangle is $\underset{\sim}{m} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$. 1 mark

Points lying on the line passing through A and M have a position vector given by $\underset{\sim}{\mathrm{r}}_A = \underset{\sim}{a} + \overrightarrow{AM}\,t$, $t \in R$.

b. Show that $\underset{\sim}{\mathrm{r}}_A = (1-t)\underset{\sim}{a} + \frac{t}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$. 2 marks

SECTION B – Question 4- continued

TURN OVER

c. Hence show that $\underset{\sim}{r}_A = \left(1-\frac{3}{2}t\right)\underset{\sim}{a}+\frac{t}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)$. 1 mark

d. State in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ the position vector:

i. $\underset{\sim}{r}_B$ of points lying on the line passing through *B* and the midpoint of side *AC*. 1 mark

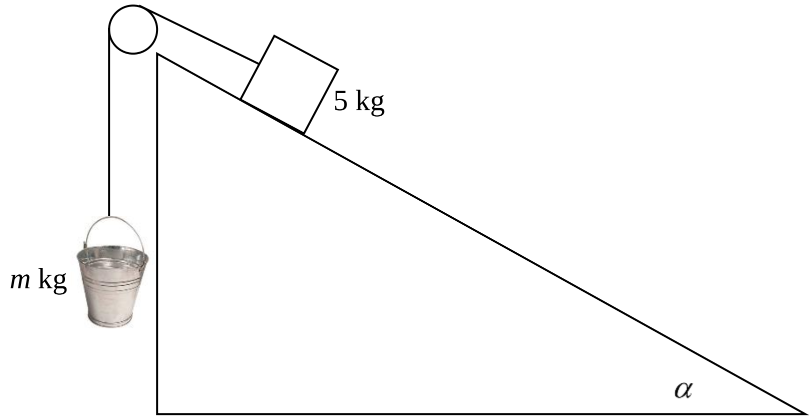
ii. $\underset{\sim}{r}_C$ of points lying on the line passing through *C* and the midpoint of side *AB*. 1 mark

e. Hence show that the medians of a triangle intersect in a point and state the position vector of this point in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. 2 marks

SECTION B - continued

Question 5 (11 marks)

An object of mass 5 kg is initially held at rest on a smooth plane inclined at an angle α to the horizontal. The mass is connected by a light inextensible string passing over a light smooth pulley to a freely hanging bucket of water of mass m kg.



a. Show that after it is released, the 5 kg object moves down the plane if $m < 5\sin(\alpha)$. 2 marks

b. Find, correct to two decimal places, the value of *m* if $\tan(\alpha) = \frac{12}{5}$ and the 5 kg object moves up the plane with an acceleration of 0.5 ms^{-2} after it is released. 2 marks

SECTION B – Question 5 - continued

Take $\tan(\alpha) = \frac{12}{5}$.

c. The mass of the bucket of water is initially 6 kg but once the 5 kg object is released, water begins to leak from the bottom of the bucket at a rate of 0.1 kg per second.

i. At what times after the release of the 5 kg object does the bucket have an acceleration of magnitude 0.2 ms^{-2}? Give your answers correct to two decimal places. 4 marks

SECTION B – Question 5 - continued

TURN OVER

ii. Find, correct to two decimal places, the distance travelled by the 5 kg object six seconds after it is released. 3 marks

Question 6 (11 marks)

Let the complex function $w = \cos(x) + i\sin(x)$, where $x \in R$ and $i^2 = -1$.

a. **i.** Show that $\frac{dw}{dx} = iw$. 1 mark

ii. Hence show that $e^{ix} = \cos(x) + i\sin(x)$. 2 marks

b. Show that $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$. 1 mark

SECTION B – Question 6 - continued

TURN OVER

Let $\sin(x) = p$ where $p \in [-1, 1]$.

c. **i.** Show that $\left(e^{ix}\right)^2 - 2ipe^{ix} - 1 = 0$. 1 mark

ii. By defining $\sin^{-1}(0) = 0$, show that $\sin^{-1}(p) = -i \log_e\left(\sqrt{1-p^2} + ip\right)$. 2 marks

SECTION B – Question 6 - continued

iii. Let $z = r\text{cis}(\theta)$ where θ is the principal argument of z.
Show that $\log_e(z) = \log_e(r) + i\theta$. 1 mark

d. Hence show that $\sin^{-1}(1) = \frac{\pi}{2}$. 1 mark

e. Show that $-i\ \log_e\left(\sqrt{1-p^2}+ip\right)$ is real for $p \in [-1,\ 1]$ and non-real otherwise. 2 marks

END OF QUESTION AND ANSWER BOOK