Instructions

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

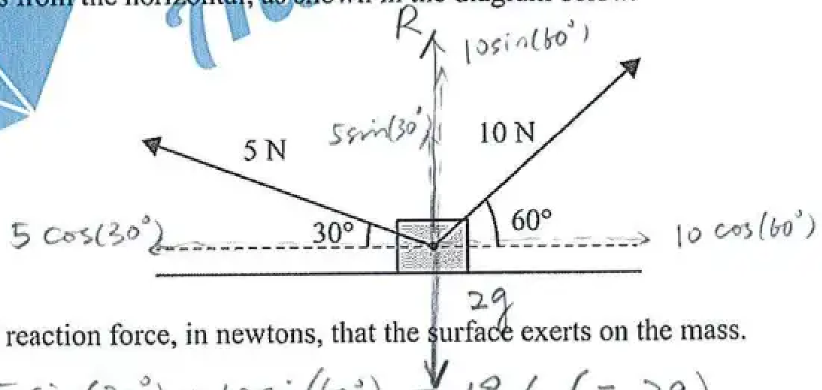
In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (5 marks)

A 2 kg mass is initially at rest on a smooth horizontal surface. The mass is then acted on by two constant forces that cause the mass to move horizontally. One force has magnitude 10 N and acts in a direction 60° upwards from the horizontal, and the other force has magnitude 5 N and acts in a direction 30° upwards from the horizontal, as shown in the diagram below.



a. Find the normal reaction force, in newtons, that the surface exerts on the mass. 2 marks

$R + 5\sin(30°) + 10\sin(60°) = 19.6 \; (= 2g)$ Not acceptable?

$R + \frac{5}{2} + 5\sqrt{3} = 19.6 \; (= 2g)$ (or $R = 2g - \frac{5}{2} - 5\sqrt{3}$?) newtons

$R = 17.1 - 5\sqrt{3} \; \left(= \frac{1710}{100} - 5\sqrt{3} = \frac{1710 - 500\sqrt{3}}{100}\right)$ Newtons

b. Find the acceleration of the mass, in ms^{-2}, after it begins to move. 2 marks

$10\cos(60°) - 5\cos(30°) = 2a$ (To the right)

$5 - \frac{5\sqrt{3}}{2} = 2a, \quad 2a = \frac{10 - 5\sqrt{3}}{2} = \frac{5(2-\sqrt{3})}{2}$

$a = \frac{5(2-\sqrt{3})}{4}$

c. Find how far the mass travels, in metres, during the first four seconds of motion. 1 mark

$x = ut + \frac{1}{2}at^2$

$u = 0, t = 4, a = \frac{5}{4}(2 - \sqrt{3})$

$x = \frac{1}{2}\left(\frac{5}{4}(2-\sqrt{3})\right) \times 16$

$= 10(2 - \sqrt{3})$ metres

Question 2 (4 marks)

Evaluate $\int_{-1}^{0} \frac{1+x}{\sqrt{1-x}}\,dx$. Give your answer in the form $a\sqrt{b}+c$, where $a, b, c \in R$.

$$\int_{-1}^{0} \frac{1}{\sqrt{1-x}}\,dx + \int_{-1}^{0} \frac{x}{\sqrt{1-x}}\,dx$$

let $u = 1-x$, $x = 1-u$

$\frac{du}{dx} = -1$, $1 = -\left(\frac{du}{dx}\right)$

$x=-1, u=2$

$x=0, u=1$

$$= \left[-2\sqrt{1-x}\right]_{-1}^{0} + \int_{x=-1}^{x=0} \frac{1-u}{\sqrt{u}}\left(-\frac{du}{dx}\right)dx$$

$$= \left[-2\sqrt{1-x}\right]_{-1}^{0} + \int_{u=2}^{u=1} \left(\frac{1}{\sqrt{u}} - \sqrt{u}\right)(-du)$$ swap orders

$$= (-2\sqrt{1}) - (-2\sqrt{2}) + \left[2\sqrt{u} - \frac{2}{3}\sqrt{u^3}\right]_1^2$$

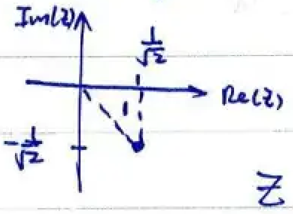
$$= -2 + 2\sqrt{2} + \left(2\sqrt{2} - \frac{2}{3}\times 2\sqrt{2}\right) - \left(2 - \frac{2}{3}\right)$$

$$= -2 - \frac{4}{3} + 4\sqrt{2} - \frac{4\sqrt{2}}{3} = \frac{8\sqrt{2}}{3} - \frac{10}{3}$$

Question 3 (3 marks)

Find the cube roots of $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. Express your answers in polar form using principal values of the argument.

$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \text{cis}\left(\frac{-\pi}{4}\right)$ let $z^3 = \text{cis}\left(-\frac{\pi}{4} + 2k\pi\right)$, $k \in Z$



$z^3 = \text{cis}\left(\frac{-\pi + 8k\pi}{4}\right) = \text{cis}\left(\frac{\pi(8k-1)}{4}\right)$

$\Rightarrow z = \text{cis}\left(\frac{\pi}{12}(8k-1)\right)$ by De Moivre's Theorem

z needs 3 distinct values

$k=0$, $z = \text{cis}\left(\frac{-\pi}{12}\right)$

$k=1$, $z = \text{cis}\left(\frac{7\pi}{12}\right)$

$k=-1$, $z = \text{cis}\left(\frac{-9\pi}{12}\right) = \text{cis}\left(-\frac{3\pi}{4}\right)$

Answer:

$$\begin{cases} z = \text{cis}\left(-\frac{\pi}{12}\right) \\ z = \text{cis}\left(\frac{7\pi}{12}\right) \\ z = \text{cis}\left(\frac{-3\pi}{4}\right) \end{cases}$$

Question 4 (4 marks) Features graphical approach

Solve the inequality $3-x>\frac{1}{|x-4|}$ for x, expressing your answer in interval notation.

Using absolute value properties : $|x-4| = x-4,\ x>4$

$|x-4| = 4-x,\ x<4$

Case 1:

$3-x > \frac{1}{x-4} \Rightarrow (3-x)(x-4)>1 \Rightarrow 3x-12-x^2+4x>1$

$\Rightarrow -x^2+7x-13>0$ No need to keep solving!

As for $x>4$, $(y_1=3-x)>(y_2=\frac{1}{x-4})$ is above

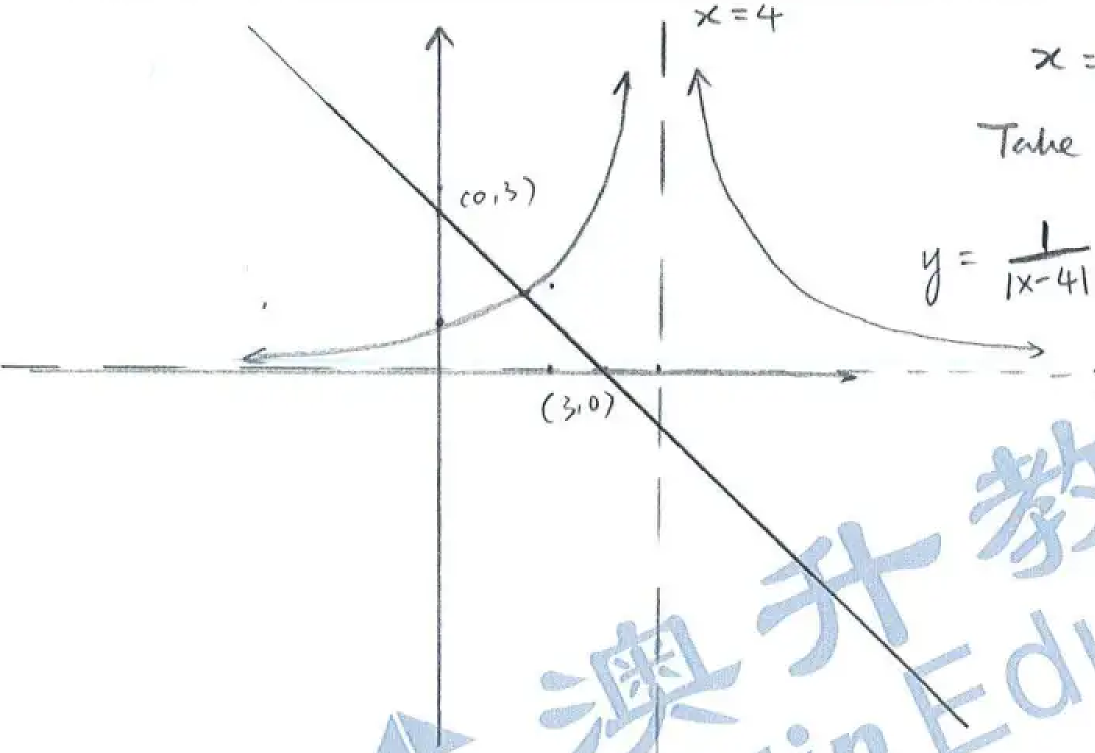
Case 2:

$3-x>\frac{1}{4-x} \Rightarrow (3-x)(4-x)>1 \Rightarrow 12-7x+x^2>1$

$\Rightarrow x^2-7x+11>0$

Solving for x first, i.e: let $3-x=\frac{1}{4-x}$ (left branch only)

$x^2-7x+11=0 \quad x=\frac{7\pm\sqrt{49-44}}{2}$



$x=\frac{7\pm\sqrt{5}}{2}$

Take the smaller as in the diagram $(x<4)$

$\therefore x\in\left(-\infty, \frac{7-\sqrt{5}}{2}\right)$

Question 5 (4 marks)

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}$, where m is an integer.

The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $-\frac{11}{18}\left(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}\right)$.

a. Find the value of m. 3 marks

$\underset{\sim}{a}_{//} = \left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\right)\cdot\underset{\sim}{b} = \frac{-11}{18}(i + mj - k)$

① $\underset{\sim}{a}\cdot\underset{\sim}{b} = -11 \Rightarrow 2\times1 + (-3)\times m + 1\times(-1) = -11$

$-3m + 1 = -11,\ -3m = -12,\ m = 4$

② check if $\underset{\sim}{b}\cdot\underset{\sim}{b} = 18$?

$1^2 + m^2 + (-1)^2 = 18,\ m^2 = 16,\ m = \pm 4$

Considering ① & ② together: $m = 4$ only

b. Find the component of $\underset{\sim}{a}$ that is perpendicular to $\underset{\sim}{b}$. 1 mark

$\underset{\sim}{a}_{\perp} = \underset{\sim}{a} - \underset{\sim}{a}_{//}$

$= (2i - 3j + k) - \left(\frac{-11}{18}(i + 4j - k)\right)$

$= 2i - 3j + k + \frac{11}{18}(i + 4j - k)$

$= \left(2 + \frac{11}{18}\right)i + \left(-3 + \frac{44}{18}\right)j + \left(1 - \frac{11}{18}\right)\underset{\sim}{k}$

$= \frac{47}{18}i - \frac{5}{9}j + \frac{7}{18}k$

(Equivalent ans: OK $\frac{1}{18}(47i - 10j + 7k)$)

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Question 6 (5 marks)

Let $f(x) = \arctan(3x - 6) + \pi$.

a. Show that $f'(x) = \dfrac{3}{9x^2 - 36x + 37}$. 1 mark

$f'(x) = \dfrac{1}{(3x-6)^2 + 1} \times 3$

$\Rightarrow f'(x) = \dfrac{3}{9x^2 - 36x + 36 + 1} = \dfrac{3}{9x^2 - 36x + 37}$

b. Hence, show that the graph of f has a point of inflection at $x = 2$. 2 marks

$f''(x) = \dfrac{-3(18x-36)}{(9x^2 - 36 + 37)^2} = 0$

Note: (Not needed, but good)
for $x > 2$, $f''(x) < 0$
$(9x^2 - 36x + 37)^2 > 0$
for $x < 2$, $f''(x) > 0$
$\therefore$ change in concavity guaranteed

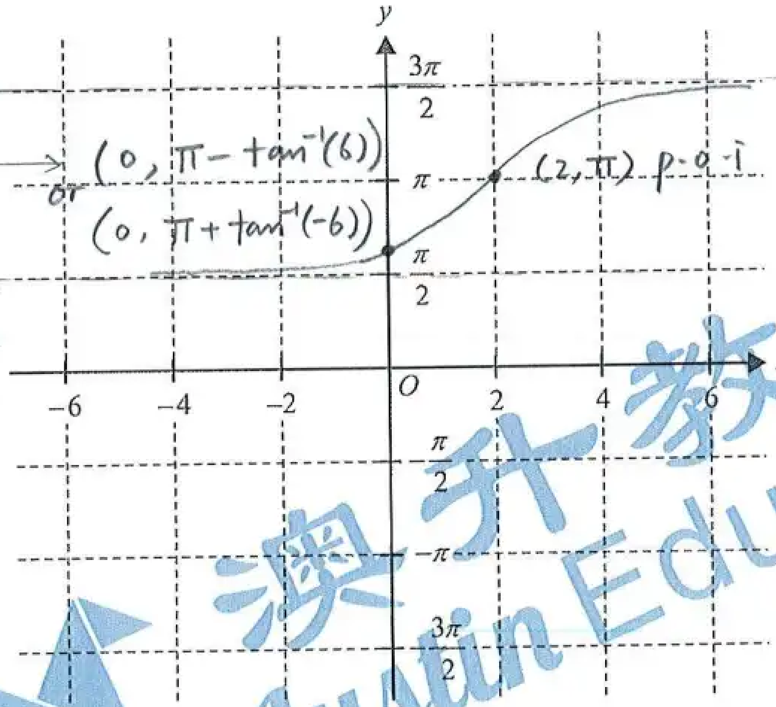
Numerator $= 0$ only $\Rightarrow 18x - 36 = 0$, $18(x-2) = 0$

$\therefore x = 2$ for the only inflection point

(Note: don't accept subing $x=2$! That's verifying)

c. Sketch the graph of $y = f(x)$ on the axes provided below. Label any asymptotes with their equations and the point of inflection with its coordinates. 2 marks

Not required but position should be accurate



$y = \frac{3\pi}{2}$ (HA)

$y = \frac{\pi}{2}$ (HA)

Question 7 (5 marks)

Consider the function defined by

$$f(x)=\begin{cases} mx+n, & x<1 \\ \dfrac{4}{1+x^2}, & x\geq 1 \end{cases}$$

$$f(x)=\begin{cases} -2x+4, & x<1 \\ \dfrac{4}{1+x^2}, & x\geq 1 \end{cases}$$

$$f'(x)=\begin{cases} m, & x<1 \\ -\dfrac{4(2x)}{(1+x^2)^2}, & x\geq 1 \end{cases}$$

continuous

where m and n are real numbers.

a. Given that $f(x)$ and $f'(x)$ are continuous over R, show that $m=-2$ and $n=4$. 2 marks

Continuity: $f(1^+)=f(1^-) \Rightarrow m+n=\frac{4}{2}$

$m+n=2$ ①

~~Smoothness~~

$f'(1^+)=f'(1^-) \Rightarrow m=\frac{-8}{4}=-2$ ②

$\therefore m=-2$, and $-2+n=2$

$\therefore n=4$

There are also other methods, but not sub in m & n first

b. Find the area enclosed by the graph of the function, the x-axis and the lines $x=0$ and $x=\sqrt{3}$. 3 marks

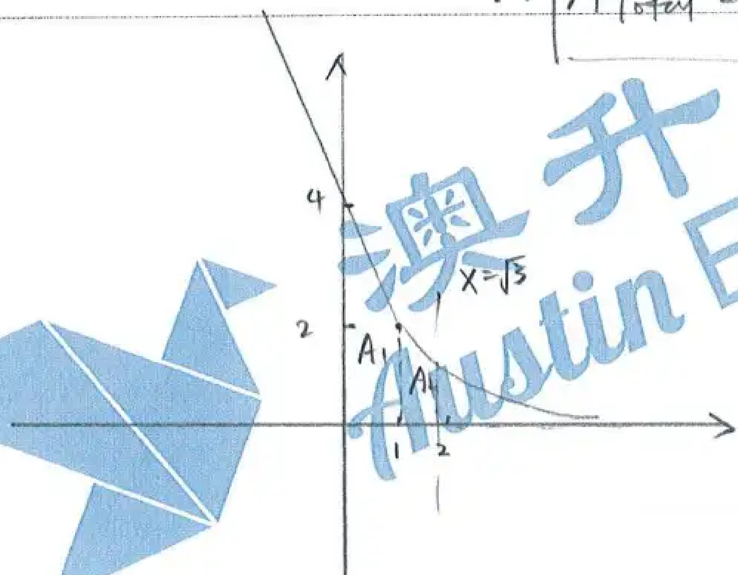
$A_1=\frac{1}{2}\times(2+4)\times 1=3$

$A_2=\int_1^{\sqrt{3}}\frac{4}{1+x^2}\,dx=4\left[\tan^{-1}(x)\right]_1^{\sqrt{3}}$

$=4\left[\tan^{-1}(\sqrt{3})-\tan^{-1}(1)\right]$

$=4\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=4\times\frac{\pi}{12}=\frac{\pi}{3}$.

$\therefore \boxed{A_{Total}=3+\frac{\pi}{3}}$



Note.

$g(x)=\frac{4}{1+x^2}$ look like

careful! after $(\ )^2$, $2 \to 4$!

Question 8 (5 marks)

Find the volume, V, of the solid of revolution formed when the graph of $y = 2\sqrt{\dfrac{x^2+x+1}{(x+1)(x^2+1)}}$ is rotated about the x-axis over the interval $\left[0, \sqrt{3}\right]$. Give your answer in the form $V = 2\pi\left(\log_e(a) + b\right)$, where $a, b \in R$.

$$V = \pi \int_0^{\sqrt{3}} 4\left(\frac{x^2+x+1}{(x+1)(x^2+1)}\right) dx$$

$$= 4\pi \int_0^{\sqrt{3}} \left(\frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{x^2+1}\right) dx$$

$$= 2\pi \int_0^{\sqrt{3}} \left(\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$

$$= 2\pi\left[\log_e|x+1| + \frac{1}{2}\log_e|x^2+1| + \tan^{-1}(x)\right]_0^{\sqrt{3}}$$

$$= 2\pi\left(\log_e(\sqrt{3}+1) + \frac{1}{2}\log_e(4) + \frac{\pi}{3}\right) - 2\pi(0+0+0)$$

$$= 2\pi\left(\log_e\left(2(\sqrt{3}+1)\right) + \frac{\pi}{3}\right)$$

(where $a = 2(\sqrt{3}+1)$, $b = \frac{\pi}{3}$)

Partial Fractions

$$\frac{x^2+x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$A(x^2+1) + (x+1)(Bx+C) = x^2+x+1$$

$x = 0$, $A + C = 1$

$x = 1$, $2A + 2B + 2C = 3$

$x = -1$, $2A = 1$, $A = \frac{1}{2}$, $C = \frac{1}{2}$

$B = \frac{1}{2}$

$\therefore$ P.F:

$$\frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{x^2+1}$$

Question 9 (5 marks)

Consider the curve defined parametrically by

$$x = \arcsin(t)$$

$$y = \log_e(1+t) + \frac{1}{4}\log_e(1-t)$$

where $t \in [0, 1)$.

a. $\left(\frac{dy}{dt}\right)^2$ can be written in the form $\frac{1}{a(1+t)^2} + \frac{1}{b(1-t^2)} + \frac{1}{c(1-t)^2}$, where a, b and c are real numbers.

Show that $a = 1$, $b = -2$ and $c = 16$. 2 marks

$$\frac{dy}{dt} = \frac{1}{1+t} + \frac{1}{4} \times \frac{-1}{1-t}$$

$$= \frac{1}{(1+t)} - \frac{1}{4(1-t)}$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+t)^2} - 2 \times \frac{1}{(1+t)} \times \frac{1}{4(1-t)} + \left(\frac{1}{4(1-t)}\right)^2$$

$$= \frac{1}{(1+t)^2} - \frac{1}{2(1-t^2)} + \frac{1}{16(1-t)^2} \Rightarrow \therefore \begin{cases} a = 1 \\ b = -2 \\ c = 16 \end{cases}$$

b. Find the arc length, s, of the curve from $t = 0$ to $t = \frac{1}{2}$. Give your answer in the form $s = \log_e(m) + n\log_e(p)$, where $m, n, p \in Q$. 3 marks

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{1}{1-t^2}, \quad \text{Need } \int_0^{\frac{1}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+t)^2} + \frac{1}{2(1-t^2)} + \frac{1}{16(1-t)^2}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{1}{1+t} + \frac{1}{4(1-t)}\right)^2} = \frac{1}{t+1} + \frac{1}{4(1-t)}$$

$$\therefore \text{Arclength } S = \int_0^{\frac{1}{2}} \left(\frac{1}{t+1} + \frac{1}{4(1-t)}\right) dt$$

$$= \left[\log_e|t+1| - \frac{1}{4}\log_e|1-t|\right]_0^{\frac{1}{2}}$$

$$= \log_e\left(\frac{3}{2}\right) - \frac{1}{4}\log_e\left(\frac{1}{2}\right) - (0 - 0)$$

$$= \log_e\left(\frac{3}{2}\right) + \frac{1}{4}\log_e(2)$$

Many other possible acceptable answers

END OF QUESTION AND ANSWER BOOK