VCAA 2019 Specialist Mathematics
Examination 1 Provisional Solutions

TWM Publications

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Question 1 (4 marks)

$$\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}, \quad y(0) = \pi$$

$$\Rightarrow \int_{\pi}^{y} \frac{1}{t}\,dt = \int_{0}^{x} \frac{2e^{2t}}{1+e^{2t}}\,dt$$

$$\Rightarrow \Big[\log_e|t|\Big]_{\pi}^{y} = \Big[\log_e(1+e^{2t})\Big]_{0}^{x} \qquad (1+e^{2t} > 0)$$

$$\Rightarrow \log_e\left(\frac{y}{\pi}\right) = \log_e\left(\frac{1+e^{2x}}{2}\right) \qquad (y(0) = \pi > 0)$$

$$\Rightarrow y(x) = \frac{\pi}{2}(1+e^{2x}).$$

Question 2 (3 marks)

Here, we need to case-break $|x-4| = \frac{x}{2} + 7$

For $x \geq 4$, we have $x - 4 = \frac{x}{2} + 7 \Rightarrow x = 22$

For $x < 4$, we have $4 - x = \frac{x}{2} + 7 \Rightarrow x = -2$.

Hence, $x = -2, 22$.

Question 3a (1 mark)

Let H be the length of a piece. $E(H)=3$, $sd(H)=0.1$.

$V=\pi\left(\frac{1}{2}\right)^2 H=\frac{\pi}{4}H$, so

$$E(V)=\frac{\pi}{4}E(H)=\frac{3\pi}{4}\text{ cm}^3.$$

Question 3b (1 mark)

Then, $\operatorname{Var}(V)=\frac{\pi^2}{16}\operatorname{Var}(H)=\frac{\pi^2}{1600}\text{ cm}^6$.

Question 3c (1 mark)

Let $S=2\pi\left(\frac{1}{2}\right)^2+2\pi\left(\frac{1}{2}\right)H=\frac{\pi}{2}+\pi H$ denot surface area.

$$E(S)=\frac{\pi}{2}+\pi E(H)=\frac{7\pi}{2}\text{ cm}^2.$$

Question 4 (3 marks)

Here, $\underset{\sim}{r}_A(t)=(t^2-1)\underset{\sim}{i}+\left(a+\frac{t}{3}\right)\underset{\sim}{j}$, $\underset{\sim}{r}_B(t)=(t^3-t)\underset{\sim}{i}+\arccos\left(\frac{t}{2}\right)\underset{\sim}{j}$

Equating $\underset{\sim}{i}$ components gives

$$t^2-1=t(t^2-1)\Rightarrow t=1 \quad \text{(collide after moving)}$$

Thus, we have $a+\frac{1}{3}=\arccos\left(\frac{1}{2}\right)$

$$\Rightarrow a=\frac{\pi}{3}-\frac{1}{3}.$$

Question 5a.i (1 mark)

If $f(x)=\cos^2(x)+\cos(x)+1$, then

$$f'(x)=-2\cos(x)\sin(x)-\sin(x).$$

Question 5a.ii (2 marks)

Thus, $f'(x) = -2\cos(x)\sin(x) - \sin(x) = 0$

$\Rightarrow -\sin(x)\left(2\cos(x)+1\right) = 0$

$\Rightarrow \sin(x) = 0$ or $\cos(x) = \frac{-1}{2}$

$\Rightarrow x = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}. \quad (x \in (0, 2\pi))$

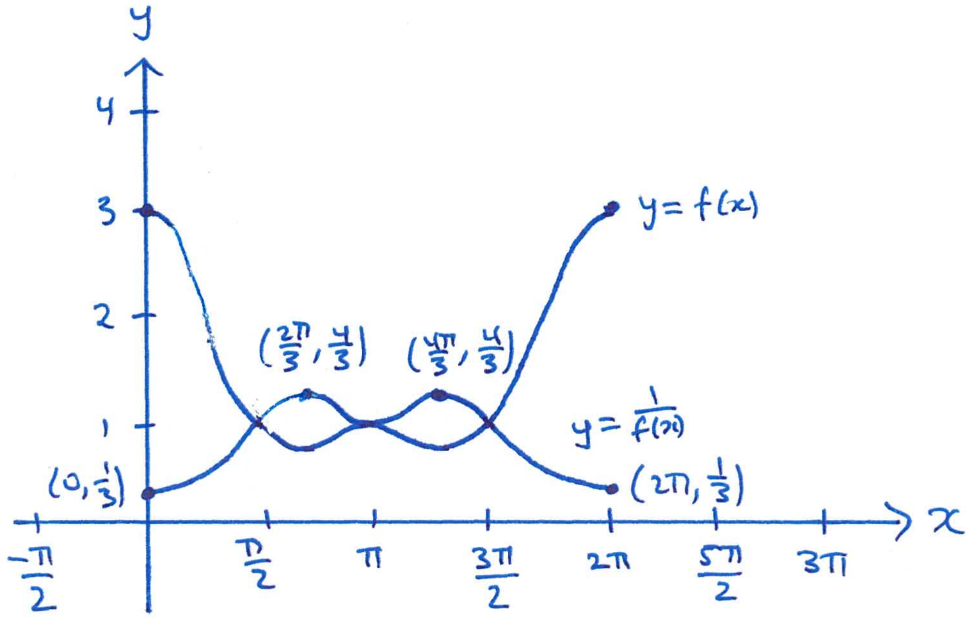
Then, $f\left(\frac{2\pi}{3}\right) = f\left(\frac{4\pi}{3}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$ and

$f(\pi) = 1 - 1 + 1 = 1.$

Thus, the turning points are at

$\left(\frac{2\pi}{3}, \frac{3}{4}\right)$, $(\pi, 1)$ and $\left(\frac{4\pi}{3}, \frac{3}{4}\right)$.

Question 5b (3 marks)



Question 6 (3 marks)

Since $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}$ are not parallel, we can form the linear system

$$\underset{\sim}{c} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + d\underset{\sim}{k} = \alpha\underset{\sim}{a} + \beta\underset{\sim}{b}, \quad \alpha, \beta \in \mathbb{R}\backslash\{0\}.$$

$$\Rightarrow \begin{cases} -6 = 2\alpha - 2\beta \text{ —— } (1) \\ 2 = -3\alpha + 4\beta \text{ —— } (2) \\ d = 4\alpha - 8\beta \text{ —— } (3) \end{cases}$$

Solving (1) and (2) gives

$$-12 + 2 = 4\alpha - 3\alpha + 4\beta - 4\beta$$

$$\Rightarrow \alpha = -10 \text{ and } \beta = -7$$

Thus, $d = -40 + 56 = 16$.

Question 7a (1 mark)

$$|3 - \sqrt{3}i| = \sqrt{3^2 + 3} = \sqrt{12} = 2\sqrt{3}.$$

Since $-\frac{\pi}{2} < \operatorname{Arg}(3 - \sqrt{3}i) < 0$,

$$\operatorname{Arg}(3 - \sqrt{3}i) = \arctan\left(\frac{-\sqrt{3}}{3}\right) = \frac{-\pi}{6}.$$

Thus, $3 - \sqrt{3}i = 2\sqrt{3}\operatorname{cis}\left(\frac{-\pi}{6}\right)$, as required.

Question 7b (2 marks)

$$(3 - \sqrt{3}i)^3 = (2\sqrt{3})^3 \operatorname{cis}\left(\frac{-\pi}{6} \times 3\right) \quad \text{(de Moivre's theorem)}$$

$$= 24\sqrt{3}\operatorname{cis}\left(\frac{-\pi}{2}\right)$$

$$= -24\sqrt{3}i.$$

Question 7c (1 mark)

$(3-\sqrt{3}i)^n = (2\sqrt{3})^n \operatorname{cis}\left(\frac{-n\pi}{6}\right), \quad n \in \mathbb{Z},$

So if $(3-\sqrt{3}i)^n \in \mathbb{R}$, then we require

$$\sin\left(\frac{-n\pi}{6}\right) = 0$$

$\Rightarrow \quad n = 6k, \quad k \in \mathbb{Z}.$

Question 7d (1 mark)

For $(3-\sqrt{3}i)^n \in \{ai \mid a \in \mathbb{R}\}$, we require

$$\cos\left(\frac{-\pi n}{6}\right) = 0$$

$\Rightarrow \quad \frac{\pi n}{6} = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$

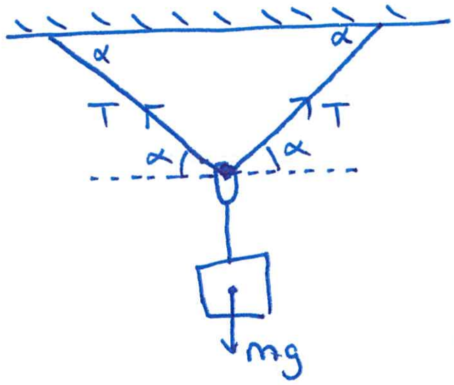
$\Rightarrow \quad n = 3 + 6k, \quad k \in \mathbb{Z}.$

Question 8 (4 marks)

Note that $y = \sqrt{\frac{1+2x}{1+x^2}} > 0 \ \forall x \in [0,1]$, so

$$\begin{aligned}
V &= \pi \int_0^1 \frac{1+2x}{1+x^2}\,dx \\
&= \pi \int_0^1 \frac{1}{1+x^2}\,dx + \pi \int_0^1 \frac{2x}{1+x^2}\,dx \\
&= \pi\left[\arctan(x)\right]_0^1 + \pi\left[\log_e(x^2+1)\right]_0^1 \qquad (x^2+1>0) \\
&= \pi\left(\frac{\pi}{4} - 0\right) + \pi\left(\log_e(2) - 0\right) \\
&= \frac{\pi^2}{4} + \pi\log_e(2) \text{ units}^3.
\end{aligned}$$

Question 9a (1 mark)

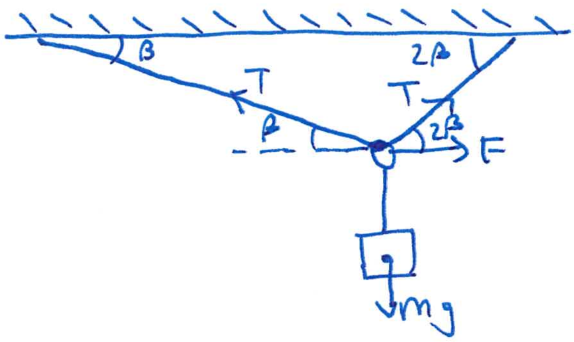


Resolving forces vertically gives

$2T\sin(\alpha) = mg$

$\Rightarrow T = \dfrac{mg}{2\sin(\alpha)}$.

Question 9b (3 marks)



Vertically, we have $mg = T\sin(\beta) + T\sin(2\beta)$

$$\Rightarrow T = \frac{mg}{\sin(\beta)+\sin(2\beta)} = \frac{mg}{\sin(\beta)(1+2\cos(\beta))}.$$

Horizontally, we have $T\cos(\beta) = T\cos(2\beta) + F$

$$\Rightarrow F = \frac{mg}{\sin(\beta)(1+2\cos(\beta))}\left(\cos(\beta)-\cos(2\beta)\right)$$

$$= \frac{mg}{\sin(\beta)(1+2\cos(\beta))}\left(\cos(\beta)-2\cos^2(\beta)+1\right)$$

$$= \frac{mg}{\sin(\beta)\cancel{(1+2\cos(\beta))}}(1-\cos(\beta))\cancel{(2\cos(\beta)+1)}$$

$$= mg\frac{1-\cos(\beta)}{\sin(\beta)},$$ as required.

Question 10 (5 marks)

$$\sin(x^2) + \cos(y^2) = \frac{3\sqrt{2}}{\pi} x y$$

$$\Rightarrow \quad 2x\cos(x^2) - 2y\sin(y^2)\frac{dy}{dx} = \frac{3\sqrt{2}}{\pi} y + \frac{3\sqrt{2}}{\pi} x \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{3\sqrt{2}}{\pi} x + 2y\sin(y^2)\right) = 2x\cos(x^2) - \frac{3\sqrt{2}}{\pi} y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x\cos(x^2) - \frac{3\sqrt{2}}{\pi} y}{\frac{3\sqrt{2}}{\pi} x + 2y\sin(y^2)} = \frac{2\pi x\cos(x^2) - 3\sqrt{2}y}{3\sqrt{2}x + 2\pi y\sin(y^2)}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}}\right)} = \frac{\frac{2\pi\sqrt{\pi}}{\sqrt{6}}\cos\left(\frac{\pi}{6}\right) - \frac{3\sqrt{2}\sqrt{\pi}}{\sqrt{3}}}{\frac{3\sqrt{2}\sqrt{\pi}}{\sqrt{6}} + \frac{2\pi\sqrt{\pi}}{\sqrt{3}}\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{\frac{2\pi\sqrt{3}}{2} - 6}{3\sqrt{2} + \frac{2\sqrt{2}\pi}{2}}$$

$$= \frac{\pi - 2\sqrt{3}}{\sqrt{2}(\pi + \sqrt{3})}.$$