TWM Publications

Victorian Certificate of Education – Free Trial Examinations

STUDENT NUMBER	S	O	L	U	T	I	O	N	Letter S

SPECIALIST MATHEMATICS

Free Trial Written Examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total: 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Version 2 (February 2019)

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

The number of asymptotes the graph of the function $f(x) = \frac{x}{x^2 - x - 2}$ has is

A. 0

B. 1

C. 2

D. 3

E. 4

$f(x) = \frac{x}{(x-2)(x+1)}$

Asymptotes: $y=0$, $x=2$, $x=-1$ ⇒ there are $\underline{3}$.

Question 2

The solutions to the inequation $\cos(x) > \frac{1}{2}\cot(x)$, where $x \in (0, 2\pi) \setminus \{\pi\}$ are given by

A. $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

B. $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

C. $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

D. $x \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

E. $x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

$\cos(x) = \frac{1}{2}\cot(x)$ where $x \in (0, 2\pi) \setminus \{\pi\}$ gives $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$. Then using a graph it is clear we have

$x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$.

Question 3

The implied domain of the function $f(x) = \arcsin\left(\frac{a}{x}\right)$, where $a > 0$, is

A. $\mathbb{R}$

B. $(-\infty, -a] \cup [a, \infty)$

C. $\left[\frac{-1}{a}, \frac{1}{a}\right] \setminus \{0\}$

D. $[-a, a] \setminus \{0\}$

E. $\mathbb{R} \setminus \{0\}$

$-1 \le \frac{a}{x} \le 1 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$.

Question 4

A polynomial with real coefficients, defined for a complex variable, has $3-i$, $3+i$, 2 and $4+3i$ amongst its roots.

The minimum degree of the polynomial is

A. 4
B. 5
C. 6
D. 7
E. 8

The polynomial has the roots

$z = 3 \pm i, 2, 4 \pm 3i$ at the very least by the conjugate root theorem. ie. 5.

Question 5

In the complex plane, which one of the graphs of the following relations passes through the point $1-\sqrt{3}i$?

A. $\left|z-\left(1-\sqrt{3}i\right)\right| = 4$
B. $z+\overline{z} = 1$
C. $\text{Re}(z+1)+\sqrt{3}\,\text{Im}(z+1) = 1$
D. $|z-4| = \left|z-\left(-2-2\sqrt{3}i\right)\right|$
E. $\left|z-\left(1-\sqrt{3}i\right)\right| = \left|z-\left(-1+\sqrt{3}i\right)\right|$

Substitute $z = 1-\sqrt{3}i$ into option D.

$|z-4| = 2\sqrt{3}$

$|z-(-2-2\sqrt{3}i)| = 2\sqrt{3}$, so

graph of option D passes through $1-\sqrt{3}i$.

Question 6

The number of solutions to the equation $z^n = \overline{z}$ for z, where n is an integer greater than 1 and $z \in \mathbb{C}$, is

A. 0
B. $n-1$
C. n
D. $n+1$
E. $n+2$

$z=0$ is a solution.

For $z \neq 0$, $z^n = \overline{z} \Rightarrow \dfrac{z^{n+1}}{z} = \overline{z} \Rightarrow z^{n+1} = |z|^2$,

which has $n+1$ solutions. All together, there are $n+2$ solutions for z.

Question 7

Given that $\dfrac{dy}{dx} = \dfrac{\arccos(y)}{\log_e(x)}$, the value of $\dfrac{d^2y}{dx^2}$ at the point $\left(e, \dfrac{\sqrt{3}}{2}\right)$ is

A. $-\left(2+\dfrac{\pi}{6e}\right)$
B. $\dfrac{\pi}{6}\left(2-\dfrac{1}{e}\right)$
C. $\dfrac{\pi}{3}\left(2-\dfrac{1}{e}\right)$
D. $\dfrac{-\pi}{6}\left(2+\dfrac{1}{e}\right)$
E. $\dfrac{-\pi}{3}\left(1+\dfrac{1}{e}\right)$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dy}[\arccos(y)]\frac{dy}{dx}\log_e(x) - \frac{d}{dx}[\log_e(x)]\arccos(y)}{\log_e^2(x)}$$

$$= \frac{-\arccos(y)\left(x+\sqrt{1-y^2}\right)}{x\log_e^2(x)\sqrt{1-y^2}}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(e,\sqrt{3}/2)} = \frac{-\pi}{3} - \frac{\pi}{6e} = \frac{-\pi}{6}\left(2+\frac{1}{e}\right)$$

Question 8

With a suitable substitution, $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^5(x)\,dx$ can be expressed as

A. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(u^4-2u^2+1\right)du$

B. $-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(u^4-2u^2+1\right)du$

C. $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left(u^4-2u^2+1\right)du$

D. $-\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left(u^4-2u^2+1\right)du$

E. $\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \left(u^4-2u^2+1\right)du$

$$= \int_{\pi/4}^{\pi/3} \sin(x)\left(1-\cos^2(x)\right)^2 dx$$

$$= -\int_{1/\sqrt{2}}^{1/2} \left(1-u^2\right)^2 du \quad \left[\begin{array}{ll} u=\cos(x) & u(\pi/4)=1/\sqrt{2} \\ -\frac{du}{dx}=\sin(x), & u(\pi/3)=1/2 \end{array}\right]$$

$$= \int_{1/2}^{1/\sqrt{2}} \left(u^4-2u^2+1\right) du$$

Question 9

A solution to the differential equation $\frac{dy}{dx} = \sin(x+y)+\cos(x-y)$ can be obtained from

A. $\int [\sin(y)+\cos(y)]\,dy = \int [\sin(x)+\cos(x)]\,dx$

B. $\int \frac{1}{\sin(y)+\cos(y)}\,dy = \int [\sin(x)+\cos(x)]\,dx$

C. $\int \frac{1}{\cos(y)-\sin(y)}\,dy = \int [\cos(x)-\sin(x)]\,dx$

D. $\int \sec(y)\,dy = \int 2\sin(x)\,dx$

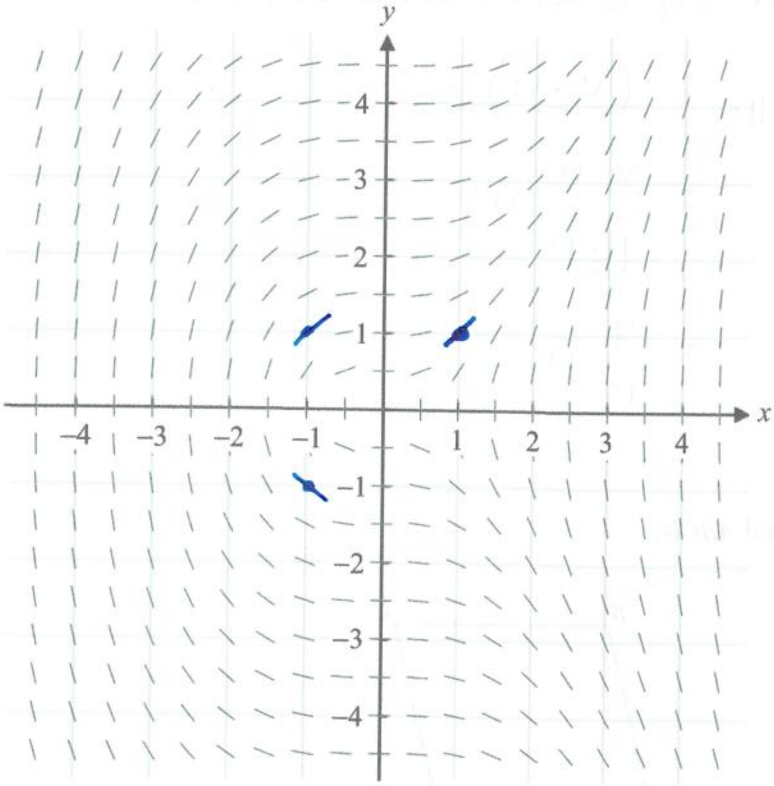
E. $\int \sec(y)\,dy = \int 2\cos(x)\,dx$

$$\frac{dy}{dx} = \sin(x)\cos(y) + \cos(x)\sin(y) + \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$= \left(\sin(x)+\cos(x)\right)\left(\sin(y)+\cos(y)\right)$$

$$\Rightarrow \int \frac{1}{\sin(y)+\cos(y)}\,dy = \int \left(\sin(x)+\cos(x)\right) dx$$

Question 10



The differential equation which best describes the direction field is

A. $\frac{dy}{dx} = \frac{x^2}{y}$

~~**B.**~~ $\frac{dy}{dx} = -\frac{x^2}{y}$

~~**C.**~~ $\frac{dy}{dx} = \frac{x}{y^2}$

~~**D.**~~ $\frac{dy}{dx} = -\frac{x}{y^2}$

~~**E.**~~ $\frac{dy}{dx} = \frac{y}{x^2}$

$\left.\frac{dy}{dx}\right|_{(1,1)} > 0$ eliminates options B and D

$\left.\frac{dy}{dx}\right|_{(0,y)} = 0$ eliminates option E.

$\left.\frac{dy}{dx}\right|_{(-1,1)} > 0$ eliminates option C.

Question 11

Consider the differential equation $\frac{dy}{dx} = \frac{1}{\arctan(x)}$, where $y(2) = y_0 = 1$.

Using Euler's method with a step size of 0.1, $y(1.7) = y_3$, correct to four decimal places is

A. 0.7236, and this is an overestimate of $y(1.7)$.

B. 0.7236, and this is an underestimate of $y(1.7)$.

C. 1.2665, and this is an overestimate of $y(1.7)$.

D. 1.2665, and this is an underestimate of $y(1.7)$.

E. 1.2823, and this is an overestimate of $y(1.7)$.

$y(1.9) \approx y_1 = 1 - 0.1 \times \frac{1}{\arctan(2)} = 0.909678...$

$y(1.8) \approx y_2 = 0.909678... - 0.1 \times \frac{1}{\arctan(1.9)}$
$= 0.817624...$

$y(1.7) \approx y_3 = 0.817624... - 0.1 \times \frac{1}{\arctan(1.8)}$
$= 0.723612...$

Note that $\frac{d^2y}{dx^2} = \frac{-1}{(x^2+1)\arctan^2(x)} < 0$

$\Rightarrow$ solution curve is concave down $\Rightarrow y_3 = 0.7236$ overestimates

Question 12

If $\underset{\sim}{u} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, then the vector resolute of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is

A. $\frac{7}{12}\underset{\sim}{u}$

B. $\frac{7}{6}\underset{\sim}{u}$

C. $\frac{14}{9}\underset{\sim}{v}$

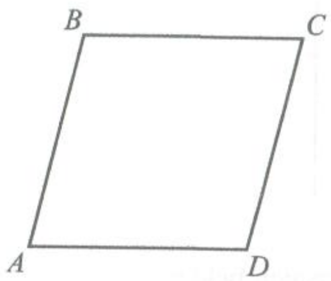
D. $\frac{7}{3}\underset{\sim}{u}$

E. $\frac{28}{9}\underset{\sim}{v}$

$$\underset{\sim}{v}_{\parallel \underset{\sim}{u}} = (\underset{\sim}{v} \cdot \hat{\underset{\sim}{u}})\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{|\underset{\sim}{u}|^2}\underset{\sim}{u} = \frac{7}{12}\underset{\sim}{u}$$

Question 13

Consider a quadrilateral $ABCD$.



To prove that $ABCD$ is a rhombus, it would be sufficient to show that

A. ~~A.~~ $\overrightarrow{AB} \parallel \overrightarrow{DC}$ and $\overrightarrow{AD} \parallel \overrightarrow{BC}$

B. ~~B.~~ $\overrightarrow{AB} + \overrightarrow{CD} = \underset{\sim}{0}$ and $\overrightarrow{BC} + \overrightarrow{DA} = \underset{\sim}{0}$

C. $\overrightarrow{AB} = \overrightarrow{DC}$ and $|\overrightarrow{AB}| = |\overrightarrow{BC}|$

D. ~~D.~~ $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

E. ~~E.~~ $\overrightarrow{AB} \parallel \overrightarrow{DC}$ and $|\overrightarrow{AB}| = |\overrightarrow{BC}|$

- Options A and B are sufficient for a parallelogram, not a rhombus.
- Option D could describe a right-angled kite.
- Option E is sufficient for a trapezium, not a rhombus.

Question 14

The vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 3\underset{\sim}{j} + \alpha\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$, where $\alpha \in \mathbb{R}$, are linearly independent if

A. $\alpha = 1$

B. $\alpha = 11$

C. $\alpha \in \mathbb{R} \setminus \{1\}$

D. $\alpha \in \mathbb{R} \setminus \{11\}$

E. $\alpha \in \mathbb{R}$

Since $\underset{\sim}{b}, \underset{\sim}{c}$ are not parallel, linear dependence for $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ occurs when α is such that

$$\underset{\sim}{a} = k_1\underset{\sim}{b} + k_2\underset{\sim}{c} \Rightarrow \begin{cases} 3 = k_1 + 2k_2 \\ -3 = k_1 - k_2 \\ \alpha = -3k_1 + 4k_2 \end{cases}$$

$\Rightarrow \alpha = 11,\ k_1 = -1,\ k_2 = 2.$

$\Rightarrow \underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are linearly independent for $\alpha \in \mathbb{R} \setminus \{11\}$

Question 15

A 20 kg object has a velocity of $-3\underset{\sim}{i}+4\underset{\sim}{j}$ ms^{-1} at a particular time. The object moves with constant acceleration, and after four seconds, its velocity is $5\underset{\sim}{i}-2\underset{\sim}{j}$ ms^{-1}.

The magnitude of the net force, in newtons, that acts on the body during the period of motion is

A. $\frac{\sqrt{10}}{2}$

B. $\frac{5}{2}$

C. $10\sqrt{10}$

D. 50

E. 200

$$\underset{\sim}{a} = \frac{5\underset{\sim}{i}-2\underset{\sim}{j} - (-3\underset{\sim}{i}+4\underset{\sim}{j})}{4} = 2\underset{\sim}{i} - \frac{3}{2}\underset{\sim}{j} \text{ ms}^{-2}.$$

$$\Rightarrow \underset{\sim}{F} = 40\underset{\sim}{i} - 30\underset{\sim}{j} \text{ N}$$

$$\Rightarrow |\underset{\sim}{F}| = 50 \text{ N}$$

Question 16

An object of mass 10 kg is initially at rest on a rough plane inclined at 30° to the horizontal. The object is pulled along the surface up the plane by a force of 144 N, acting parallel to the inclined plane. A friction force of 80 N opposes the motion.

After the pulling force has acted for 10 s, the magnitude of the momentum, in kg ms^{-1}, of the object is

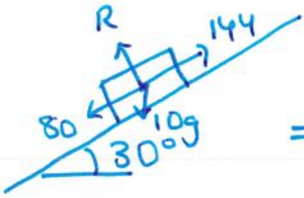
A. 15

B. 80

C. 150

D. 640

E. 950



$$F = 144 - 80 - 10g\sin(30°) = 64 - 5g \text{ N}$$

$$\Rightarrow p(10) = \int_0^{10} (64-5g)\,dt = 150 \text{ kg ms}^{-1}.$$

Question 17

A particle of mass m kg moves in a straight line due to a resultant force F newtons, where $F = f(x)$.

Given that the particle's velocity is p ms^{-1} where its position is q m from a fixed point, where $p, q > 0$, an expression for the particle's velocity, v ms^{-1}, in terms of its position, x m, is given by

A. $v = \sqrt{\frac{2}{m}\int_q^x \sqrt{f(z)}\,dz + p}$

B. $v = \sqrt{\frac{2}{m}\int_p^x f(z)\,dz + q}$

C. $v = \sqrt{\frac{2}{m}\int_q^x f(z)\,dz + p}$

D. $v = \sqrt{\frac{2}{m}\int_q^x f(z)\,dz + p^2}$

E. $v = \sqrt{\frac{2}{m}\int_q^x \left(f(z)+p^2\right)dz}$

$$mv\frac{dv}{dx} = f(x), \quad v(q) = p$$

$$\Rightarrow m\int_p^v z\,dz = \int_q^x f(z)\,dz$$

$$\Rightarrow \frac{m}{2}(v^2 - p^2) = \int_q^x f(z)\,dz$$

$$\Rightarrow v^2 = \frac{2}{m}\int_q^x f(z)\,dz + p^2$$

$$\Rightarrow v = \sqrt{\frac{2}{m}\int_q^x f(z)\,dz + p^2} \quad (\text{since } v(q) = p > 0)$$

Question 18

The heights of males in Australia vary normally with a mean of μ cm and a standard deviation of $\sigma = 8$ cm. The mean height of 100 randomly selected Australian males is found to be 176 cm.

Based on this sample, a 99% confidence interval for μ, correct to one decimal place, is

A. $(175.8, 176.2)$
B. $(160.3, 191.7)$
C. $(174.4, 177.6)$
D. $(155.4, 196.6)$
E. $(173.9, 178.1)$

$CI = \left(176 - 2.5758\ldots\frac{8}{\sqrt{100}},\ 176 + 2.5758\ldots\frac{8}{\sqrt{100}}\right)$

$= (173.939\ldots, 178.061\ldots)$

Question 19

Study scores given to students in a VCE subject with a large cohort vary normally with a mean of 30 and a standard deviation of 7. There are 16 VCE Specialist Mathematics students in a particular class.

The probability that the average study score of this class exceeds 32 is closest to

A. 0.0163
B. 0.1265
C. 0.2498
D. 0.3854
E. 0.3875

$S \sim N(30, 7^2) \Rightarrow \bar{S} \mathrel{\dot\sim} N\left(30, \left(\frac{7}{4}\right)^2\right)$

$\Rightarrow \Pr(\bar{S} \geq 32) = 0.126549\ldots$

Question 20

Consider the following statistical test.

$H_0 : \mu = 20$
$H_1 : \mu \neq 20$

A sample of size 36 is found to have a mean of $\bar{x} = 19$ and a standard deviation of $s = 3$.

Assuming that s is a sufficiently accurate estimate of the population standard deviation σ, then the p value of the test, in terms of the standard normal variable, Z, is given by

A. $\Pr(Z < -2)$
B. $\Pr\left(Z < -\frac{1}{3}\right)$
C. $\Pr(|Z| < 2)$
D. $2\Pr\left(Z < -\frac{1}{3}\right)$
E. $\Pr(|Z| > 2)$

$\bar{X} \mathrel{\dot\sim} N\left(\mu, \left(\frac{1}{2}\right)^2\right)$

$\Rightarrow p = 2 \times \Pr(\bar{X} < 19 \mid \mu = 20)$

$= 2\Pr\left(Z < \frac{19-20}{1/2}\right) \quad (Z \sim N(0,1))$

$= 2\Pr(Z < -2)$

$= \Pr(|Z| > 2)$ (by symmetry of Z)

END OF SECTION A

CONTINUES OVER PAGE

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (9 marks)

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{12(2x+1)}{x^2+12}$.

a. **i.** State the equations of any asymptotes of the graph of f. 1 mark

$y = 0.$

ii. Find the coordinates of any stationary points of the graph of f. 1 mark

$f'(x) = \dfrac{-24(x^2-x-12)}{(x^2+12)^2} = 0$

$\Rightarrow x = -4,\ 3$

$\Rightarrow$ Stationary points: $(-4,-3)$, $(3,4)$

iii. Find the coordinates of any points of inflection of the graph of f, correct to one decimal place. 2 marks

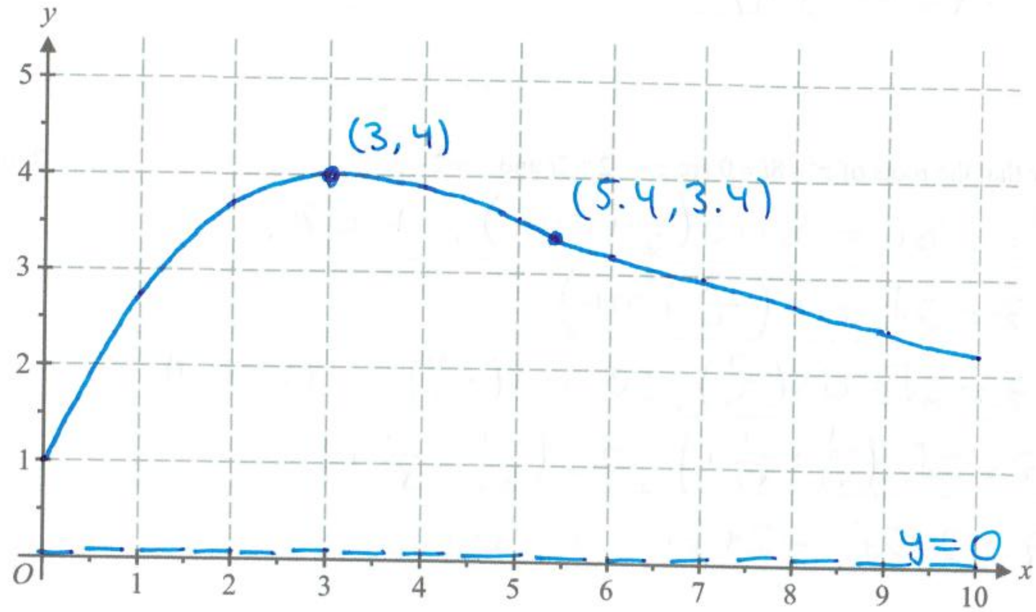
$f''(x) = \dfrac{24(2x^3+3x^2-72x-12)}{(x^2+12)^3} = 0$

$\Rightarrow x = -6.722\ldots,\ -0.165\ldots,\ 5.388\ldots,$

and these are all points of inflection.

Points of inflection: $(-6.7,-2.6)$, $(-0.2, 0.7)$, $(5.4, 3.4)$.

b. Sketch the graph of f on the axes provided below from $x = 0$ to $x = 10$, marking stationary points, points of inflection and intercepts with the axes, labelling them with their coordinates. Show any asymptotes and label them with their equations. 3 marks



c. The part of the graph of f from $x = 0$ to $x = 10$ is rotated about the x-axis to form a volume of revolution that is to model part of a wine glass, where units are in centimetres.

i. Write down a definite integral expression that when evaluated, gives the total volume of the wine glass. 1 mark

$$V = \pi \int_0^{10} \left(\frac{12(2x+1)}{x^2+12} \right)^2 dx$$

ii. Hence, find the total volume of the wine glass, in cm^3, correct to one decimal place. 1 mark

$V = 318.1 \text{ cm}^3.$

Question 2 (11 marks)

a. Express $u = 2 + 2i$ in polar form. 1 mark

$u = 2\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right)$

b. **i.** Show that the roots of $z^2 + 8i = 0$ are $z = -2 + 2i$ and $z = 2 - 2i$. 2 marks

$z^2 = -8i = 8\,\mathrm{cis}\left(\frac{-\pi}{2} + 2\pi k\right),\ k \in \mathbb{Z}.$

$\Rightarrow z = 2\sqrt{2}\,\mathrm{cis}\left(\frac{-\pi}{4} + \pi k\right)$

$\Rightarrow z = 2\sqrt{2}\,\mathrm{cis}\left(\frac{-\pi}{4}\right),\ 2\sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right) \quad (k = 0, 1)$

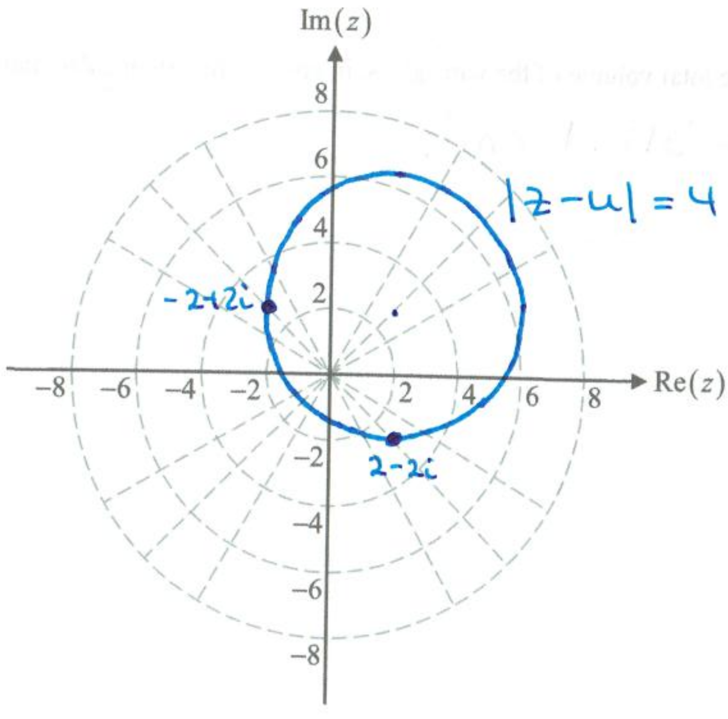
$\Rightarrow z = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right),\ 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$\Rightarrow z = 2 - 2i,\ -2 + 2i$, as required.

ii. Express the roots of $z^2 + 8i = 0$ in terms of u. 1 mark

$z = \overline{u},\ -\overline{u}$

c. Sketch the circle given by $|z - u| = 4$ and plot the roots of $z^2 + 8i = 0$ on the Argand diagram provided below. 2 marks



d. The equation of the line passing through the roots of $z^2 + 8i = 0$ can be expressed in the form $z = a\overline{z}$, where $a \in \mathbb{C}$.

Find a. 1 mark

The line corresponds to $y = -x$.

Let $z = x + yi$.

$x - xi = a(x + xi)$

$\Rightarrow a = \dfrac{x - xi}{x + xi} = -i$

e. The line passing through the roots of $z^2 + 8i = 0$ cuts the circle given by $|z - u| = 4$ into two segments.

Find the area of the major segment. 2 marks

$A = \frac{3}{4}\pi \times 4^2 + \frac{1}{2} \times 4 \times 4 \times \sin\left(\frac{\pi}{2}\right)$

$= 12\pi + 8$ units^2

f. State the range of values of α, where $-\pi < \alpha \le \pi$, for which the ray given by $\text{Arg}(z + u) = \alpha$ does **not** intersect or touch the circle given by $|z - u| = 4$. 2 marks

The ray is tangent to the circle if $\alpha = 0$ or $\alpha = \frac{\pi}{2}$. Thus, it does not intersect nor touch for $\alpha \in (-\pi, 0) \cup (\frac{\pi}{2}, \pi]$.

Question 3 (10 marks)

A large tank initially contains 100 L of pure water. A solution of salt and water with a salt concentration of 0.05 kg/L flows into the tank at 4 L/min. The mixture, which is kept uniform with stirring, flows out of the tank at 2 L/min.

a. Let x kg be the amount of salt in the tank at time t minutes.

i. Write down an expression for the concentration of salt in the tank, in kg/L, in terms of x and t. 1 mark

$$C = \frac{x}{100+(4-2)t} = \frac{x}{100+2t}$$

ii. Show that the differential equation relating x and t is given by $\frac{dx}{dt} + \frac{x}{t+50} = \frac{1}{5}$. 2 marks

$$\frac{dx}{dt} = 0.05 \times 4 - 2 \times \frac{x}{100+2t}$$

$$= 0.2 - \frac{x}{t+50}$$

$$\Rightarrow \frac{dx}{dt} - \frac{x}{t+50} = \frac{1}{5}, \text{ as required.}$$

b. It can be shown that $x(t) = \frac{t(t+100)}{10(t+50)}$.

Verify by substitution that the given solution for x satisfies both the differential equation given in **part a.ii.** and the initial condition. 3 marks

$$x(t) = 5 + \frac{t}{10} - \frac{250}{t+50} \Rightarrow x'(t) = \frac{1}{10} + \frac{250}{(t+50)^2}$$

$$\Rightarrow \frac{dx}{dt} + \frac{x}{t+50} = \frac{1}{10} + \frac{250}{(t+50)^2} + \frac{t^2+100t}{10(t+50)^2}$$

$$= \frac{1}{10} + \frac{(t+50)^2}{10(t+50)^2}$$

$$= \frac{1}{10} + \frac{1}{10}$$

$$= \frac{1}{5}.$$

Also, $x(0) = \frac{0 \times 100}{10 \times 50} = 0$ kg.

Hence, the given solution for $x(t)$ satisfies the ODE and the initial condition, as required.

c. Find the time it takes, in minutes, for the **concentration** of salt in the tank to reach 0.02 kg/L. 2 marks

$C(t) = \frac{t(t+100)}{20(t+50)^2} = 0.02 \text{ kg/L}$

$\Rightarrow t = \frac{50}{3}(\sqrt{15} - 3)$ min.

d. Find the total amount of salt that **flows out of the tank** in the first 20 minutes. 2 marks

$x_{OUT} = \int_0^{20} \frac{x(t)}{t+50}\, dt$

$= 4/7$ kg.

Alternatively,

$x_{OUT} = x_{IN} - x(20)$

$= 20 \times \frac{1}{5} - x(20) = 4/7$ kg.

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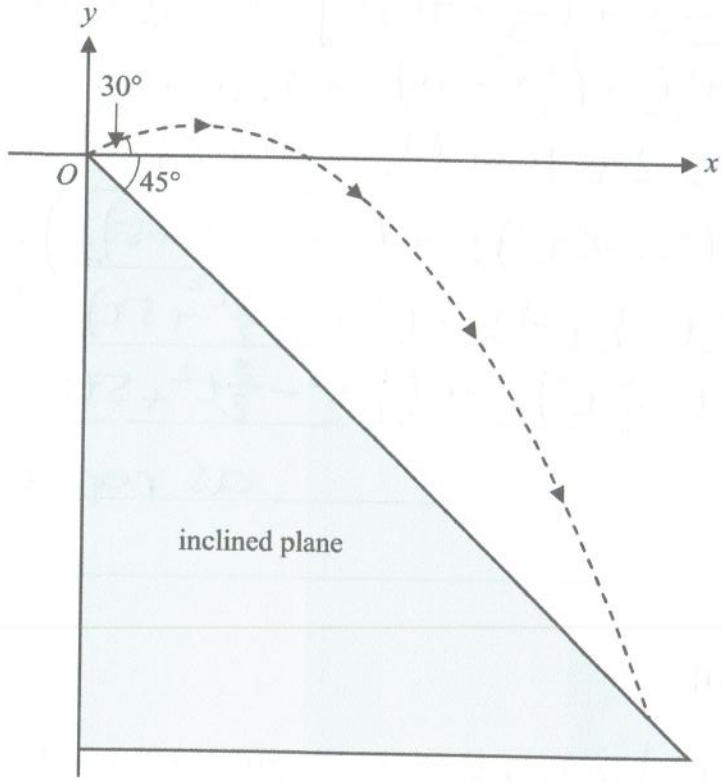
Question 4 (11 marks)

The diagram below shows an inclined plane that makes an angle of 45° with the horizontal. A projectile is fired at time $t = 0$ seconds from the top of the inclined with an initial speed of 10 ms^{-1} at an angle of 30° to the horizontal. The projectile travels in the air for T seconds before landing on the inclined plane.

Let the origin, O, of a cartesian coordinate system be the point where the projectile is fired.

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the positive x and y directions respectively.

All displacements are measured in metres and time is in seconds.



a. Show that the initial velocity of the projectile is $5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}\ \text{ms}^{-1}$. 1 mark

$\underset{\sim}{r}'(0) = 10\left(\cos(30°)\underset{\sim}{i} + \sin(30°)\underset{\sim}{j}\right)$

$= 10\left(\frac{\sqrt{3}}{2}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}\right)$

$= 5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}\ \text{ms}^{-1}$, as required.

b. The acceleration vector of the projectile at time t seconds is given by $\ddot{\underset{\sim}{r}}(t) = -kt\underset{\sim}{i} + (kt - g)\underset{\sim}{j}$, where $k > 0$ and $0 \le t \le T$.

Show that the position vector of the projectile at time t seconds is given by

$$\underset{\sim}{r}(t) = \left(5\sqrt{3}t - \frac{k}{6}t^3\right)\underset{\sim}{i} + \left(\frac{k}{6}t^3 - \frac{g}{2}t^2 + 5t\right)\underset{\sim}{j}.$$

3 marks

$$\underset{\sim}{r}'(t) = \int_0^t \left(-k\tau\underset{\sim}{i} + (k\tau - g)\underset{\sim}{j}\right) d\tau + 5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}$$
$$= \left[\frac{-k\tau^2}{2}\underset{\sim}{i} + \left(\frac{k\tau^2}{2} - g\tau\right)\underset{\sim}{j}\right]_0^t + 5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}$$
$$= -\frac{k}{2}t^2\underset{\sim}{i} + \left(\frac{k}{2}t^2 - gt\right)\underset{\sim}{j} + 5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}$$
$$= \left(5\sqrt{3} - \frac{k}{2}t^2\right)\underset{\sim}{i} + \left(\frac{k}{2}t^2 - gt + 5\right)\underset{\sim}{j}$$
$$\Rightarrow \underset{\sim}{r}(t) = \int_0^t \left(\left(5\sqrt{3} - \frac{k}{2}\tau^2\right)\underset{\sim}{i} + \left(\frac{k}{2}\tau^2 - g\tau + 5\right)\underset{\sim}{j}\right) d\tau$$
$$= \left[\left(5\sqrt{3}\tau - \frac{k}{6}\tau^3\right)\underset{\sim}{i} + \left(\frac{k}{6}\tau^3 - \frac{g\tau^2}{2} + 5\tau\right)\underset{\sim}{j}\right]_0^t$$
$$= \left(5\sqrt{3}t - \frac{k}{6}t^3\right)\underset{\sim}{i} + \left(\frac{k}{6}t^3 - \frac{g}{2}t^2 + 5t\right)\underset{\sim}{j},$$

as required.

c. Show that $T = \frac{10}{g}\left(\sqrt{3} + 1\right)$.

2 marks

The projectile lands on the slope $y = -x$.

$$\Rightarrow \cancel{\frac{k}{6}t^3} - \frac{g}{2}t^2 + 5t = -5\sqrt{3}t + \cancel{\frac{k}{6}t^3}$$
$$\Rightarrow -\frac{g}{2}t^2 + (5 + 5\sqrt{3})t = 0$$

Now, $T > 0$, so

$$-\frac{g}{2}T + 5 + 5\sqrt{3} = 0$$
$$\Rightarrow -gT = -10(\sqrt{3} + 1)$$
$$\Rightarrow T = \frac{10}{g}(\sqrt{3} + 1), \text{ as required.}$$

The value of k varies day-to-day depending on the wind conditions.

d. Suppose that on a particular day, $k = 0.2$.

i. Find the distance from O to the point on the inclined plane where the projectile lands. Give your answer in metres, correct to two decimal places. 1 mark

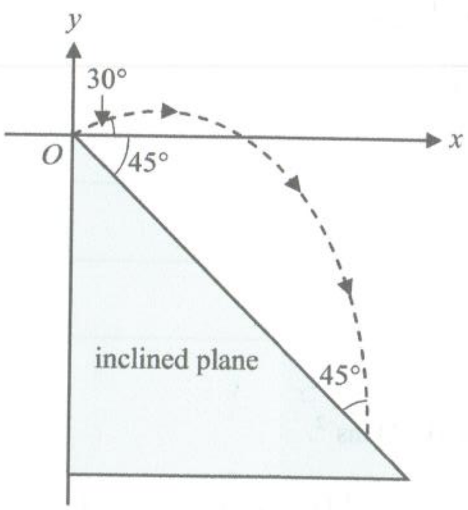
$|\underset{\sim}{r}(T)| = 33.12 \text{ m}$

ii. Find the speed at which the particle lands on the inclined plane. Give your answer in ms^{-1}, correct to two decimal places. 1 mark

$|\underset{\sim}{r}'(T)| = 22.94 \text{ ms}^{-1}$

The projectile is fired under the same launch conditions on another day where the value of k has changed.

e. After travelling in the air for T seconds, the projectile lands at an angle of 45° to the inclined plane, as shown below.



Find the value of k on this particular day, correct to two decimal places. 3 marks

$\underset{\sim}{r}'(T) = \left(5\sqrt{3} - \frac{100k}{g^2}(\sqrt{3}+2)\right)\underset{\sim}{i} + \left(\frac{100k}{g^2}(\sqrt{3}+2) - 10\sqrt{3} - 5\right)\underset{\sim}{j}$

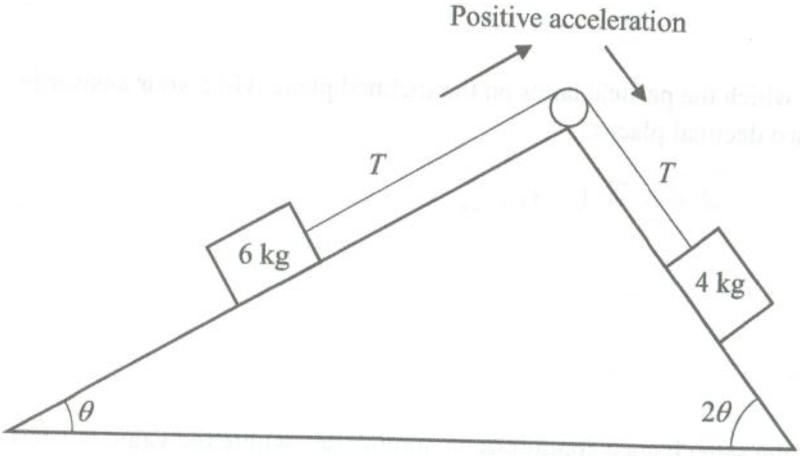
Here, the projectile lands with 0 horizontal velocity.

$\Rightarrow 5\sqrt{3} - \frac{100k}{g^2}(\sqrt{3}+2) = 0$

$\Rightarrow k = 2.22 \ (\text{ms}^{-3})$

Question 5 (10 marks)

The diagram below shows a block of mass 6 kg on a frictionless plane inclined at θ to the horizontal. This block is connected by a light, inextensible rope via a frictionless pulley at the top of the inclined plane to another block of mass 4 kg, placed on another frictionless surface inclined at 2θ to the horizontal. Take acceleration, a ms^{-2}, to be positive in the direction as indicated and assume that $0° < \theta < 45°$. The 6 kg block is released from rest and allowed to accelerate up the plane.



a. **i.** Write an equation of motion, in the direction of motion, for each block. 2 marks

$6a = T - 6g\sin(\theta)$ —— (1)

$4a = 4g\sin(2\theta) - T$ —— (2)

ii. Hence, show that $a = \dfrac{g\sin(\theta)}{5}(4\cos(\theta) - 3)$ ms^{-2}. 1 mark

(1)+(2) gives $10a = 4g\sin(2\theta) - 6g\sin(\theta)$

$\Rightarrow 5a = g(4\sin(\theta)\cos(\theta) - 3\sin(\theta))$

$\Rightarrow a = \dfrac{g\sin(\theta)}{5}(4\cos(\theta) - 3)$, as required.

b. Find the value of θ, in degrees, for which the system is in equilibrium, correct to one decimal place. 1 mark

$a = 0 \Rightarrow \theta = \arccos\left(\frac{3}{4}\right) = 41.4°.$

c. Consider the situation where θ is fixed at 30°.

i. Find the tension in the rope, in newtons, expressing your answer in the form $b\left(\sqrt{c}+d\right)$, where $b,c,d \in \mathbb{Q}$. 1 mark

$T = 6a + 6g\sin(\theta)$

$= \frac{49}{25}\left(\sqrt{3} + \frac{27}{2}\right)$ N (many forms possible)

ii. Find the magnitude of the momentum of the 6 kg block, in kg ms^{-1}, after it has moved $2\sqrt{3}+3$ metres up the plane. Express your answer in the form $\frac{p\sqrt{q}}{r}$, where $p,q,r \in \mathbb{N}$. 2 marks

$v\frac{dv}{dx} = \frac{g\sin(30°)}{5}(4\cos(30°) - 3) = \frac{g}{10}(2\sqrt{3}-3)$

$\Rightarrow \int_0^{v(6)} z\,dz = \int_0^{2\sqrt{3}+3} \frac{g}{10}(2\sqrt{3}-3)\,dx$

$\Rightarrow v(6) = \frac{7\sqrt{3}}{5}$ ms^{-1} $(v>0)$

$\Rightarrow p(6) = \frac{42\sqrt{3}}{5}$ kg ms^{-1}.

The 6 kg block and the 4 kg block are replaced by blocks of mass m kg and n kg respectively.

d. If the system accelerates in the positive direction of motion **for all** $0° < \theta < 45°$, it can be shown that $n \geq km$, where $k > 0$.

Find the smallest possible value of k. 2 marks

$ma = T - mg\sin(\theta)$ and $na = ng\sin(2\theta) - T$

gives $a = \frac{g\sin(\theta)}{m+n}(2n\cos(\theta) - m)$.

The smallest value of k arises when $a \to 0$ as $\theta \to 45°^-$, and so

$2n\cos(45°) - m \geq 0 \Rightarrow n \geq \frac{1}{\sqrt{2}}m.$

$\Rightarrow k_{min} = \frac{1}{\sqrt{2}}.$

Question 6 (9 marks)

A neurologist is investigating the effects of a new drug designed to improve reaction times. It is known that the reaction times of humans responding to visual stimuli vary normally with a mean of 280 ms and a standard deviation of 60 ms.

To test whether the drug results in lower reaction times, a one-tailed statistical test is carried out using a random sample of 100 patients. The mean reaction time of the patients in the sample is found to be 272 ms. Let $\bar{X}$ denote the mean reaction time of a random sample of 100 patients.

a. Write down suitable null and alternative hypotheses for this test. 1 mark

$H_0: \mu = 280$ ms.

$H_1: \mu < 280$ ms.

b. Find the p value of the test, correct to four decimal places, and hence, state **with a reason** whether the null hypothesis should be rejected at the 5% level of significance. 2 marks

$p = \Pr(\bar{X} < 272 \mid \mu = 280)$ $\quad X \sim N\left(\mu, \left(\frac{60}{\sqrt{100}}\right)^2\right)$

$= \Pr\left(Z < \frac{272-280}{6}\right)$

$= \Pr(Z < -4/3)$

$= 0.0912 \Rightarrow$ Do not reject H_0 as $p > 0.05$.

c. Find the largest value of the sample mean that could be observed for the null hypothesis to be rejected, correct to two decimal places. 1 mark

$\Pr(\bar{X} < c \mid \mu = 280) = 0.05$

$\Rightarrow c = 270.13$.

d. If the true mean reaction time of a patient who has taken the drug is in fact 265 ms, find the probability that the null hypothesis will be accepted at the 5% level of significance, correct to three decimal places. Explain whether this result indicates a type I error, a type II error, or no error. 2 marks

$\Pr(\text{Accept } H_0 \mid \mu = 265) = \Pr(\bar{X} > c \mid \mu = 265)$

$= \Pr(Z > 0.855146\ldots)$

$= 0.196$.

This represents $\Pr(\text{Accept } H_0 / H_0 \text{ is false})$, which is a type II error.

e. In future clinical trials of sufficiently large sample sizes, it was found that the reaction times of patients who took the drug **were in fact lower**, and varied with a mean of 265 ms and a standard deviation of 50 ms.

Find the probability that the reaction time of a randomly selected patient, who took the drug, is lower than the reaction time of a randomly selected patient who did not take the drug. Give your answer correct to three decimal places.

3 marks

$D \sim N(265, 50^2)$, $N \sim N(280, 60^2)$

Then, $E(D-N) = 265 - 280 = -15$, and

$Var(D-N) = 50^2 + 60^2 = 6100$

$\Rightarrow sd(D-N) = 10\sqrt{61}$.

Thus, $Pr(D < N) = Pr(D - N < 0)$

$= 0.576$.

END OF QUESTION AND ANSWER BOOK