SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1

2019 Trial Examination

SOLUTIONS

Question 1 (3 marks)

Answer:

$$\frac{d}{dx}(3x^2y + 2x - y^3) = 6xy + 3x^2.\frac{d}{dx}(y) + 2 - \frac{d}{dx}(y^3) = 0$$
$$6xy + 3x^2.\frac{d}{dx}(y) + 2 - \frac{d}{dx}(y^3) = 0$$
$$6xy + 3x^2.1\frac{dy}{dx} + 2 - 3y^2.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3y^2 - 3x^2) = 6xy + 2$$

$$\frac{dy}{dx}(27 - 12) = 36 + 2$$

$$\frac{dy}{dx} = \frac{38}{15}$$

Explanation:

Use implicit differentiation.

Question 2 (2 marks)

Answer:

$\underset{\sim}{a}$ *parallel to* $\underset{\sim}{b} = \left(\underset{\sim}{a}.\,\underset{\sim}{\hat{b}}\right).\,\underset{\sim}{\hat{b}}$

$\left(\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right).\left(\tfrac{1}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)\right).\quad\left(\tfrac{1}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)\right)\right)$

$= (-1)\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

$= -\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

Question 3 (4 marks)

a.

Answer:

$\underset{\sim}{M} + \underset{\sim}{N} = 5\underset{\sim}{i} + 12\underset{\sim}{k}$

$5\underset{\sim}{i} + 12\underset{\sim}{k} = 3.25\,\underset{\sim}{a}$

$3.25\,a = 13$

$a = 4\,ms^{-2}$

b.

Answer:

$a = 4. u = 0, v = 12$

$v = u + at$

$12 = 0 + 4t$

$t = 3\ sec$

c.

Answer:

$5\underset{\sim}{i} + 12\underset{\sim}{k} = 3.25\,\underset{\sim}{a}$

$\underset{\sim}{a} = \frac{4}{13}\left(5\underset{\sim}{i} + 12\underset{\sim}{k}\right)$

$\underset{\sim}{v} = \frac{4}{13}\left(5t\,\underset{\sim}{i} + 12t\,\underset{\sim}{k}\right)$

$\underset{\sim}{r} = \frac{4}{13}\left(\left(\frac{5t^2}{2} + 3\right)\underset{\sim}{i} + (6t^2 + 2)\underset{\sim}{k}\right)$

$\underset{\sim}{r}(2) = 4\underset{\sim}{i} + 8\underset{\sim}{k}$

Question 4 (3 marks)

a.

$E(A) = E(2X - Y) = 2E(X) - E(Y)$

$= 2 \times 10 - 25$

$= -5$

$Var(A) = Var(2X - Y) = 4Var(X) + Var(Y)$

$= 4 \times 4 + 9$

$= 25$

b.

$\Pr(A \geq 1.4)$

$z = \frac{1.4 - -5}{5} = \frac{6.4}{5} = 1.28$

$\Pr(z \geq 1.28) \approx 1 - 0.9 \approx 0.1$

Question 5 (4 marks)

a.

Answer:

$\Pr(z \leq -1.00) \approx 0.16 = 16\%$

b.

Answer:

$H_0: \mu = 20$ kg

Null hypothesis: The average weight of the cement bags in the sample is 20.0 kg.

$H_1: \mu < 20$ kg
Alternative hypothesis: The average weight of the cement bags in the sample is significantly less than 20.0 kg.

c.

Answer:

$$E(\bar{x}) = 20$$
$$sd(\bar{x}) = \frac{0.4}{\sqrt{25}} = 0.08$$

$\Pr(z \le -1.64) \approx 0.05$

$\frac{\bar{x}-20.0}{0.08} = -1.64$

$\bar{x} = 19.87$

So, since the mean weight of the sample (19.8) is less than 19.87, we reject the null hypothesis at the 0.05 level of significance.

Question 6 (3 marks)

Answer:

$\operatorname{cosec} 2x = 1.25$

$\sin 2x = \frac{4}{5}$

$\cos 2x = \frac{3}{5}$

$2\cos^2 x - 1 = \frac{3}{5}$

$\cos^2 x = \frac{4}{5}$

$\cos x = \frac{2}{\sqrt{5}}$

$\tan x = \frac{1}{2}$

$\tan(x + \frac{\pi}{4}) = \frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x . \tan\frac{\pi}{4}}$

$\tan(x + \frac{\pi}{4}) = \frac{1\frac{1}{2}}{\frac{1}{2}} = 3$

Question 7 (5 marks)

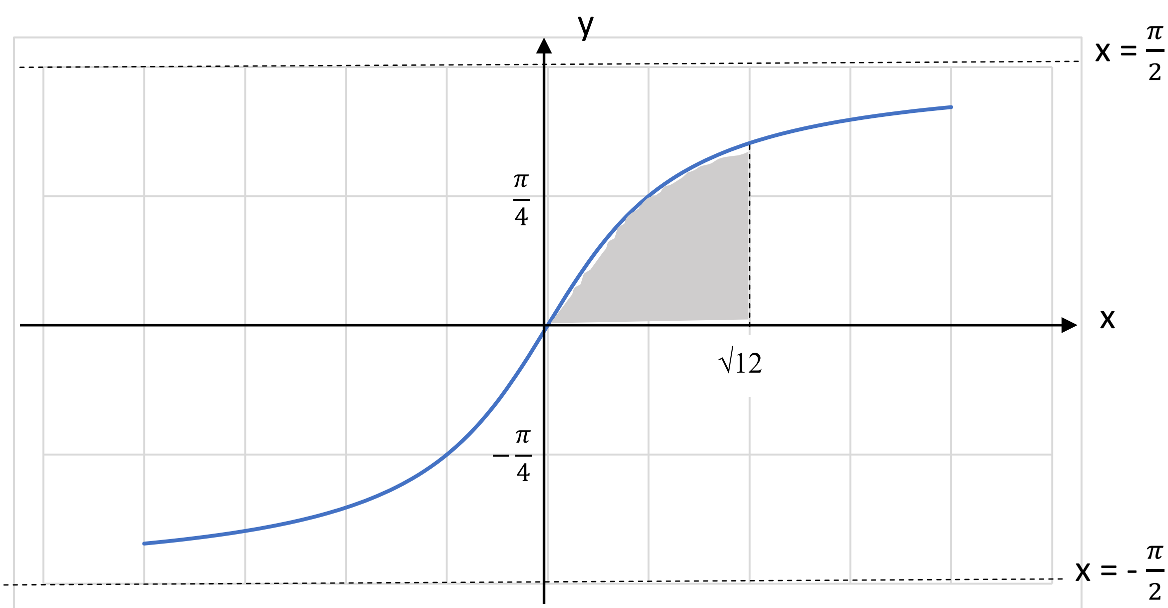
a.

Answer:

$f^{-1}: R \to R \;\; where \; f^{-1}(x) = \tan^{-1}\left(\frac{x}{2}\right)$

b.

Answer:



c.
Answer:

Consider the original function $y = 2\,tan\,x$
The required area will be the difference between the area of the rectangle width $= \frac{\pi}{3}$, height $= \sqrt{12}$ and the area under $y = 2\tan x$ from $x = 0\ to\ x = \frac{\pi}{3}$

$$= \frac{\pi}{3} \times \sqrt{12} - \int_0^{\frac{\pi}{3}} 2\tan x\, dx$$

$$= \frac{2\pi\sqrt{3}}{3} - 2\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$$

$$= \frac{2\pi\sqrt{3}}{3} + 2[log_e(\cos x)]_0^{\frac{\pi}{3}}$$

$$= \frac{2\pi\sqrt{3}}{3} + 2(log_e(\cos\frac{\pi}{3}) - log_e(\cos 0))$$

$$= \frac{2\pi\sqrt{3}}{3} + 2(log_e(½) - log_e 1)$$

$$= \frac{2\pi\sqrt{3}}{3} - 2log_e 2$$

Question 8 (5 marks)

a.

Answer:
$P(2i) = 0$
$(z - 2i)$ is a factor of $P(z)$
Due to the conjugate root theorem: $(z + 2i)$ is also a factot of $P(z)$

b.

Answer:
$P(z) = (z - 2i)(z + 2i)(z^2 + cz + d) = z^4 + 10z^3 + 31z^2 + az + b$

$P(z) = (z^2 + 4))(z^2 + cz + d) = z^4 + 10z^3 + 31z^2 + az + b$

$P(z) = z^4 + cz^3 + (d + 4)z^2 + 4cz + 4d = z^4 + 10z^3 + 31z^2 + az + b$

Equating coefficients
$c = 10, d + 4 = 31, 4c = a, 4d = b$
$a = 40, b = 108, c = 10, d = 27$

c.

Answer:
$P(z) = (z - 2i)(z + 2i)(z^2 + 10z + 27) = 0$

$P(z) = (z - 2i)(z + 2i)(z^2 + 10z + 25) + 2) = 0$

$P(z) = (z - 2i)(z + 2i)\left((z + 5)^2 - (\sqrt{2}i)^2\right) = 0$

$P(z) = (z - 2i)(z + 2i)(z + 5 - \sqrt{2}i)(z + 5 + \sqrt{2}i) = 0$

$z_1 = 2i, \quad z_2 = -2i, \quad z_3 = -5 - \sqrt{2}i, \quad z_4 = -5 + \sqrt{2}i$

Question 9 (5 marks)

a.

Answer:

$V = 480h$

b. (3 marks)

Answer:

$\frac{dV}{dt} = -4\sqrt{h}, \quad \frac{dV}{dh} = 480$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dh}{dt} = \frac{-4\sqrt{h}}{480} = \frac{-\sqrt{h}}{120}$

$$\frac{dt}{dh} = \frac{-120}{\sqrt{h}} = -120h^{-½}$$

$$t = -120 \int h^{-½}\, dh$$

$$t = -120 \times 2h^{½} + c$$

$$t = 0, \qquad h = 100$$

$$0 = -120 \times 2(100)^{½} + c$$

$$c = 2400$$

$$t = -120 \times 2h^{½} + 2400$$

$$h = (\frac{2400 - t}{240})^2$$

c. (1 mark)

Answer:

$$h = \left(\frac{2400 - t}{240}\right)^2 = 0$$

$$h = 2400\, minutes = 40\, hours$$

Question 10 (6 marks)

a.

Answer:

$u = 0, v = 4, x = 16$

$v^2 = u^2 + 2ax$

$4^2 = 0^2 + 32a$

$a = 0.5\ ms^{-2}$

Combine into a single mass of 30 kg

Let R = resultant force

$R = ma$

$R = 30\ \times 0.5 = 15\ newtons$

b.

Answer:

Back carriage (number 4)

$T_4 = 5 \times 0.5 = 2.5\ newtons$

Second back carriage (number 3)

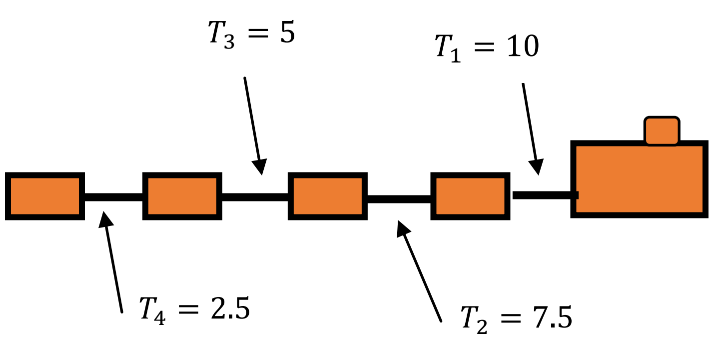
$T_3 - 2.5 = 5 \times 0.5 = 5\ newtons$

Second carriage (number 2)

$T_2 - 5 = 5 \times 0.5 = 7.5\ newtons$

First carriage (number 1)

$T_1 - 7.5 = 5 \times 0.5 = 10\ newtons$



c.

Answer:

R = resultant force on system

$R = ma$

$R = 30a$

$-12v = 30a$

$a = -0.4v$

$v.\dfrac{dv}{dx} = -0.4v$

$\dfrac{dv}{dx} = -0.4$

$v = \int -0.4\,dx$

$v = -0.4x + 4$
$v = 0$

$x = \dfrac{4}{0.4}$

$x = 10\ m$