The Mathematical Association of Victoria

Trial Examination 2019

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 27 pages.
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet
- Unless otherwise indicated, the diagrams are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A - Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

If $f(x) = x^2 - 5x - 6$, then the graph of $y = -\dfrac{2}{f(x)}$ has

A. asymptotes with equation $x = 6$ and $x = -1$ only

B. a local minimum at the point $\left(\dfrac{5}{2}, \dfrac{8}{49}\right)$

C. a maximal range of $R \setminus \{0\}$

D. x-intercepts at $x = 6$ and $x = -1$.

E. only one tangent that makes an acute angle of 45^0 with the x-axis.

Question 2

Consider the function f with rule $f(x) = \sqrt{\dfrac{\pi}{2} - \cos^{-1}\left(x - \dfrac{1}{2}\right)}$. The maximal domain of f is

A. $\left[\dfrac{1}{2}, \dfrac{3}{2}\right]$

B. $\left(\dfrac{1}{2}, \dfrac{3}{2}\right)$

C. $\left[-\dfrac{1}{2}, \dfrac{3}{2}\right]$

D. $\left(-\dfrac{1}{2}, \dfrac{3}{2}\right)$

E. $\left[\dfrac{1}{2}, \infty\right)$

SECTION A – continued
TURN OVER

Question 3

Given that $\sin(2x) = \frac{24}{25}$, the value of $\sin^4(x) + \cos^4(x)$ is equal to

A. $\frac{581}{625}$

B. $\frac{144}{625}$

C. $\frac{288}{625}$

D. $\frac{337}{625}$

E. $\frac{481}{625}$

Question 4

One of the sixth roots of the complex number *z* is equal to $3\text{cis}\left(\frac{5\pi}{3}\right)$.

Which of the following is **not** a sixth root of *z*?

A. -3

B. $-3\text{cis}\left(\frac{4\pi}{3}\right)$

C. $3\text{cis}(0)$

D. $3\text{cis}\left(\frac{5\pi}{6}\right)$

E. $3\text{cis}(2\pi)$

Question 5

The relation $\left\{z : \text{Im}\left(z + \overline{z}^2\right) = 2\right\}$ in the complex plane defines a

A. line

B. circle

C. hyperbola

D. parabola

E. ellipse

SECTION A – continued

Question 6

A solution to the differential equation $\frac{dy}{dx} = y^2 + 2x$ is $y = f(x)$ and the graph of $y = f(x)$ passes through the points $(-2,-2)$ and $(2,-5)$. Which of the following must be true of the graph of $y = f(x)$?

A. $(2,-5)$ is a local maximum

B. $(2,-5)$ is a point of inflection of the graph

C. $(-2,-2)$ is a local maximum

D. $(-2,-2)$ is a local minimum

E. $(-2,-2)$ is a point of inflection of the graph

Question 7

The length of the curve $y = \tan(x)$ between $x = a$ and $x = b$, where $0 < a < b < \frac{\pi}{2}$, is given by

A. $\int_a^b \sqrt{x^2 + \tan^2(x)}\, dx$

B. $\int_a^b \sqrt{x + \tan(x)}\, dx$

C. $\int_a^b \sqrt{1 + \sec^2(x)}\, dx$

D. $\int_a^b \sqrt{1 + \tan^2(x)}\, dx$

E. $\int_a^b \sqrt{1 + \sec^4(x)}\, dx$

Question 8

Consider the relation with equation $y = x^2 - \cos(y)$ where $x \in [-2, 0]$. For what value of x is a tangent drawn to this curve parallel to the y-axis?

A. 0

B. $-\sqrt{\frac{\pi}{2}}$

C. $-\sqrt{\frac{3\pi}{2}}$

D. $-\frac{\pi}{2}$

E. No value of x exists

SECTION A - continued

TURN OVER

Question 9

The population P of koalas on an island is being studied. It is found that the rate of growth of the koala population can be modelled by the differential equation

$$\frac{dP}{dt} = (k_1 - k_2)P$$

where t is measured in months and k_1, k_2 are constants related to the birth and death rates resepctively of the koalas.

The koala population would be halved in 8 months if there were no births.

The value of k_2 is

A. $-\dfrac{\log_e(2)}{6}$

B. $-\dfrac{\log_e(2)}{8}$

C. $\dfrac{\log_e(2)}{24}$

D. $\dfrac{\log_e(2)}{12}$

E. $\dfrac{\log_e(2)}{8}$

Question 10

Let $f:[0,\pi] \to R$, $f(x) = \sin(x)$.

Let A be the area bounded by the graph of f and the x-axis. The lines $x = a$ and $x = b$, where $0 < a < b < \pi$, divide A into three equal regions.

The value of $\sin(b-a)$ is equal to:

A. 0

B. $\dfrac{4\sqrt{2}}{9}$

C. $\dfrac{2\sqrt{5}}{3}$

D. $\dfrac{\sqrt{6}-\sqrt{3}}{4}$

E. $\dfrac{\sqrt{6}-\sqrt{3}}{2}$

SECTION A – continued

Question 11

The vectors $\overrightarrow{PQ} = -3\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\overrightarrow{PR} = 5\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ form two of the sides of the triangle *PQR*.

The length of the median from *P* is equal to

A. $\sqrt{14}$

B. $\sqrt{15}$

C. $\sqrt{17}$

D. $\sqrt{18}$

E. $\sqrt{19}$

Question 12

The triangle formed by the three points whose position vectors are $2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$, $4\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k}$ and $3\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}$ is

A. an equilateral triangle

B. a right angled triangle which is not isosceles

C. an isosceles triangle which is not right angled

D. a scalene triangle

E. a right angled isosceles triangle

Question 13

John is riding a jet ski. The jet ski is level with the pier (origin) and is travelling at 5 ms^{-1} due North when John decides to accelerate uniformly. After accelerating for 40 seconds he is travelling due East at 4 ms^{-1}. Given the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are directed East and North respectively, the position vector of the jet ski at the end of the 40 second time period is

A $80\underset{\sim}{i} + 100\underset{\sim}{j}$

B $100\underset{\sim}{i} + 80\underset{\sim}{j}$

C $80\underset{\sim}{i} - 100\underset{\sim}{j}$

D $100\underset{\sim}{i} - 80\underset{\sim}{j}$

E $0.1\underset{\sim}{i} - 0.125\underset{\sim}{j}$

SECTION A – continued

TURN OVER

Question 14

A particle moves in such a way that its position vector at time t is given by $\underset{\sim}{r}(t) = \cos(2t)\,\underset{\sim}{i} + \left(2 - \sin^2(t)\right)\underset{\sim}{j}$.

The maximum speed of the particle and the first time that this speed is reached is respectively

A. $\sqrt{5}$, $\frac{\pi}{2}$

B. $\sqrt{5}$, $\frac{\pi}{4}$

C. $\sqrt{3}$, $\frac{\pi}{2}$

D. $\sqrt{3}$, $\frac{\pi}{4}$

E. $2\sqrt{2}$, $\frac{\pi}{4}$

Question 15

A particle is moving along a straight line with an acceleration given by $a(t) = \dfrac{4+t}{\sqrt{1+t^5}}$, $t \geq 0$.

If the initial velocity of the particle is 5 ms^{-1}, the velocity in ms^{-1} of the particle at $t = 3$ is closest to:

A. 0.913

B. 1.134

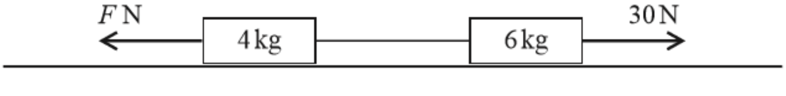
C. 6.134

D. 6.913

E. 11.913

Question 16

The diagram below shows blocks of mass 4 kg and 6 kg on a smooth horizontal surface. The blocks are connected by a light, inextensible string. A horizontal force of size F Newtons acts on the 4 kg block and a horizontal force of size 30 Newtons acts on the 6 kg block. The magnitude of the acceleration of the system is 2 ms^{-2}.



Given the tension in the string is T Newtons, which of the following statements is correct?

A. $F = 10,\ T = 18$

B. $F = 10,\ T = 18$ or $F = 50,\ T = 42$

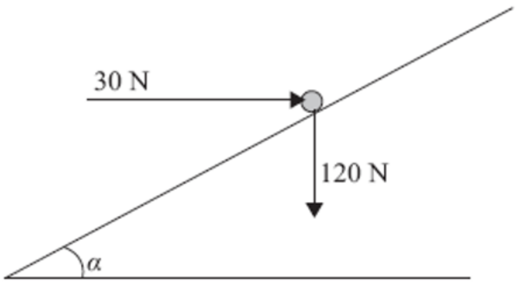
C. $F = 50,\ T = 42$

D. $F = 50,\ T = 62$

E. $F = 10,\ T = 42$ or $F = 30,\ T = 42$

Question 17

A particle of weight 120 Newtons is placed on a fixed plane which is inclined at an angle α to the horizontal, where $\alpha = \arctan\left(\frac{3}{4}\right)$. The particle is held at rest in equilibrium by a horizontal force of magnitude 30 Newtons, as shown in the diagram below.



The normal reaction force, in Newtons, between the particle and the plane has magnitude

A. 78

B. 96

C. 114

D. 124

E. 150

Question 18

The mass of rubbish in filled large bins in the tuck shop at a school is normally distributed with a mean of 6.8 kg and a standard deviation of 1.5 kg.

The mass of rubbish in filled small bins in the tuck shop is normally distributed with a mean of 3.2 kg and a standard deviation of 0.6 kg.

There are 8 large bins and 3 small bins in the tuck shop. The probability, correct to 3 decimal places, that the total mass of rubbish exceeds 70 kg on a particular day when all the bins are filled is equal to

A. 0.084

B. 0.085

C. 0.376

D. 0.377

E. 0.999

SECTION A – continued

TURN OVER

Question 19

A car manufacturer claims that the average fuel consumption of a new model car is 8.2 litres per 100 km. A consumer organisation is sceptical of this claim and thinks that the manufacturer is underestimating the number of litres needed per 100 km travelled.

If μ represents the average fuel consumption for the population of new model cars, which of the following gives the null and alternative hypotheses that the consumer organisation should test?

A. $H_0 : \mu < 8.2$
$H_1 : \mu \geq 8.2$

B. $H_0 : \mu \leq 8.2$
$H_1 : \mu > 8.2$

C. $H_0 : \mu = 8.2$
$H_1 : \mu > 8.2$

D. $H_0 : \mu = 8.2$
$H_1 : \mu < 8.2$

E. $H_0 : \mu = 8.2$
$H_1 : \mu \neq 8.2$

Question 20

A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \dfrac{2x}{9} & 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

A sample of forty independent measurements of X are taken and the mean of the sample is calculated. The probability that the mean of the sample is greater than 2.10 is closest to

A. 0.1855

B. 0.4438

C. 0.3828

D. 0.5562

E. 0.8145

END OF SECTION A

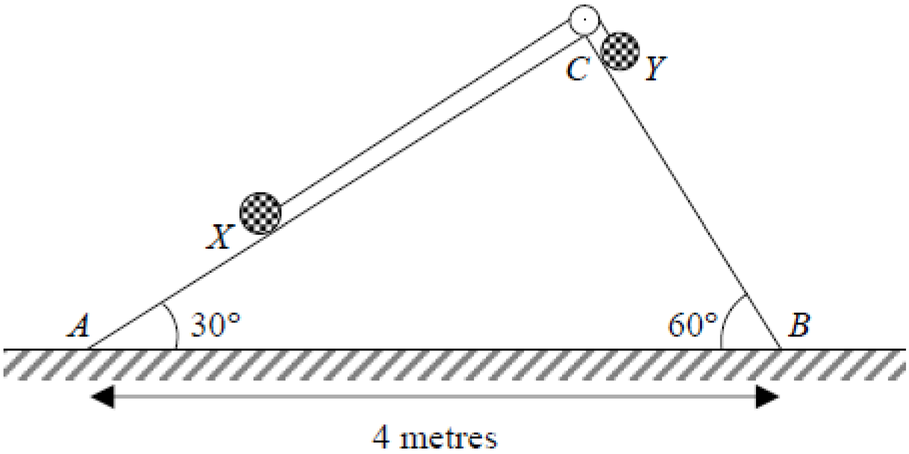
– continued

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (10 marks)



The diagram above shows an object X of mass 3 kg on a smooth inclined plane at an angle of 30° to the horizontal. It is connected by a light, inextensible string of length 2.5 m to an object Y, of mass 2 kg, on a smooth inclined plane at an angle of 60° to the horizontal. The string connecting the two objects is taut and passes over a smooth pulley at C.

Initially object Y is at a point just below C, touching the pulley, with the string being taut. When the system is released from rest object X moves up the plane and object Y moves down the plane.

A and B are at ground level with AB horizontal and of length 4 metres.

a. On the diagram above, mark all forces acting on objects X and Y. 2 marks

b. Show that the acceleration of the system is $a = \dfrac{g\left(2\sqrt{3}-3\right)}{10}$. 3 marks

SECTION B – Question 1 - continued

TURN OVER

c. Find the tension in the string joining objects X and Y. Give your answer in Newtons and correct to one decimal place. 2 marks

d. Object Y strikes the ground at B with momentum of magnitude $P \text{ kg ms}^{-1}$.
Calculate the value of P, correct to two decimal places. 3 marks

Question 2 (11 marks)

An object starts moving from an origin O with an initial velocity of 8 ms^{-1} and acceleration $a\ \text{ms}^{-2}$ where $a = \dfrac{-v\left(1+v^2\right)}{40}$ and $v\ \text{ms}^{-1}$ is the velocity of the object t seconds after it starts moving.

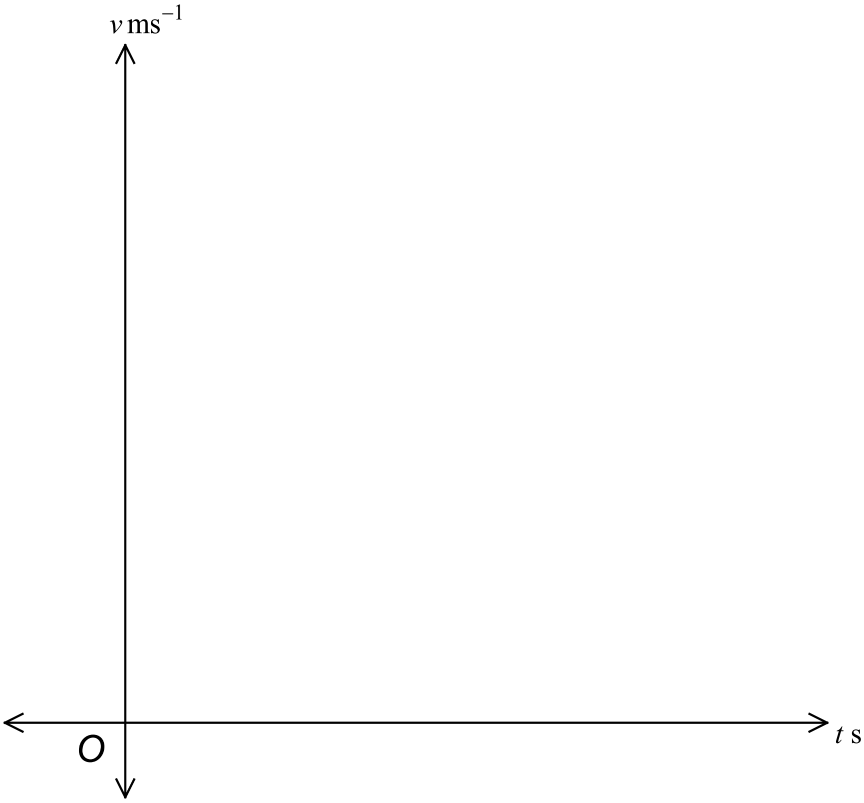
a. **i.** State a definite integral that gives the time it takes, in seconds, for the velocity of the object to decrease from 8 ms^{-1} to 4 ms^{-1}. 1 mark

ii. **Hence** find, correct to three decimal places, the time in seconds it takes for the velocity of the object to decrease from 8 ms^{-1} to 4 ms^{-1}. 1 mark

iii. Find an expression for t in the form $k\log_e\left(\dfrac{m(v^2+1)}{nv^2}\right)$ where k, m, n are positive integers. 1 mark

SECTION B – Question 2 – continued
TURN OVER

b. Sketch a graph of $v = v(t)$ on the axes below, labelling all important features. 2 marks



Working space

SECTION B – Question 2 - continued

c. Let x be the displacement in metres of the object t seconds after it starts moving.

i. Show that $x = 40\tan^{-1}(8) - 40\tan^{-1}(v)$. 2 marks

ii. Hence show that $v = \dfrac{8 - \tan\left(\dfrac{x}{40}\right)}{1 + 8\tan\left(\dfrac{x}{40}\right)}$. 3 marks

SECTION B – Question 2 - continued

TURN OVER

d. Briefly explain whether or not the displacement of the object can equal $40\tan^{-1}(8)$. 1 mark

SECTION B - continued

Question 3 (12 marks)

Consider the function f $f:\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ where $f(x) = a\tan(x) + b\sec(x)$ and a and b are positive real numbers.

a. Consider $a > b$.

i. Show that f has at least one x-intercept. 2 marks

ii. Show that the equation $a\tan(x) + b\sec(x) = y$ has a solution for all real values of y. 3 marks

SECTION B – Question 3 – continued
TURN OVER

b. Find in terms of a and b the value of $f(x)$ when $\operatorname{cosec}(2x) = -\frac{3\sqrt{2}}{4}$ where $-\frac{\pi}{2} < x < -\frac{\pi}{4}$. 3 marks

SECTION B – Question 3 – continued

c. Consider $a < b$.

i. Show that f has at least one stationary point and find in terms of a and b the x-coordinate of all stationary points. 2 marks

ii. Hence find the coordinates of all stationary points of f. 2 marks

Question 4 (11 marks)

Let $z = x + yi$ where $x, y \in R$.

a. Find the Cartesian equations of the following relations:

i. $A = \left\{ z : \; i\bar{z} - iz = 3\sqrt{2} \right\}$. 1 mark

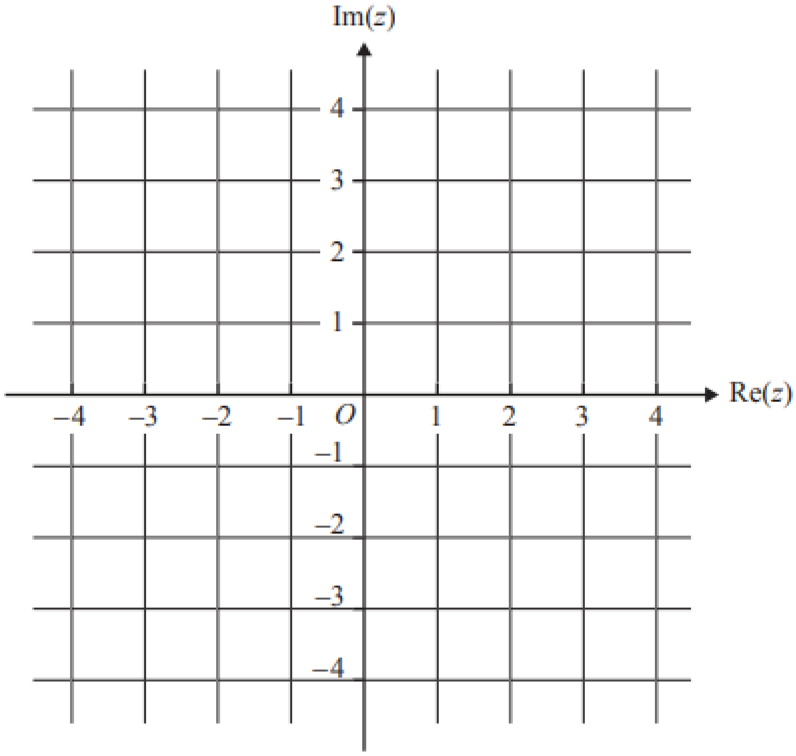
ii. $B = \{ z : \; z\bar{z} = 9 \}$. 1 mark

iii. $C = \left\{ z : \; |z - i| = \left|z - \sqrt{3}\right| \right\}$. 1 mark

SECTION B – Question 4 - continued

Let S define a region bounded by the relations A, B and C such that $\frac{5}{2}i \in S$.

b. Sketch S on the Argand diagram below. Label any axes intercepts of the boundary of S with their coordinates. 2 marks



For $z \in S$:

c. State the largest value of $|z|$. 1 mark

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d. Find the largest value of $\text{Arg}(z)$. 1 mark

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SECTION B – Question 4 – continued
TURN OVER

e. Find, correct to two decimal places, the area of S. 2 marks

f. Let $P = \{z : \; |z - i| = |z - a|\}$ where $a \in R$.

Find all values of a such that the relations defined by A, B and P have a common intersection point. 2 marks

Question 5 (7 marks)

Relative to a fixed origin O, the points A, B and C have position vectors $\underset{\sim}{a} = -\underset{\sim}{i} + \frac{4}{3}\underset{\sim}{j} + 7\underset{\sim}{k}$, $\underset{\sim}{b} = 4\underset{\sim}{i} + \frac{4}{3}\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = 6\underset{\sim}{i} + \frac{16}{3}\underset{\sim}{j} + 2\underset{\sim}{k}$ respectively.

a. Find the cosine of the angle ABC. 1 mark

b. Hence find the area of the triangle ABC. 1 mark

SECTION B – Question 5 – continued
TURN OVER

c. Use a vector method to find the shortest distance between the point A and the line passing through the points B and C. 3 marks

Let D be the point such that the quadrilateral $ABCD$ is a kite, where the length of each pair of adjacent sides is equal, that is, $BC = CD$ and $BA = AD$.

d. Find the position vector of the point D. 2 marks

SECTION B - continued

Question 6 (9 marks)

When fully charged, an Infinity Gauntlet is claimed by the dwarves of Nidavellir to have a mean lifetime of $\mu = 150$ hours before a recharge is needed. To test this claim, a statistical test of the hypotheses

$H_0: \ \mu = 150$

$H_1: \ \mu < 150$

is carried out at the 5% level of significance.

A random sample of *n* Infinity Gauntlets is collected and found to have a mean lifetime of 146 hours before a recharge is needed and a standard deviation of 12 hours. The *p* value is found to be 0.012, correct to three decimal places.

a. Find the value of *n*.

Assume that *n* is sufficiently large that reasonable approximations for the standard deviation of the population (of Infinity Gauntlet lifetimes) and shape of the distribution of the sample means can be used. 3 marks

A second random sample of Infinity Gauntlets is collected. The sample mean was found to be 138 hours and the standard deviation 12 hours.

Assume that the sample size is sufficiently large that reasonable approximations for the standard deviation of the population (of Infinity Gauntlet lifetimes) and shape of the distribution of the sample means can be used.

b. Find the smallest sample size for which the endpoints of the 90% confidence interval for the population mean μ would be no greater than 2.5 either side of the sample mean. 2 marks

The size of the second random sample is in fact 47. A statistical test of the hypotheses

$H_0: \mu = 150$

$H_1: \mu < 150$

is carried out at the 5% level of significance using this sample.

c. **i.** Let $\bar{x}$ be the mean, in hours, of the lifetime of a fully charged Infinity Gaunlet in a sample of size 47 and let C^* be the smallest value of $\bar{x}$ for which the null hypothesis is not rejected.

Find, correct to two decimal places, the value of C^*. 1 mark

ii. If the probability of a type 2 error is 0.22, find the true mean lifetime $\mu_1 < 150$ of an Infinity Gauntlet. Give your answer correct to one decimal place. 3 marks

END OF QUESTION AND ANSWER BOOK