Victorian Certificate of Education – 2018 VCAA Examinations

SPECIALIST MATHEMATICS

Written Examination 2

PROVISIONAL SOLUTIONS

SECTION A

Question 1 (1 mark)

The range of $y=\frac{1}{2}\arctan(x)$ is $\left(\frac{-\pi}{4},\frac{\pi}{4}\right)$.

Hence, the graph has asymptotes at $y=\frac{\pm\pi}{4}$.

The answer is **E**.

Question 2 (1 mark)

The domain of f, where $c>0$, is

$\{x\mid -1\le cx+d\le 1\}\cap\{x\mid \arcsin(cx+d)>0\}$

$=\left\{x\mid \frac{-1-d}{c}\le x\le\frac{1-d}{c}\right\}\cap\left\{x\mid x>\frac{-d}{c}\right\}$

Since $\frac{-1-d}{c}<\frac{-d}{c}<\frac{1-d}{c}$ for all $c>0$ and $d\in\mathbb{R}$,

the domain of f is $x\in\left(\frac{-d}{c},\frac{1-d}{c}\right]$.

The answer is **B**.

Question 3 (1 mark)

$$\frac{2x^2+3x+1}{(2x+1)^3(x^2-1)}=\frac{(x+1)(2x+1)}{(2x+1)^3(x+1)(x-1)}$$

$$=\frac{1}{(2x+1)^2(x-1)}$$

The factor $(2x+1)$ is repeated twice.

The answer is **D**.

Question 4 (1 mark)

$\csc(-x)=-\csc(x)$ $(\csc(.)$ is odd$)$

$=-\frac{\cot(x)}{\cos(x)}$

$=-\frac{b}{-a}$

$=\frac{b}{a}$

The answer is **A**.

Question 5 (1 mark)

$z=a+bi$, where $a\ne 0$ and $b\ne 0$.

$z+\frac{1}{z}=\frac{a}{a^2+b^2}+a+\left(b-\frac{b}{a^2+b^2}\right)i$

If $z+\frac{1}{z}\in\mathbb{R}$, then $b-\frac{b}{a^2+b^2}=0$

$a^2+b^2=|z|=1$.

The answer is **D**.

Question 6 (1 mark)

In the complex plane, the points $0, z, iz, z+iz$ represent a square since $|z|=|iz|$ and

$\operatorname{Arg}(iz)-\operatorname{Arg}(z)\equiv\frac{\pi}{2}$.

Let $A=$ area of triangle.

$A=\frac{1}{2}|z||iz|$

$=\frac{|z|^2}{2}$

The answer is **C**.

Question 7 (1 mark)
Let $L =$ length.
$$L = \int_0^{2\pi} \sqrt{\left(\frac{d}{dt}[\sin(2t)]\right)^2 + \left(\frac{d}{dt}[2\cos(t)]\right)^2}\,dt$$
$= 12.1944...$ units.
The answer is **C**.

Question 8 (1 mark)
Let $I = \int_0^{\frac{\pi}{6}} \tan^2(x)\sec^2(x)\,dx.$
Let $u = \tan(x)$.
$\frac{du}{dx} = \sec^2(x),\ u(0) = 0$ and $u\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}.$
Hence, $I = \int_0^{\frac{1}{\sqrt{3}}} u^2\,du.$
The answer is **E**.

Question 9 (1 mark)
$$\frac{dy}{dx} = \frac{2}{\sin(x+y) - \sin(x-y)}$$
$$= \frac{1}{\cos(x)\sin(y)}$$
$$\int \sin(y)\,dy = \int \frac{1}{\cos(x)}\,dx$$
$$\int \sec(x)\,dx = \int \sin(y)\,dy$$
The answer is **D**.

Question 10 (1 mark)
$\left.\frac{dy}{dx}\right|_{(x,\,0)} < 0$ and $\left.\frac{dy}{dx}\right|_{(0,\,y)} > 0.$
By elimination, $\frac{dy}{dx} = \frac{2x+y}{y-2x}.$
The answer is **A**.

Question 11 (1 mark)
$\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}.$
$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(30^\circ)$
$m = \sqrt{3}, \frac{1}{\sqrt{3}}.$
The answer is **C**.

Question 12 (1 mark)
$|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}| \Rightarrow |\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2|\underset{\sim}{a}||\underset{\sim}{b}|$
But, from the cosine rule,
$$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)\ \ (0 \le \theta \le \pi)$$
Hence, we require $\cos(\theta) = 1$.
Therefore, $\theta = 0$, and so $\underset{\sim}{a} \,||\, \underset{\sim}{b}$.
The answer is **A**.

Question 13 (1 mark)
$\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 4\sin(t)\underset{\sim}{j}.$
$|\dot{\underset{\sim}{r}}(t)| = \sqrt{7\cos^2(t) + 9}.$
Let $\frac{d|\dot{\underset{\sim}{r}}(t)|}{dt} = 0.$
Minimum occurs when $\cos(t) = 0$.
Hence, the first time when minimum speed is
$$t = \frac{\pi}{2}.$$
The answer is **B**.

Question 14 (1 mark)
$\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}.$
$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = -\frac{9\sqrt{14}}{14}$
The answer is **C**.

Question 15 (1 mark)
$a = \frac{P}{8}$
$v\frac{dv}{dx} = \frac{P}{8}$
$\int_4^{20} v\,dv = \int_0^{15} \frac{P}{8}\,dx$
$P = 102.4$ N
The answer is **E**.

Question 16 (1 mark)
Vertically, $F_2\sin(45^\circ) = 4 + 3\sin(30^\circ)$.
$F_2 = \frac{11\sqrt{2}}{2}$
The answer is **B**.

Question 17 (1 mark)
$a = -g$
$0 = \int_0^t \left[\int_0^z -g \, dw + 2 \right] dz + 50$
$t = 3.40498...$ s.
The answer is **E**.

Question 18 (1 mark)
$2 \cdot 1.95996... \frac{\sigma}{\sqrt{36}} = 67.31 - 58.42$
$\sigma = 13.6074...$ days
The answer is **D**.

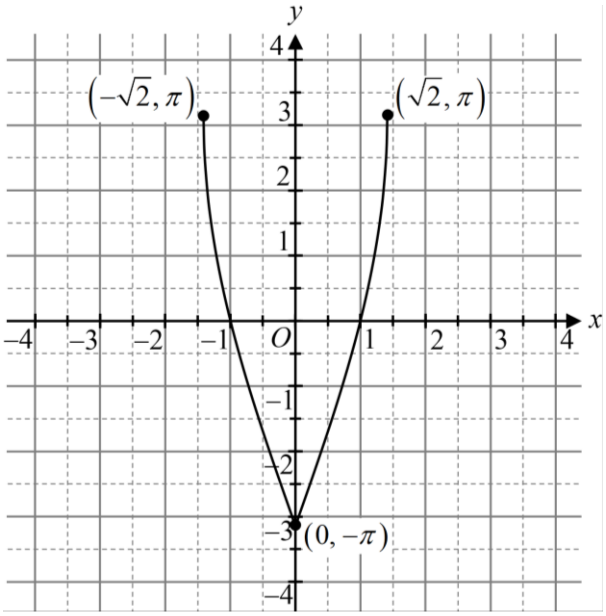
Question 19 (1 mark)
Let X = gestation period of cats.
$X \sim \mathrm{N}\left(66, \left(\frac{4}{3}\right)^2\right)$
$\bar{X} \sim \mathrm{N}\left(66, \frac{16}{45}\right)$
$\Pr\left(\bar{X} > 65\right) = 0.953234...$
The answer is **E**.

Question 20 (1 mark)
Let M = scores on maths test.
Let S = scores on statistics test.
$M \sim \mathrm{N}\left(71, 10^2\right)$
$S \sim \mathrm{N}\left(75, 7^2\right)$
$\mathrm{E}(M - S) = 71 - 75$
$= -4$
$\mathrm{sd}(M - S) = \sqrt{10^2 + 7^2}$
$= \sqrt{149}$
$\Pr(M > S) = \Pr(M - S > 0)$
$= 0.371572...$
The answer is **B**.

SECTION B

Question 1a (2 marks)
$f : D \to \mathbb{R}, \ f(x) = 2\arcsin\left(x^2 - 1\right)$
$\mathrm{Dom}(f) = \left\{x \mid -1 \le x^2 - 1 \le 1\right\}$
$= \left[-\sqrt{2}, \sqrt{2}\right]$
$\mathrm{Ran}(f) = \left[f(0), f\left(\sqrt{2}\right)\right]$
$= [-\pi, \pi]$

Question 1b (3 marks)



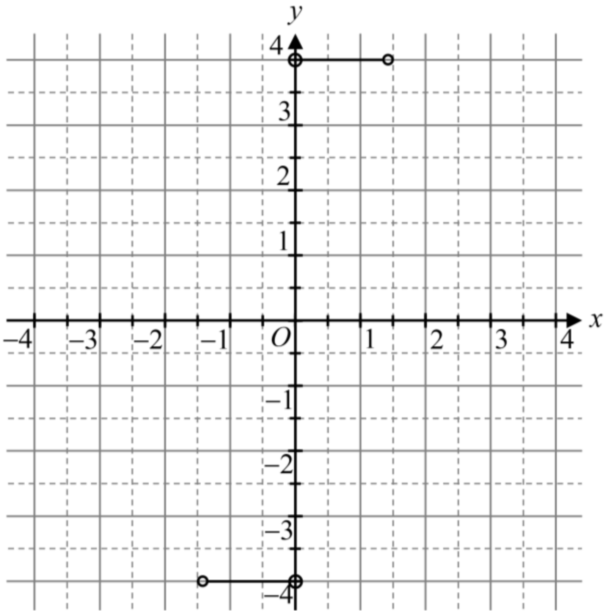
Question 1c (1 mark)
$f'(x) = \frac{4\,\mathrm{sgn}(x)}{\sqrt{2 - x^2}}$
Therefore, for $x > 0$, $f'(x) = \frac{4}{\sqrt{2 - x^2}}$.

Question 1d (1 mark)
For $x < 0$, $f'(x) = \frac{-4}{\sqrt{2 - x^2}}$.

Question 1e.i (1 mark)
$\mathrm{Dom}(f') = \left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right)$

Question 1e.ii (1 mark)
$g(x) = \begin{cases} -4 & -\sqrt{2} < x < 0 \\ 4 & 0 < x < \sqrt{2} \end{cases}$

Question 1.e.iii (2 marks)



Question 2a (1 mark)

$|z-(1+2i)|=2$

Centre: $(1,2)$

Radius: 2

Question 2b (2 marks)

$|z+1|=\sqrt{2}|z-i|$

Let $z=x+yi$.

$\sqrt{(x+1)^2+y^2}=\sqrt{2}\sqrt{x^2+(y-1)^2}$

$x^2+2x+1+y^2=2x^2+2y^2-4y+2$

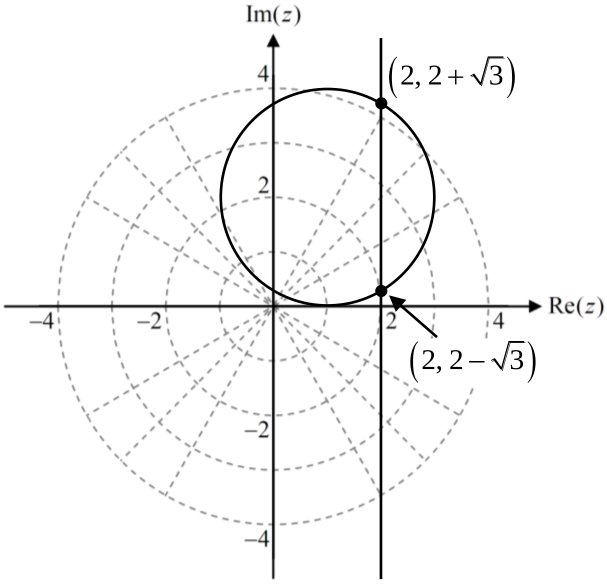
$x^2-2x+y^2-4y=-1$

$(x-1)^2-1+(y-2)^2-4=-1$

$(x-1)^2+(y-2)^2=2^2$

Hence, the circle given by $|z+1|=\sqrt{2}|z-i|$ has a centre at $(1,2)$ and a radius of 2, the same as in part **a**, as required.

Question 2c



Question 2d (2 marks)

The perpendicular bisector of $|z-1|=|z-3|$ is the line $x=2$.

Let $x=2$.

$y=2\pm\sqrt{3}$

(*see graph above in* **Q2c**)

Question 2e (3 marks)

Let $\theta=$ angle formed by line segments joining centre and intersections.

$\theta=\arctan\left(\frac{2+\sqrt{3}-2}{2-1}\right)-\arctan\left(\frac{2-\sqrt{3}-2}{2-1}\right)$

$=\frac{2\pi}{3}$

Let $A=$ area of minor segment.

$A=\frac{2^2}{2}\left(\frac{2\pi}{3}-\sin\left(\frac{2\pi}{3}\right)\right)$

$=\frac{4\pi}{3}-\sqrt{3}$ units^2

Question 3a (2 marks)

$y=\frac{1}{2}\sqrt{4x^2-1}$

$x^2=y^2+\frac{1}{4}$

$V=\pi\int_0^h\left(y^2+\frac{1}{4}\right)dy$

$=\pi\left[\frac{y^3}{3}+\frac{y}{4}\right]_0^h$

$=\pi\left(\frac{h^3}{3}+\frac{h}{4}\right)$

$=\frac{\pi}{4}\left(\frac{4}{3}h^3+h\right)$, as required.

Question 3b (2 marks)

Let $V(h)=\frac{1}{2}V\left(\frac{\sqrt{3}}{2}\right)$.

$h=0.59\text{ m }(2\text{dp})$

Question 3c.i (2 marks)

$\frac{dV}{dt}=0.04-0.05\sqrt{h}$

$=\frac{4-5\sqrt{h}}{100}$

$\frac{dV}{dh}=\frac{\pi}{4}\left(4h^2+1\right)$

$\frac{dh}{dt}=\frac{dh}{dV}\cdot\frac{dV}{dt}$

$=\frac{4}{\pi\left(4h^2+1\right)}\cdot\frac{4-5\sqrt{h}}{100}$

$=\frac{4-5\sqrt{h}}{25\pi\left(4h^2+1\right)}$, as required.

Question 3c.ii (1 mark)

$\left.\frac{dh}{dt}\right|_{h=\frac{1}{4}}=0.0153\text{ ms}^{-1}\ (4\text{dp})$

Question 3d (2 marks)

Let $T=$ required time.

$T=\int_0^{\frac{1}{4}}\frac{25\pi\left(4h^2+1\right)}{4-5\sqrt{h}}dh$

$=9.8\text{ s }(1\text{dp})$

Question 3e (2 marks)

$h(25)=0.4$

$h(30)=0.4+5\cdot\frac{4-5\sqrt{0.4}}{25\pi\left(4\cdot0.4^2+1\right)}$

$=0.43\text{ m }(2\text{dp})$

Question 3f (2 marks)

Let $\frac{dh}{dt}=0$.

$h=\frac{16}{25}$.

Let $d=$ height between equilibruim depth and full depth.

$d=\frac{\sqrt{3}}{2}-\frac{16}{25}$

$=0.23\text{ m }(2\text{dp})$

Question 4a (2 marks)

For yacht A, $x=t+1$ and $y=t^2+2t$.

Therefore, $y=(x-1)^2+2(x-1)$

$=x^2-1$.

For yacht B, $x=t^2$ and $y=t^2+3$.

Therefore, $y=x+3$.

Question 4b (2 marks)

Let $t^2=t+1,\ t\geq0$.

$t=\frac{1+\sqrt{5}}{2}$

But, $\underset{\sim}{r}_A\left(\frac{1+\sqrt{5}}{2}\right)=\frac{3+\sqrt{5}}{2}\underset{\sim}{i}+\frac{5+3\sqrt{5}}{2}\underset{\sim}{j}$ and

$\underset{\sim}{r}_B\left(\frac{1+\sqrt{5}}{2}\right)=\frac{3+\sqrt{5}}{2}\underset{\sim}{i}+\frac{9+\sqrt{5}}{2}\underset{\sim}{j}$.

Hence, the yachts cannot collide, as required.

Question 4c (2 marks)

Let $x+3=x^2-1$, where $x\geq1$.

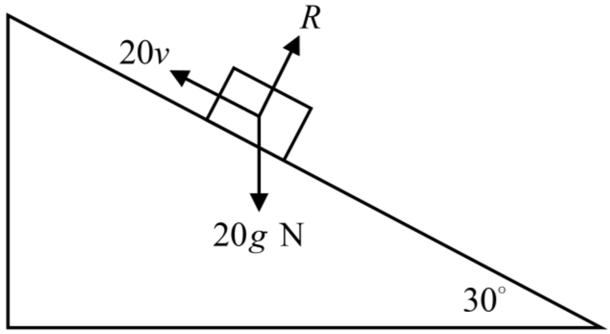
$x=\frac{1+\sqrt{17}}{2},\ y=\frac{7+\sqrt{17}}{2}$.

Hence, they cross paths at $(2.562, 5.562)$ (3dp)

Question 4d (2 marks)
Define $s(t)=\left|\dot{\underset{\sim}{r}}_{\mathrm{A}}(t)\right|-\left|\dot{\underset{\sim}{r}}_{\mathrm{B}}(t)\right|$.
Let $s(t)=0$.
$t=\dfrac{5}{2}$.
Using a graph, $s(t)>0$ for $t\in\left[0,\dfrac{5}{2}\right)$.

Question 4e (2 marks)
Let $\left|\underset{\sim}{r}_{\mathrm{A}}(t)-\underset{\sim}{r}_{\mathrm{B}}(t)\right|=\dfrac{1}{5}$.
$t=1.52883...,1.59734$.
Using a graph, $\left|\underset{\sim}{r}_{\mathrm{A}}(t)-\underset{\sim}{r}_{\mathrm{B}}(t)\right|\le\dfrac{1}{5}$ when
$t\in[1.52883...,1.59734...]$.
Hence $\left|\underset{\sim}{r}_{\mathrm{A}}(t)-\underset{\sim}{r}_{\mathrm{B}}(t)\right|\le\dfrac{1}{5}$ for
$T=(1.59734...-1.52883...)\cdot 60$
$=4.1$ minutes (1dp)

Question 5a (1 mark)



Question 5b.i (1 mark)
$20a=20g\sin\left(30^{\circ}\right)-20v$

Question 5b.ii (1 mark)
$a=g\sin\left(30^{\circ}\right)-v$
$=\dfrac{g}{2}-v$
$=\dfrac{g-2v}{2}$, as required.

Question 5c (2 marks)
$v\dfrac{dv}{dx}=\dfrac{g-2v}{2},\ x(0)=0.$

$$x=\int_0^v \frac{2z}{g-2z}\,dz$$
$$=\int_0^v\left(-1+\frac{g}{g-2z}\right)dz$$
$$=\left[-z-\frac{g}{2}\log_e(g-2z)\right]_0^v$$
$$=-v-4.9\log_e(9.8-2v)+0+4.9\log_e(9.8)$$
$$=-v+4.9\log_e\left(\frac{4.9}{4.9-v}\right)$$

Question 5d (1 mark)
Let $x=15$.
$v=4.81\ \mathrm{ms}^{-1}$ (2dp)

Question 5e.i (1 mark)
Let T = time to 4.5 ms^{-1}.
$$T=\int_0^{4.5}\frac{2}{g-2v}\,dv$$

Question 5e.ii (1 mark)
$T=2.51$ s (2dp)

Question 6a (1 mark)
$H_0:\mu=150$
$H_1:\mu<150$

Question 6b (1 mark)
$\mathrm{sd}\left(\bar{X}\right)=\dfrac{15}{\sqrt{50}}$
$=\dfrac{3\sqrt{2}}{2}$

Question 6c (2 marks)
$p=\Pr\left(\bar{X}<145\mid\mu=150\right)$
$=0.0092$ (4dp)

Question 6d (1 mark)
H_0 should be rejected since $p<0.05$.

Question 6e (1 mark)

Define $c =$ smallest test statistic for which H_0 is is not rejected.

$\Pr\left(\bar{X} < c\right) = 0.05$

$c = 146.5107... = 146.51 \text{ cm } (2\text{dp})$

Question 6f (1 mark)

$\Pr\left(\bar{X} > 146.5107... \mid \mu = 145\right) = 0.24\ (2\text{dp})$

Question 6g (1 mark)

99% CI is given by

$$\left(145 - 2.57583... \cdot \frac{3\sqrt{2}}{2}, 145 + 2.57583... \cdot \frac{3\sqrt{2}}{2}\right)$$

$= (139.5, 150.5)\ (1\text{dp})$

END OF PROVISIONAL SOLUTIONS