SPECIALIST MATHEMATICS

Written examination 2

2018 Trial Examination

SOLUTIONS

SECTION A: Multiple-choice questions (1 mark each)

Question 1

Answer: **B**

Explanation:

The range for $y = \tan^{-1} x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
(which is given on the VCAA formula sheet).

$y = \tan^{-1}(x+1) - \frac{\pi}{4}$ has been translated $\frac{\pi}{4}$ down,
so the range is $\left(\frac{-3\pi}{4}, \frac{3\pi}{4}\right)$

Question 2

Answer: **D**

Explanation:

Options A to D are all ellipses, but the direction of motion for options A to C is anti-clockwise. Only option D gives motion in a **clockwise** direction.

This can be verified by substituting some values for t, or (preferably) by sketching the parametric equations using CAS.

Question 3

Answer: **A**

Explanation:

$$\frac{w^2}{\bar{w}} = \frac{(1-i)^2}{1+i}$$
$$= \frac{\left(\sqrt{2}\text{cis}\left(\frac{-\pi}{4}\right)\right)^2}{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}$$
$$= \frac{2\text{cis}\left(\frac{-\pi}{2}\right)}{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}$$
$$= \sqrt{2}\text{cis}\left(\frac{-\pi}{2} - \frac{\pi}{4}\right)$$
$$= \sqrt{2}\text{cis}\left(\frac{-3\pi}{4}\right)$$

This can also be evaluated quickly on CAS:

TI-nspire:

Define $w=1-i$	*Done*
$\text{angle}\left(\frac{w^2}{\text{conj}(w)}\right)$	$\frac{-3\cdot\pi}{4}$

Question 4

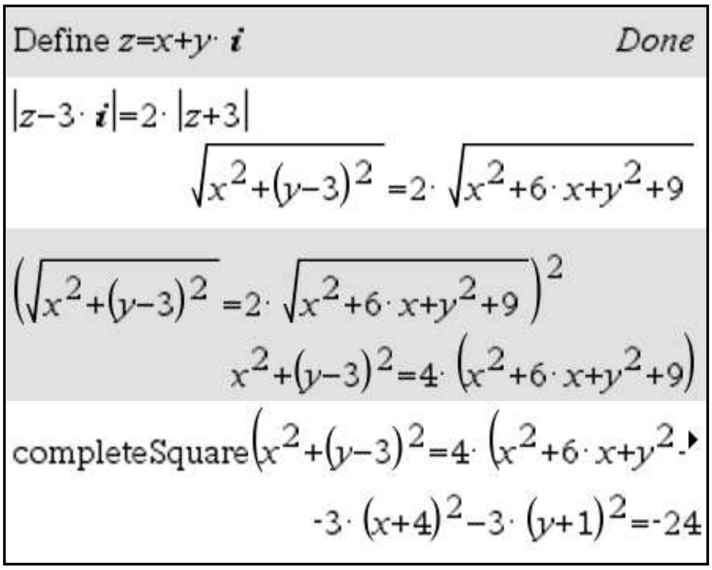
Answer: **E**

Explanation:

$$|z-3i| = 2|z+3|$$
$$|x+yi-3i| = 2|x+yi+3|$$
$$\sqrt{x^2+(y-3)^2} = 2\sqrt{(x+3)^2+y^2}$$
$$x^2+(y-3)^2 = 4((x+3)^2+y^2)$$
$$x^2+y^2-6y+9 = 4x^2+24x+4y^2+36$$

Rearranging and completing the square gives:
$$(x+4)^2+(y+1)^2 = 8$$

TI-nspire:



Question 5

Answer: **D**

Explanation:

$$\begin{aligned} f(z) &= z^5 - az^3 - 2z^2 + 2a \\ &= z^3(z^2 - a) - 2(z^2 - a) \\ &= (z^3 - 2)(z^2 - a) \end{aligned}$$

Now $(z^3 - 2)$ has two non-real roots (and one real root),
and $(z^2 - a)$ will have a maximum of two non-real roots (if $a < 0$).

So the maximum number of non-real roots is four.

Question 6

Answer: **C**

Explanation:

$g'(x) = \sin^{-1} x$
$g''(x) = \frac{1}{\sqrt{1-x^2}}$
$g''(x) > 0$ for all $x \in (-1,1)$

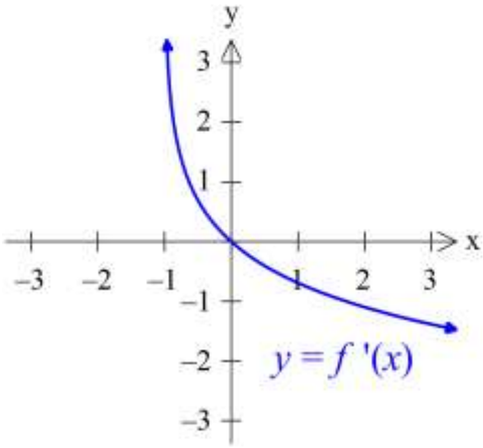
So the function g is never concave down.

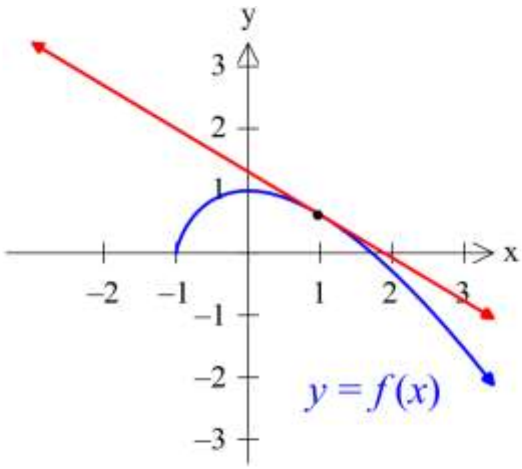
Question 7

Answer: **D.**

Explanation:

In option D, $f'(1) < 0$ and $f''(1) < 0$.
Therefore, using the tangent to the original function at $f(1)$ would be overestimate $f(1.1)$.
See graphs of $y = f'(x)$ and $y = f(x)$ for option D below:





Question 8

Answer: **A**

Explanation:

Let $u = e^x$
Then $\frac{du}{dx} = e^x$
and $e^x = u$

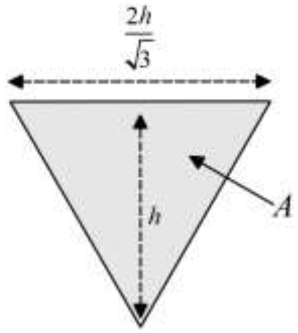
So $\int_0^1 e^{2x}\sqrt{e^x + 1}\ dx$
$= \int_{e^0}^{e^1} u \times \frac{du}{dx} \times \sqrt{u+1}\ dx$
$= \int_1^e u\sqrt{u+1}\ du$

Question 9

Answer: **D**

Explanation:

Let the area of the equilateral triangle with height h be A.
Using trigonometry, the "base" of the equilateral triangle must be $2 \times \frac{h}{\tan(60°)} = \frac{2h}{\sqrt{3}}$



So $A = \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h$
$= \frac{h^2}{\sqrt{3}}$

And the volume of the prism is:
$V = A \times 2$
$= \frac{2h^2}{\sqrt{3}}$

So $\frac{dV}{dh} = \frac{4h}{\sqrt{3}}$

Now, $\frac{dV}{dt} = \frac{1}{2}$ and
$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$
$= \frac{\sqrt{3}}{4h} \times \frac{1}{2}$
$= \frac{\sqrt{3}}{4}$ when $h = \frac{1}{2}$

Question 10

Answer: **E**

Explanation:

Graph each pair of functions on CAS, trying a few different values for k (or using a slider).

Option A is incorrect, because $y = \sec(x)$ does not intersect with $y = \mathrm{cosec}(x - \frac{\pi}{2})$
Option B is incorrect, because $y = \sec(x)$ does not intersect with $y = \cot(x - \frac{\pi}{2})$
Option C is incorrect, because $y = \sec(x)$ does not intersect with $y = \cos(x - \frac{\pi}{2})$
Option D is incorrect, because $y = \sec(x)$ does not intersect with $y = \cos^{-1}(x + \frac{\pi}{2})$
Option E is correct, because $y = \sec(x)$ does intersect with $y = \tan^{-1}(x + k)$ for all $k \in R$

Question 11

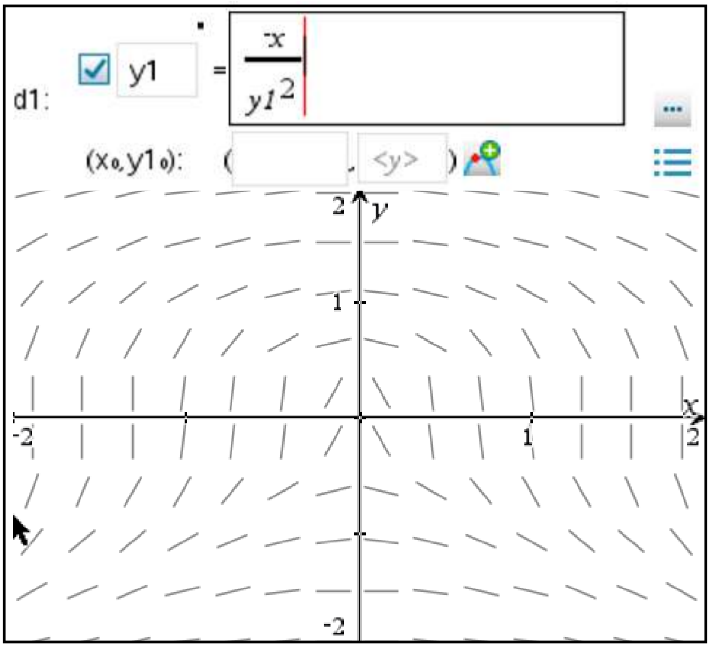
Answer: **E**

Explanation:

When $= 0$, $\frac{dy}{dx}$ is undefined (this is satisfied by all options)
When $= 0$, $\frac{dy}{dx} = 0$ (this is satisfied by all options)
When $x > 0$ and > 0 , $\frac{dy}{dx} < 0$ (this is only satisfied by options B and E)
When $x > 0$ and < 0 , $\frac{dy}{dx} < 0$ (this is only satisfied by option E)

To check, we can sketch option E using the CAS:

TI-nspire:



Question 12

Answer: **C**

Explanation:

Let $\underset{\sim}{u} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{k}$,
then the resolute of $\underset{\sim}{u}$ parallel to $\underset{\sim}{v}$ is $\left(\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}}\right)\hat{\underset{\sim}{v}}$
Now, $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{2}}(\underset{\sim}{i} + \underset{\sim}{k})$
and $\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}} = \frac{3}{\sqrt{2}}$
so $\left(\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}}\right)\hat{\underset{\sim}{v}} = \frac{3}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\left(\underset{\sim}{i} + \underset{\sim}{k}\right)$
$= \frac{3}{2}\underset{\sim}{i} + \frac{3}{2}\underset{\sim}{k}$

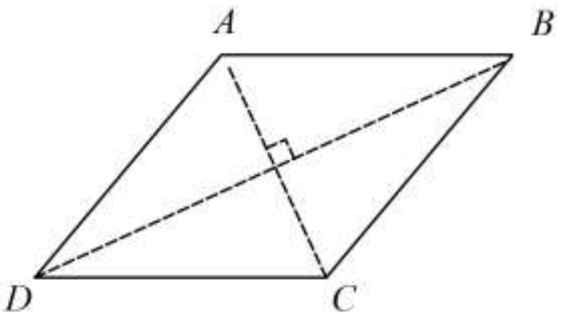
Question 13

Answer: **A**

Explanation:

The condition $\overrightarrow{AB} \cdot \overrightarrow{AD} \neq 0$ means that vertex A is not a right angle.
$\overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BC} + \overrightarrow{CD}$ are the diagonals of the quadrilateral.
The condition $\left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(\overrightarrow{BC} + \overrightarrow{CD}\right) = 0$ means that the diagonals are perpendicular.

Drawing a diagram, we can see that the quadrilateral could be a rhombus, but not a square.



Question 14

Answer: **D**

Explanation:

The vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ must be linearly independent.
This means that there must **not** be constants p and q such that $p\underset{\sim}{a} + q\underset{\sim}{b} = \underset{\sim}{c}$.
We could test each option, but if we notice that option A is $\underset{\sim}{a} + \underset{\sim}{b}$ and option B is $\underset{\sim}{a} - \underset{\sim}{b}$, we can rule these out and start testing from option C:

Solving $p\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) + q\left(3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right) = \underset{\sim}{i} + 6\underset{\sim}{j} - \underset{\sim}{k}$
gives $p = -2$ and $q = 1$ (vectors are linearly dependent).

Try option D:
Solving $p\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) + q\left(3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right) = -2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$
gives no solutions (vectors are linearly independent).

TI-nspire:

```
solve(p·[1  -2  1]+q·[3  2  1]=[1  6  -1],
                                p=-2 and q=1
solve(p·[1  -2  1]+q·[3  2  1]=[-2  1  1],
                                       false
```

Question 15

Answer: **B**

Explanation:

$$a = \sqrt{x-3}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \sqrt{x-3}$$
$$\frac{1}{2}v^2 = \frac{2}{3}(x-3)^{\frac{3}{2}} + c$$

Substituting $x = 7$, $v = 1$ and solving gives $c = -\frac{29}{6}$

$$\frac{1}{2}v^2 = \frac{2}{3}(x-3)^{\frac{3}{2}} - \frac{29}{6}$$

Substituting $x = 12$ and solving gives $v = \frac{\sqrt{237}}{3}$
(note that $v > 0$ because of the initial condition)

Question 16

Answer: **B**

Explanation:

Resolving forces parallel to the plane, given that the objects are in equilibrium, we have:

$M_1 \sin(45°) = M_2 \sin(30°)$

$\frac{\sqrt{2}}{2} M_1 = \frac{1}{2} M_2$

$\frac{M_1}{M_2} = \frac{1}{\sqrt{2}}$

$\frac{M_1}{M_2} = \frac{\sqrt{2}}{2}$

So $M_1 : M_2 = \sqrt{2} : 2$

As a logical check, we can notice that as M_1 is on a steeper slope, M_2 must be the heavier object to keep the system in equilibrium.

Question 17

Answer: **D**

Explanation:

Let $\underset{\sim}{i}$ be a unit vector in the direction of the $5N$ force, and $\underset{\sim}{j}$ be a unit vector perpendicular to the $5N$ force (90° anti-clockwise).

We can then represent the net force as:

$$(5 + 3\cos(130°) + 4\cos(-140°))\underset{\sim}{i} + (3\sin(130°) + 4\sin(-140°))\underset{\sim}{j}$$

$$\approx 0.008\underset{\sim}{i} - 0.273\underset{\sim}{j}$$

which gives a direction between the $4N$ and $5N$ force

Question 18

Answer: **A**

Explanation:

The sample mean must be in the centre of the confidence interval, so $\bar{x} = \frac{35.32 + 41.58}{2} = 38.45$

For a 99% confidence interval, $z = 2.5758$

(using the standard normal $Z \sim N(0,1)$ to solve $\Pr(Z < z) = 0.995$ for z)

Now, $38.45 + 2.5758 \times \frac{s}{\sqrt{50}} = 41.58$

$s = 8.59$

Question 19

Answer: **D**

Explanation:

$$SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$
$$= \frac{15}{\sqrt{25}}$$
$$= 3$$

Calculating the (two-tailed) p values for each option:

A. $\bar{x} = 107,\ p = 2 \times \Pr\left(Z < \frac{107-120}{3}\right) \approx 0.00001$
B. $\bar{x} = 109,\ p = 2 \times \Pr\left(Z < \frac{109-120}{3}\right) \approx 0.0002$
C. $\bar{x} = 111,\ p = 2 \times \Pr\left(Z < \frac{111-120}{3}\right) \approx 0.003$
D. $\bar{x} = 113,\ p = 2 \times \Pr\left(Z < \frac{113-120}{3}\right) \approx 0.020$
E. $\bar{x} = 115,\ p = 2 \times \Pr\left(Z < \frac{115-120}{3}\right) \approx 0.096$

Option D is the only option with a p value between 0.01 and 0.05

Question 20

Answer: **D**

Explanation:

The distance the ball has travelled is given by:
$X = Ut + \frac{1}{2}at^2$,
where $U \sim N(5, 0.5^2)$
and $-0.1 = 0.1a$
$a = -1$

So X $= Ut - \frac{1}{2}t^2$,
and after 2 seconds:
$X = 2U - 2$

For the ball to travel at least 10m,
$2U - 2 > 10$
$U > 6$
Using the normal random variable $U \sim N(5, 0.5^2)$:

$Pr(U > 6) = 0.023$

SECTION B: Extended response questions

Question 1

a. Using CAS:

$f'(x) = \frac{-2x}{(x-1)^3}$ (1 mark)

For stationary points:

$f'(x) = 0$

$x = 0$

Coordinates are $(0, -1)$ (1 mark)

b. $f''(x) = \frac{2(2x+1)}{(x-1)^4}$ (1 mark)

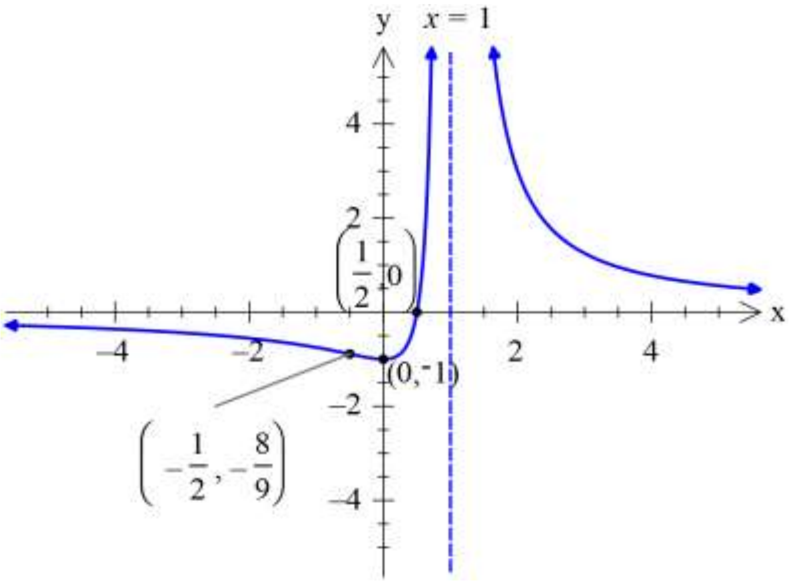
For points of inflection:

$f''(x) = 0$

$x = -\frac{1}{2}$

Coordinates are $(-\frac{1}{2}, -\frac{8}{9})$ (1 mark)

c.



Graph shape (1 mark)
Intercept and asymptote (1 mark)
Stationary point and inflection point (1 mark)

d. $f(x) = \frac{2}{(x-1)} + \frac{1}{(x-1)^2}$ (1 mark)
(Using the 'expand' function on CAS)

e. f is below the y axis in the required interval.

Area = $\int_{\frac{1}{2}}^{-2} \frac{2}{(x-1)} + \frac{1}{(x-1)^2}\, dx$ (1 mark)

$= 2\log_e 6 - \frac{5}{3}$ square units (1 mark)

Question 2

a. $z^5 = 32$

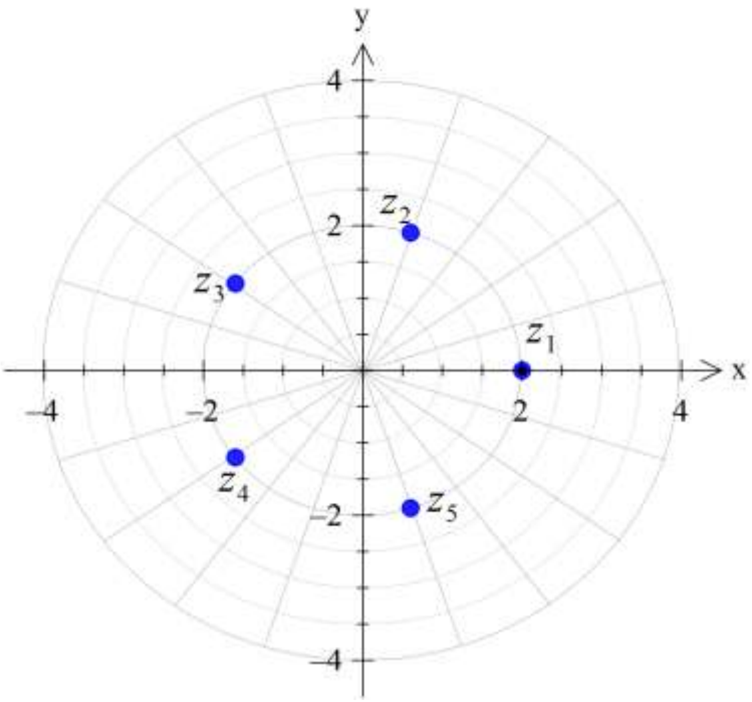
$= 2^5 \text{cis}(0 + 2k\pi)\ ,\ k \in Z$

$z = 2\text{cis}(\frac{2k\pi}{5})\ ,\ k \in Z$

$= 2\text{cis}(0)\ ,\ 2\text{cis}\left(\frac{2\pi}{5}\right)\ ,\ 2\text{cis}\left(\frac{4\pi}{5}\right)\ ,\ 2\text{cis}\left(\frac{6\pi}{5}\right)\ ,\ 2\text{cis}(\frac{8\pi}{5})$

(1 mark for at least 2 correct solutions)
(2 marks for all 5 correct solutions)

b.



(1 mark)

c. $\theta = \tan^{-1}\left(\frac{2\pi}{5}\right)$ (1 mark)

d. <u>Method 1 – Geometrically</u>

Notice that the required line is the perpendicular bisector of z_2 and z_3,
Which will also pass through the origin and z_5. (1 mark)

This line has equation $y = mx$, where the gradient $m = \dfrac{\sin\left(\frac{8\pi}{5}\right)}{\cos\left(\frac{8\pi}{5}\right)}$

$y = \frac{-1}{4}x\left(\sqrt{5}+1\right)\sqrt{2\left(\sqrt{5}+5\right)}$ (1 mark)

<u>Method 2 – Algebraically</u>

$\left|z - 2\text{cis}\left(\frac{2\pi}{5}\right)\right| = \left|z - 2\text{cis}\left(\frac{4\pi}{5}\right)\right|$ (1 mark)

$$\left|x + yi - 2\cos\left(\frac{2\pi}{5}\right) - 2i\sin\left(\frac{2\pi}{5}\right)\right| = \left|x + yi - 2\cos\left(\frac{4\pi}{5}\right) - 2i\sin\left(\frac{4\pi}{5}\right)\right|$$

$$\left(x - 2\cos\left(\frac{2\pi}{5}\right)\right)^2 + \left(y - 2\sin\left(\frac{2\pi}{5}\right)\right)^2 = \left(x - 2\cos\left(\frac{4\pi}{5}\right)\right)^2 + \left(y - 2\sin\left(\frac{4\pi}{5}\right)\right)^2$$

Solving for y gives:

$y = \dfrac{\sqrt{10}x}{\sqrt{5-\sqrt{5}}-\sqrt{5+\sqrt{5}}}$ (1 mark)

(Note that this is equivalent to the equation in the Method 1 solution)

The algebraic method is made easier by defining $cis(\theta) = \cos(\theta) + i\sin(\theta)$ on CAS:

TI-nspire:

Define $z=x+y\cdot i$	*Done*
Define $cis(\theta)=\cos(\theta)+i\cdot\sin(\theta)$	*Done*
$\text{solve}\left(\left\|z-2\cdot cis\left(\frac{2\cdot\pi}{5}\right)\right\|=\left\|z-2\cdot cis\left(\frac{4\cdot\pi}{5}\right)\right\|,y\right)$	$y=\dfrac{\sqrt{10}\cdot x}{\sqrt{5-\sqrt{5}}-\sqrt{\sqrt{5}+5}}$

or by using the exponential notation $cis(\theta) = e^{i\theta}$:

Define $z=x+y\cdot i$	*Done*
$\text{solve}\left(\left\|z-2\cdot e^{\frac{2\cdot\pi\cdot i}{5}}\right\|=\left\|z-2\cdot e^{\frac{4\cdot\pi\cdot i}{5}}\right\|,y\right)$	$y=\dfrac{\sqrt{10}\cdot x}{\sqrt{5-\sqrt{5}}-\sqrt{\sqrt{5}+5}}$

e. There are 5 possible lines (1 mark)

(Notice that each of the required lines is a perpendicular bisector of z_a and z_b, which will also pass through the origin and one of the other 5 roots).

f. The centre of the circle must be equally distant from z_2 and z_5, so the centre must lie on the x axis.

Now find the point $(w, 0)$ on the x axis that is equally distant from z_2 and the origin:

$$w = \sqrt{\left(w - \cos\left(\frac{2\pi}{5}\right)\right)^2 + \left(\sin\left(\frac{2\pi}{5}\right)\right)^2}$$ (1 mark)

$$w = \frac{\sqrt{5}+1}{2}$$

Now, w is also the radius of the circle, so the equation is:

$$\left|z - \frac{\sqrt{5}+1}{2}\right| = \frac{\sqrt{5}+1}{2}$$ (1 mark)

g. The points z_1, z_2, z_3, z_4 and z_5 lie on a circle with radius 2.
To transform them to a circle with radius $\frac{\sqrt{5}+1}{2}$,
we require $p = \frac{\sqrt{5}+1}{4}$. (1 mark)

The points z_1, z_2, z_3, z_4 and z_5 lie on a circle with centre $(0,0)$.
To transform them to a circle with centre $\left(\frac{\sqrt{5}+1}{2}, 0\right)$,
we require $q = \frac{\sqrt{5}+1}{2}$. (1 mark)

h. The factorised form of the polynomial is:

$$(z - z_1)(z - z_2)(z - z_5)$$
$$= (z - 2)\left(z - 2\text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - 2\text{cis}\left(\frac{8\pi}{5}\right)\right)$$ (1 mark)

Expanding on CAS gives:

$$z^3 - \left(\sqrt{5} + 1\right)z^2 + 2\sqrt{5}z + 2z - 8$$ (1 mark)

i. There is **not** a cubic polynomial with real coefficients whose roots are z_2, z_3, and z_4.
For polynomials with real coefficients, roots must occur in conjugate pairs.

(1 mark for answer with justification)

Question 3

a. Let $X \sim N(3000, 250^2)$
$\Pr(X < 2920) = 0.3745$ (1 mark)

b. H_0: $\mu = 3000$
H_1: $\mu < 3000$ (1 mark for both)

c. Let $\bar{X} \sim N\left(3000, \left(\frac{250}{\sqrt{50}}\right)^2\right)$
$p = \Pr(\bar{X} < 2920)$
$= 0.0118$ (1 mark)

$p > 0.01$, so accept advertiser's claim (1 mark)

d. Require $p \geq 0.01$

Using the inverse normal function on CAS:
$\Pr(Z < a) = 0.01$
$\Rightarrow \quad a = -2.3263$ (1 mark)
(where Z is the standard normal variable)

Using the standardisation formula:

$$Z = \frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$

$$-2.3263 = \frac{2920 - 3000}{\frac{250}{\sqrt{n}}}$$

$$n = 52.85$$

n must be a whole number, so $n = 52$ is the largest sample size for which the student would accept the advertised claim. (1 mark)

e. For a Type I error, the null hypothesis must be true, so $\mu = 3000$, and the null hypothesis must be rejected, so $p < 0.01$ (1 mark)

$\Pr(\text{type I error}) = 0.01$ (1 mark)
(because she is testing at the 1% significance level)

f. If the true population mean is actually $\mu_2 = 2900$, the student will make a Type II error if she does not reject the null hypothesis (ie. if $p \geq 0.01$).

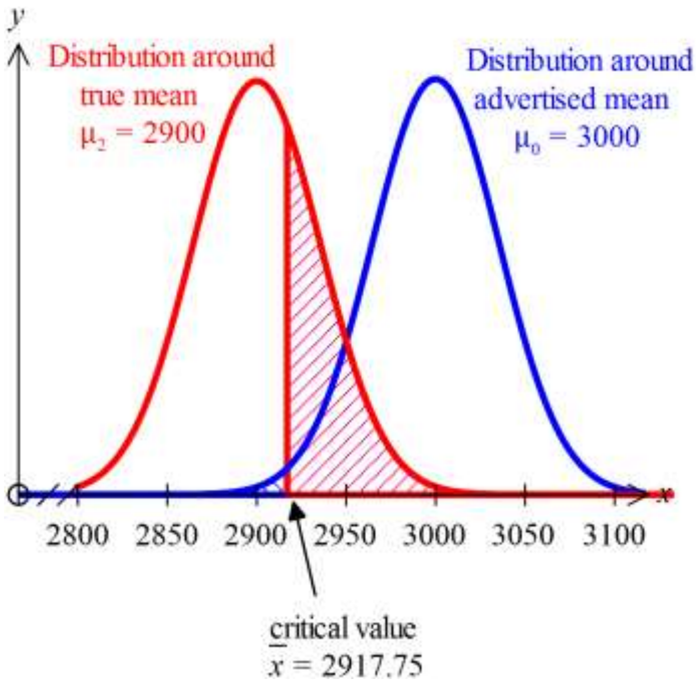
This will happen if $\bar{X}$ is above the "critical value" $\bar{X}_k$, which can be found by:

$$\Pr(Z < a) = 0.01$$
$$\Rightarrow \quad a = -2.3263$$

$$-2.3263 = \frac{\bar{X}_k - 3000}{\frac{250}{\sqrt{50}}}$$
$$\bar{X}_k = 2917.75$$ (1 mark)

Now we find $\Pr(\bar{X} > 2917.75)$, using the true mean $\mu_2 = 2900$:

$$\Pr(\bar{X} > 2917.75) = \Pr\left(Z > \frac{2917.75 - 2900}{\frac{250}{\sqrt{50}}}\right)$$
$$= 0.3078$$ (1 mark)



g. Yes, the student could still perform the statistical test, because even if the original distribution of lightbulb lifetimes is not normal, the distribution of sample means for a sample of 50 will still be normal (using the central limit theorem).

(1 mark)

Question 4

a. $\frac{dP}{dt} = \frac{100-P}{10}$

$\frac{dt}{dP} = \frac{10}{100-P}$

$t = -10\log_e|100-P| + c$ (1 mark)

$\frac{t-c}{-10} = \log_e|100-P|$

$e^{\frac{t-c}{-10}} = |100-P|$

$100 - P = \pm e^{\frac{t-c}{-10}}$ (1 mark)

$100 - P = \pm e^{\frac{-t}{10}} \times e^{\frac{c}{10}}$

$100 - P = Ae^{\frac{-t}{10}}$, where $A = \pm e^{\frac{c}{10}}$ (1 mark)

$P = Ae^{\frac{-t}{10}} + 100$

b. $4 = Ae^{\frac{-0}{10}} + 100$

$A = -96$ (1 mark)

c. $P = \frac{100}{1+24e^{\frac{-t}{10}}}$

$= 100\left(1 + 24e^{\frac{-t}{10}}\right)^{-1}$

Using differentiation:

$\frac{dP}{dt} = -100\left(1 + 24e^{\frac{-t}{10}}\right)^{-2} \times \left(-\frac{24}{10}e^{\frac{-t}{10}}\right)$

$= 240e^{\frac{-t}{10}}\left(1 + 24e^{\frac{-t}{10}}\right)^{-2}$ (1 mark)

We need to show this satisfies the differential equation $\frac{dP}{dt} = \frac{P}{10}\left(\frac{100-P}{100}\right)$

Substituting P into the right-hand side:

$RHS = \frac{P}{10}\left(\frac{100-P}{100}\right)$

$= \frac{10}{1+24e^{\frac{-t}{10}}} \times \frac{100 - \frac{100}{1+24e^{\frac{-t}{10}}}}{100}$

$= \frac{10}{1+24e^{\frac{-t}{10}}} \times \frac{100\left(1+24e^{\frac{-t}{10}}\right) - 100}{100\left(1+24e^{\frac{-t}{10}}\right)}$

$= \frac{10}{1+24e^{\frac{-t}{10}}} \times \frac{24e^{\frac{-t}{10}}}{\left(1+24e^{\frac{-t}{10}}\right)}$ (1 mark for adequate proof)

$= 240e^{\frac{-t}{10}}\left(1 + 24e^{\frac{-t}{10}}\right)^{-2}$

$= \frac{dP}{dt}$ as required

To show the equation satisfies the initial condition:

$P(0) = \frac{100}{1+24e^0}$ (1 mark)

$= 4$ as required

d. $P(30) = \frac{100}{1+24e^{-\frac{30}{10}}}$

≈ 45.56

$\approx 46\%$ (1 mark)

e. $\frac{dP}{dt} = \frac{P}{10}\left(\frac{100-P}{100}\right)$

$\frac{dt}{dP} = \frac{10}{P}\left(\frac{100}{100-P}\right)$

$t = \int_4^{75} \frac{10}{P}\left(\frac{100}{100-P}\right) dP$ (1 mark)

$= 42.77$

It takes 43 days for at least 75% to see the video (1 mark)

f. <u>Method 1 – Using implicit differentiation</u>

Subscribers are increasing at a maximum rate when $\frac{d^2P}{dt^2} = 0$

Using implicit differentiation:

$\frac{d^2P}{dt^2} = \left(\frac{1}{10}\left(\frac{100-P}{100}\right) + \frac{P}{10}\left(\frac{-1}{100}\right)\right)\frac{dP}{dt}$

$0 = \frac{1}{10}\left(\frac{100-P}{100}\right) + \frac{P}{10}\left(\frac{-1}{100}\right)$

$P = 50$ (1 mark)

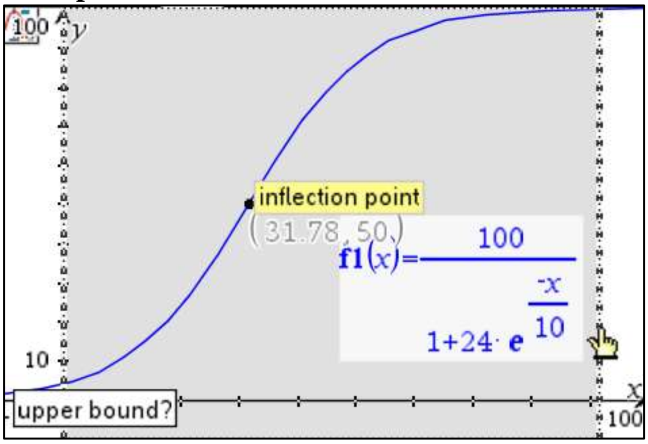
Then solve $P(t) = 50$ for t:

$\frac{100}{1+24e^{\frac{-t}{10}}} = 50$

$t = 31.78$ (1 mark)

Method 2 – Using CAS to find inflection point of $P(t)$
Subscribers are increasing at a maximum rate at point of inflection.

TI-nspire:



$t = 31.78$ (1 mark)
$P = 50$ (1 mark)

Question 5

a. $\underset{\sim}{A}(0) = 25\underset{\sim}{i}$
Starting point is $(25,0)$ (1 mark)

b. Let $\underset{\sim}{A}(t) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$,
which gives two equations:

$$\begin{cases} 25\cos\left(\frac{\pi t}{30}\right) = 0 \\ 10\sin\left(\frac{\pi t}{15}\right) = 0 \end{cases}, \text{ for } t \in [0,60]$$

Solving on CAS gives
$t = 15\text{s}$ and $t = 45\text{s}$ (1 mark for each solution)

c. Speed = $\left|\underset{\sim}{A}'(t)\right|$

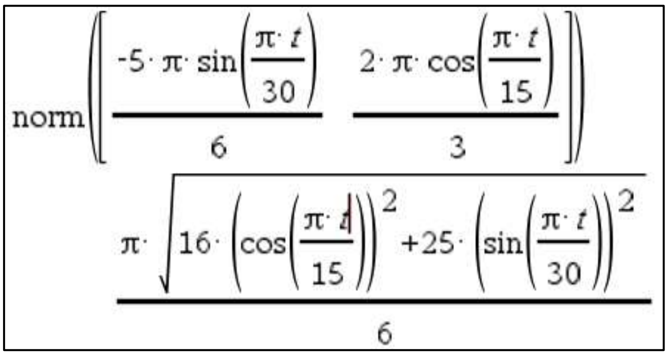
$$= \left|\left(\frac{-5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\right)\underset{\sim}{i} + \left(\frac{2\pi}{3}\cos\left(\frac{\pi t}{15}\right)\right)\underset{\sim}{j}\right| \quad \text{(1 mark)}$$

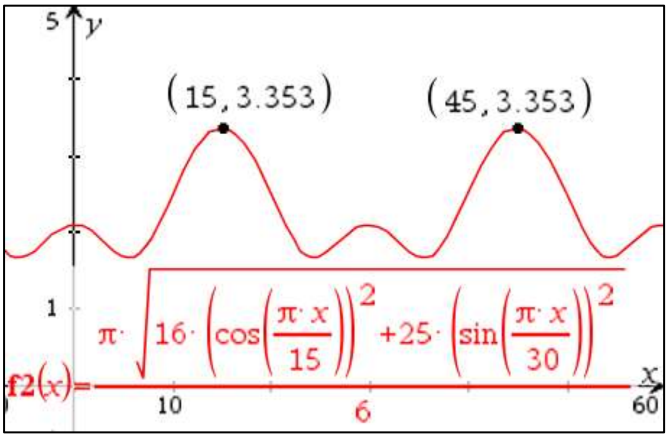
$$= \sqrt{\left(\frac{-5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\right)^2 + \left(\frac{2\pi}{3}\cos\left(\frac{\pi t}{15}\right)\right)^2}$$

Sketching this function on the CAS,
we find a maximum speed of 3.53m/s, (1 mark)
when $t = 15$ s and $t = 45$ s (1 mark)

The "norm" function on CAS can be used to find the length of a vector.

TI-nspire:





d. Distance = $\int_0^{60} \sqrt{\left(\frac{-5\pi}{6}\sin\left(\frac{\pi t}{30}\right)\right)^2 + \left(\frac{2\pi}{3}\cos\left(\frac{\pi t}{15}\right)\right)^2}\, dt$ (1 mark)

$= 137.97$ m (1 mark)

e. $\underset{\sim}{M}'(t) = -2\underset{\sim}{i} + 4\underset{\sim}{j} + \frac{-49}{20}(4t - 145)\underset{\sim}{k}$

$\underset{\sim}{M}'(35) = -2\underset{\sim}{i} + 4\underset{\sim}{j} + 12.25\underset{\sim}{k}$

Initial speed $= \left|\underset{\sim}{M}'(35)\right|$

$= 13.04$ m/s (1 mark)

Angle of elevation $= \sin^{-1}\left(\frac{12.25}{13.04}\right)$

$\approx 70°$ (1 mark)

f. Missile hits the ground when

$\frac{49}{20}(35 - t)(2t - 75) = 0$ for $t > 35$

$t = 37.5$ s

So $a = 37.5$ (1 mark)

Landing position is:

$\underset{\sim}{M}(37.5) = -15\underset{\sim}{i} + 10\underset{\sim}{j}$ (1 mark)

Position of Tank A is:

$\underset{\sim}{A}(37.5) = \frac{-25\sqrt{2}}{2}\underset{\sim}{i} + 10\underset{\sim}{j}$

So the distance from Tank A is:

$\frac{25\sqrt{2}}{2} - 15 = 2.68$ m (1 mark)