SPECIALIST MATHEMATICS

Written examination 1

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2018 Trial Examination

SOLUTIONS

Question 1

a. $\sin\theta = \frac{1}{\sqrt{1^2+7^2}}$

$= \frac{1}{\sqrt{50}}$

$= \frac{1}{5\sqrt{2}}$ 1 mark

$= \frac{\sqrt{2}}{10}$

b. Equation of motion parallel to the plane:

$F - 10g\sin\theta = 10\sqrt{2}$ 1 mark

$F = 10\sqrt{2} + 10g\left(\frac{\sqrt{2}}{10}\right)$

$= \sqrt{2}(10 + g)$ 1 mark

Question 2

$\sin(x)\sin(y) = \frac{1}{2}$

To find y value when $= \frac{\pi}{4}$:

$\sin\left(\frac{\pi}{4}\right)\sin(y) = \frac{1}{2}$

$\frac{1}{\sqrt{2}}\sin(y) = \frac{1}{2}$

$\sin(y) = \frac{\sqrt{2}}{2}$

$y = \frac{\pi}{4} + 2n\pi$ or $\frac{3\pi}{4} + 2n\pi$, $n \in Z$ (1 mark)

(accept $y = \frac{\pi}{4}$ or $\frac{3\pi}{4}$)

Implicit differentiation (using product rule):

$\cos(x)\sin(y) + \sin(x)\cos(y)\,\frac{dy}{dx} = 0$

Sub $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$:

$\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\frac{dy}{dx} = 0$

$\frac{1}{2} + \frac{1}{2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -1$ (1 mark)

Sub $x = \frac{\pi}{4}$ and $y = \frac{3\pi}{4}$:

$\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right)\frac{dy}{dx} = 0$

$\frac{1}{2} - \frac{1}{2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 1$ (1 mark)

So the possible values of the gradient are 1 or -1

Question 3

$z^3 = (1-i)^6$

$z^3 = \left(\sqrt{2}\text{cis}\left(\frac{-\pi}{4}\right)\right)^6$

$z^3 = 2^3\text{cis}\left(\frac{-6\pi}{4} + 2\pi k\right)$, k ∈ Z (1 mark)

$z^3 = 2^3\text{cis}\left(\frac{-3\pi}{2}\right)$, $z^3 = 2^3\text{cis}\left(\frac{\pi}{2}\right)$, $z^3 = 2^3\text{cis}\left(\frac{5\pi}{2}\right)$

$z = 2\text{cis}\left(\frac{-\pi}{2}\right)$, $z = 2\text{cis}\left(\frac{\pi}{6}\right)$, $z = 2\text{cis}\left(\frac{5\pi}{6}\right)$ (1 mark)

$z = -2i$, $z = \sqrt{3} + i$, $z = -\sqrt{3} + i$ (1 mark)

Question 4

a. $\overrightarrow{AQ} = \overrightarrow{AO} + \overrightarrow{OQ}$

$= -\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$ (1 mark)

b. $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \underset{\sim}{a} + \frac{1}{2}\overrightarrow{AB}$

$= \underset{\sim}{a} + \frac{1}{2}(-\underset{\sim}{a} + \underset{\sim}{b})$

$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$ (1 mark)

c. $\overrightarrow{AX} = k\overrightarrow{AQ}$

$= k(-\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b})$

$= -k\underset{\sim}{a} + \frac{1}{3}k\underset{\sim}{b}$ (1 mark)

Also $\overrightarrow{AX} = \overrightarrow{AO} + m\overrightarrow{OP}$ for some $m \in R$

$= -\underset{\sim}{a} + m(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b})$

$= \left(\frac{1}{2}m - 1\right)\underset{\sim}{a} + \frac{1}{2}m\underset{\sim}{b}$ (1 mark)

Equating the two vectors, we have:

$$-k\underset{\sim}{a} + \frac{1}{3}k\underset{\sim}{b} = \left(\frac{1}{2}m - 1\right)\underset{\sim}{a} + \frac{1}{2}m\underset{\sim}{b}$$

Because $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent, we can equate $\underset{\sim}{a}$ and $\underset{\sim}{b}$ components:

$-k = \frac{1}{2}m - 1$ …… (1)

and $\frac{1}{3}k = \frac{1}{2}m$ …… (2)

Equation (2) gives $k = \frac{3}{2}m$,

Then equation (1) gives:

$-\frac{3}{2}m = \frac{1}{2}m - 1$

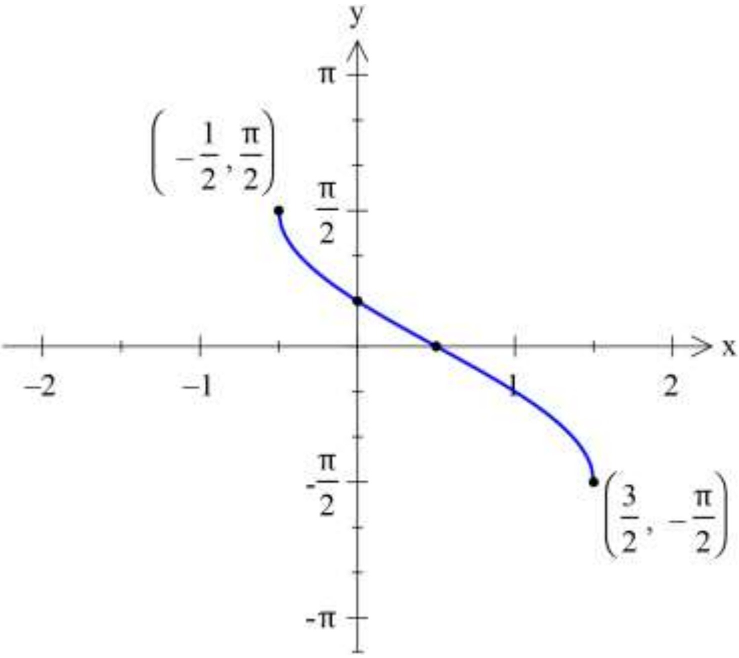
$m = \frac{1}{2}$

So $k = \frac{3}{2} \times \frac{1}{2}$

$k = \frac{3}{4}$ (1 mark)

Question 5

a.



Endpoints labelled (1 mark)
Graph shape & position of intercepts (1 mark)

b. Volume $= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} x^2 \, dy$

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{1}{2} - \sin(y)\right)^2 dy$ (1 mark)

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{1}{4} - \sin(y) + \sin^2(y)\right) \, dy$

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{1}{4} - \sin(y) + \frac{1}{2} - \frac{1}{2}\cos(2y)\right) \, dy$ (1 mark)

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{3}{4} - \sin(y) - \frac{1}{2}\cos(2y)\right) \, dy$

$= \pi \left[\frac{3y}{4} + \cos(y) - \frac{1}{4}\sin(2y)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$

$= \pi \left(\left(\frac{\pi}{8} + \frac{\sqrt{3}}{2} - \frac{1}{4} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{-3\pi}{8} + 0 - 0\right)\right)$

$= \pi \left(\frac{\pi}{8} + \frac{4\sqrt{3}}{8} - \frac{\sqrt{3}}{8} + \frac{3\pi}{8}\right)$

$= \frac{\pi^2}{2} + \frac{3\sqrt{3}\pi}{8}$ (cubic units) (1 mark)

Question 6

a. D is normally distributed with

$$E(D) = E(A) - E(B) = 14.7 - 13.7 = 1$$

and $Var(D) = Var(A) + Var(B) = 0.8^2 + 0.6^2 = 1$

so $SD(D) = 1$ (1 mark)

For the Team A player to be faster:

$$\Pr(D < 0) = \Pr(Z < -1) \approx 0.16$$ (1 mark)

b. For a random sample of four players from Team A:

$$E(\bar{A}) = 14.7$$

$$SD(\bar{A}) = \frac{0.8}{\sqrt{4}} = 0.4$$

For a random sample of four players from Team B:

$$E(\bar{B}) = 13.7$$

$$SD(\bar{B}) = \frac{0.6}{\sqrt{4}} = 0.3$$

Now let $\bar{D} = \bar{A} - \bar{B}$

$$E(\bar{D}) = 14.7 - 13.7 = 1$$

and $Var(\bar{D}) = 0.4^2 + 0.3^2 = 0.25$

So $SD(\bar{D}) = 0.5$ (1 mark)

Alternatively, take the variable D from part **a**, then for a sample of four differences:

$$E(\bar{D}) = E(D) = 1$$

and $SD(\bar{D}) = \frac{SD(D)}{\sqrt{2}} = 0.5$

$$\Pr(\bar{D} < 0) = \Pr(Z < -2) \approx 0.025$$ (1 mark)

Question 7

$\frac{dy}{dx} = \frac{y-2}{1+4x^2}$

Using separation of variables:

$\int \frac{1}{y-2}\, dy = \int \frac{1}{1+4x^2}\, dx$ (1 mark)

$\log_e|y-2| = \frac{1}{2}\tan^{-1}(2x) + c$ (1 mark)

Method 1. Substitute initial condition before rearranging:

Substitute $x = -\frac{1}{2}\ ,\ y = 1$

$\log_e|-1| = \frac{1}{2}\tan^{-1}(-1) + c$

$0 = \frac{-\pi}{8} + c$

$c = \frac{\pi}{8}$ (1 mark)

$\log_e(2-y) = \frac{1}{2}\tan^{-1}(2x) + \frac{\pi}{8}$ (1 mark)

(LHS is $\log_e(2-y)$ because $y < 2$ in the initial condition)

$y = 2 - e^{\left(\frac{1}{2}\tan^{-1}(2x)+\frac{\pi}{8}\right)}$ (1 mark)

Method 2. Rearrange before substituting initial condition:

$\log_e|y-2| = \frac{1}{2}\tan^{-1}(2x) + c$

$y - 2 = \pm\, e^{\frac{1}{2}\tan^{-1}(2x)+c}$

$y = 2 + Ae^{\frac{1}{2}\tan^{-1}(2x)}$ (1 mark)

Substitute $x = -\frac{1}{2}\ ,\ y = 1$

$1 = Ae^{\frac{1}{2}\tan^{-1}(-1)} + 2$

$-1 = Ae^{\frac{-\pi}{8}}$

$A = -e^{\frac{\pi}{8}}$ (1 mark)

$y = 2 - e^{\frac{\pi}{8}}\, e^{\frac{1}{2}\tan^{-1}(2x)}$

$y = 2 - e^{\left(\frac{1}{2}\tan^{-1}(2x)+\frac{\pi}{8}\right)}$ (1 mark)

Question 8

a. $f(x) = \frac{\sqrt{x}}{x-2}$

Using the quotient rule:

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x-2) - \sqrt{x}}{(x-2)^2} \quad \text{(1 mark)}$$

$$= \frac{(x-2)-2x}{2\sqrt{x}(x-2)^2}$$

$$= \frac{-x-2}{2\sqrt{x}(x-2)^2}$$

For stationary points:

$$0 = \frac{-x-2}{2\sqrt{x}(x-2)^2}$$

This has no solutions, because numerator is 0 when $x = -2$ only,
but the denominator is not defined for $x < 0$.
So there are no stationary points. (1 mark)

b. $f'(x) = \frac{-x-2}{2\sqrt{x}(x-2)^2}$

Using the quotient and product rules:

$$f''(x) = \frac{-2\sqrt{x}(x-2)^2 - (-x-2)\left(\frac{1}{\sqrt{x}}(x-2)^2 + 4\sqrt{x}(x-2)\right)}{4x(x-2)^4} \quad \text{(1 mark)}$$

Multiply by $\sqrt{x}$ on numerator and denominator:

$$f''(x) = \frac{-2x(x-2)^2 - (-x-2)\left((x-2)^2 + 4x(x-2)\right)}{4x\sqrt{x}(x-2)^4}$$

Divide by $(x-2)$ on numerator and denominator:

$$f''(x) = \frac{-2x(x-2) - (-x-2)\left((x-2) + 4x\right)}{4x\sqrt{x}(x-2)^3}$$

$$= \frac{-2x^2 + 4x + (x+2)(5x-2)}{4x\sqrt{x}(x-2)^3}$$

$$= \frac{-2x^2 + 4x + 5x^2 + 8x - 4}{4x\sqrt{x}(x-2)^3}$$

$$= \frac{3x^2 + 12x - 4}{4x\sqrt{x}(x-2)^3} \quad \text{(1 mark)}$$

$f''(x) = 0$ when numerator = 0

$3x^2 + 12x - 4 = 0$ (1 mark)

$x^2 + 4x - \frac{4}{3} = 0$

$(x+2)^2 - \frac{16}{3} = 0$

$x + 2 = \pm\frac{4}{\sqrt{3}}$

$x = -2 \pm \frac{4\sqrt{3}}{3}$

We require $x > 0$, so the only point of inflection occurs at

$x = -2 + \frac{4\sqrt{3}}{3}$ (1 mark)

To verify that the concavity changes at $x = -2 + \frac{4\sqrt{3}}{3}$ (which is between 0 and 2)

we can notice that $f''(x) = \frac{3x^2+12x-4}{4x\sqrt{x}(x-2)^3}$ has a negative denominator for all $x \in (0,2)$,

but the numerator changes sign at $x = -2 + \frac{4\sqrt{3}}{3}$, so $f''(x)$ changes sign at $x = -2 + \frac{4\sqrt{3}}{3}$.

Question 9

a. $\int \frac{\log_e|x-1|}{x-1}\,dx$

$= \int u\,du$ where $u = \log_e|x-1|$, $\frac{du}{dx} = \frac{1}{x-1}$

$= \frac{1}{2}u^2 + c$

$= \frac{1}{2}(\log_e|x-1|)^2$ (1 mark)

(where $c = 0$)

b. $\frac{\log_e|x-1|}{x-1} = 0$

$\log_e|x-1| = 0$

$x - 1 = \pm 1$

$x = 0$ or $x = 2$ (1 mark)

c. Area = $\int_0^{-2} \frac{\log_e|x-1|}{x-1} dx$

(reverse the terminals because the area is below the x axis)

Area = $\left[\frac{1}{2}\left(\log_e|x-1|\right)^2\right]_0^{-2}$

$= \frac{1}{2}\left(\log_e 3\right)^2 - \frac{1}{2}\left(\log_e 1\right)^2$

$= \frac{1}{2}\left(\log_e 3\right)^2$ (1 mark)

d.

$$\int_0^a \frac{\log_e|x-1|}{x-1} dx = \int_2^k \frac{\log_e|x-1|}{x-1} dx$$

$$\left[\frac{1}{2}\left(\log_e|x-1|\right)^2\right]_0^a = \left[\frac{1}{2}\left(\log_e|x-1|\right)^2\right]_2^k$$

$$\frac{1}{2}\left(\log_e|a-1|\right)^2 - \frac{1}{2}\left(\log_e 1\right)^2 = \frac{1}{2}\left(\log_e|k-1|\right)^2 - \frac{1}{2}\left(\log_e 1\right)^2$$

$$\frac{1}{2}\left(\log_e|a-1|\right)^2 = \frac{1}{2}\left(\log_e|k-1|\right)^2$$

$$\left(\log_e(1-a)\right)^2 = \left(\log_e(k-1)\right)^2$$ (1 mark)

$$\log_e(1-a) = \pm\log_e(k-1)$$

$$\log_e(1-a) = \log_e(k-1) \quad or \quad \log_e(1-a) = -\log_e(k-1)$$

$$1-a = k-1 \quad or \quad 1-a = \frac{1}{k-1}$$

$$k = 2-a \quad or \quad k = 1 + \frac{1}{1-a}$$ (2 marks)

(1 for each solution)