Neap:

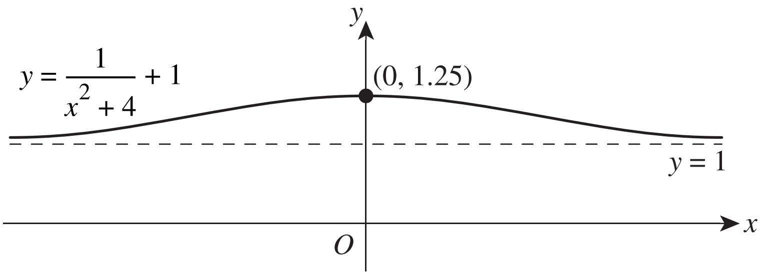
Trial Examination 2018

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (3 marks)



correct shape (concavity and asymptotic behaviour) A1
y-coordinate and stationary point is (0, 1.25) A1
horizontal asymptote is y = 1 A1

Question 2 (3 marks)

The parametric equations are $x = 2 - t^2$ and $y = 4t$.

Use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ to obtain $\frac{dy}{dx} = -\frac{2}{t}$. M1

Let the gradient of the normal be m_N.

When $t = 1$, $\frac{dy}{dx} = -2$ and so, $m_N = \frac{1}{2}$. M1

When $t = 1$, $x = 1$ and $y = 4$.

So, $y = \frac{1}{2}x + \frac{7}{2}$. A1

Question 3 (3 marks)

Let X represent the weight of the lemons.

$X \sim \text{N}(58, 9^2)$

$\bar{X} \sim \text{N}\left(58, \frac{9^2}{36}\right)$, that is, $\text{E}(\bar{X}) = 58$ and $\text{sd}(\bar{X}) = \frac{9}{6}$ (= 1.5) A1

$\Pr(\bar{X} > 61) = \Pr(\bar{X} > 58 + 2 \times 1.5)$ M1

$= 0.025$ A1

Question 4 (3 marks)

The equations of motion are $T = 3ma$ and $2mg - T = 2ma$. A1

$3mg - \frac{3T}{2} = 3ma \Rightarrow 3mg - \frac{3T}{2} = T$ M1

Solving for T gives $T = \frac{6mg}{5}$ (N). A1

Question 5 (5 marks)

It is given that $x^3 + xy^2 - y^3 = 2$.

Using implicit differentiation obtains $3x^2 + y^2 + 2xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$. M1 A1

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 + y^2 = 0$ A1

So, $x = y = 0$. A1

Formal verification (that is the substitution of $x = 0$ and $y = 0$ into $x^3 + xy^2 - y^3 = 2$) shows that (0, 0) is not on C. M1

Hence, there is no point on C at which $\frac{dy}{dx} = 0$.

Question 6 (5 marks)

The equation to solve is $z^3 = -2 + 2i$.

$z^3 = 2\sqrt{2}\text{cis}\left(\frac{3\pi}{4} + 2k\pi\right), k = 0, 1, 2$ M1

$z = \sqrt{2}\text{cis}\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right), k = 0, 1, 2$ A1

$z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ A1

Adding or subtracting $\frac{2\pi}{3}$ to obtain the other two solutions:

$z_2 = \sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right), z_3 = \sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right)$ A2

Question 7 (3 marks)

The question asks to prove that $\underset{\sim}{p} \cdot \underset{\sim}{q} = |\underset{\sim}{p}|^2$.

$\angle OPQ = 90°$ and so $\overrightarrow{PO} \cdot \overrightarrow{PQ} = |\overrightarrow{PO}||\overrightarrow{PQ}|\cos(\angle OPQ) = 0$ M1

$\overrightarrow{PO} = -\underset{\sim}{p}$ and $\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$, so $-\underset{\sim}{p} \cdot (\underset{\sim}{q} - \underset{\sim}{p}) = 0$. A1

$-\underset{\sim}{p} \cdot \underset{\sim}{q} + \underset{\sim}{p} \cdot \underset{\sim}{p} = 0$

$\underset{\sim}{p} \cdot \underset{\sim}{q} = \underset{\sim}{p} \cdot \underset{\sim}{p}$ (dot product is distributive) A1

So $\underset{\sim}{p} \cdot \underset{\sim}{q} = |\underset{\sim}{p}|^2$.

Question 8 (5 marks)

a. It is given that $a = 2x - \frac{1}{6}x^2,\ 0 \le x \le 18$.

Method 1:

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and so $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - \frac{1}{6}x^2$

$\int \frac{d}{dx}\left(\frac{1}{2}v^2\right)dx = \int \left(2x - \frac{1}{6}x^2\right)dx$ M1

$\frac{1}{2}v^2 = x^2 - \frac{1}{18}x^3 + c$ A1

When $x = 0$, $v = 0$ and so $c = 0$.

So $v^2 = 2x^2 - \frac{1}{9}x^3$. A1

Method 2:

$a = v\frac{dv}{dx}$ and so $v\frac{dv}{dx} = 2x - \frac{1}{6}x^2$

$\int v\,dv = \int \left(2x - \frac{1}{6}x^2\right)dx$ M1

$\frac{1}{2}v^2 = x^2 - \frac{1}{18}x^3 + c$ A1

When $x = 0$, $v = 0$ and so $c = 0$.

So $v^2 = 2x^2 - \frac{1}{9}x^3$. A1

b. $v = 0 \Rightarrow 2x^2 - \frac{1}{9}x^3 = 0$ M1

$x^2\left(2 - \frac{1}{9}x\right) = 0$

$x = 18$ (m) $(x > 0)$ A1

Question 9 (5 marks)

a. $\sin^2(\theta) = \frac{1}{4}(1 - \cos(2\theta))$

$\sin^4(\theta) = \frac{1}{4}(1 - \cos(2\theta))^2$ M1

$= \frac{1}{4}(1 - 2\cos(2\theta) + \cos^2(2\theta))$ A1

$= \frac{1}{4}\left(1 - 2\cos(2\theta) + \frac{1 + \cos(4\theta)}{2}\right)$ (or equivalent) A1

Leading to $\sin^4(\theta) = \frac{1}{8}(3 - 4\cos(2\theta) + \cos(4\theta))$.

b. $\int_0^{\frac{\pi}{4}} \sin^4(\theta)\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{8}(3 - 4\cos(2\theta) + \cos(4\theta))\,d\theta$ (from **part a.**)

$= \frac{1}{8}\left[3\theta - 2\sin(2\theta) + \frac{\sin(4\theta)}{4}\right]_0^{\frac{\pi}{4}}$ M1

$= \frac{1}{8}\left(\frac{3\pi}{4} - 2\sin\left(\frac{\pi}{2}\right) + \frac{\sin(\pi)}{4} - \left(0 - 2\sin(0) + \frac{\sin(0)}{4}\right)\right)$

$= \frac{3\pi - 8}{32}$ A1

Question 10 (5 marks)

Solve $\arccos(x) + \arccos(2x) = \frac{\pi}{2}$ for x where $x \geq 0$.

Method 1:

Apply cos to both sides of the equation $\arccos(2x) = \frac{\pi}{2} - \arccos(x)$ to obtain:

$$2x = \cos\left(\frac{\pi}{2} - \arccos(x)\right)$$ A1

Use of $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ on the RHS to obtain:

$$2x = \cos\left(\frac{\pi}{2}\right)\cos(\arccos(x)) + \sin\left(\frac{\pi}{2}\right)\sin(\arccos(x))$$ M1

So, the equation becomes $2x = \sqrt{1 - x^2}$. A1

Squaring both sides gives: $4x^2 = 1 - x^2$. M1

Solving this equation gives:

$$x = \frac{1}{\sqrt{5}} \quad (x \geq 0)$$ A1

Method 2:

Apply cos to both sides of the equation to obtain:

$$\cos(\arccos(x) + \arccos(2x)) = \cos\left(\frac{\pi}{2}\right)$$ A1

Use of $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ on the LHS to obtain:

$$\cos(\arccos(x) + \arccos(2x)) = \cos(\arccos(x))\cos(\arccos(2x)) - \sin(\arccos(x))\sin(\arccos(2x))$$

$$= 2x^2 - \left(\sqrt{1 - x^2}\right)\left(\sqrt{1 - 4x^2}\right)$$ M1

So, the equation becomes $2x^2 - \left(\sqrt{1 - x^2}\right)\left(\sqrt{1 - 4x^2}\right) = 0$. A1

Rearranging and squaring both sides gives: $4x^4 = 4x^4 - 5x^2 + 1$. M1

Solving this equation gives:

$$x = \frac{1}{\sqrt{5}} \quad (x \geq 0)$$ A1