The Mathematical Association of Victoria

Trial Examination 2018

SPECIALIST MATHEMATICS

Trial Written Examination 1 - SOLUTIONS

Question 1

• It is convenient to define the following notation (this can be quickly done by annotating the Question and Answer book):

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - m\underset{\sim}{j} - 3\sqrt{2}\underset{\sim}{k}$

$\Rightarrow \left|\underset{\sim}{a}\right| = \sqrt{2^2 + (-m)^2 + \left(-3\sqrt{2}\right)^2} = \sqrt{4 + m^2 + 18} = \sqrt{22 + m^2}$. **[M1]**

Let $\underset{\sim}{b} = \underset{\sim}{i} - \sqrt{2}\underset{\sim}{k}$

$\Rightarrow \left|\underset{\sim}{b}\right| = \sqrt{1^2 + \left(-\sqrt{2}\right)^2} = \sqrt{1+2} = \sqrt{3}$.

Let $\theta = \cos^{-1}\left(\frac{2}{5}\right) \qquad \Rightarrow \cos(\theta) = \frac{2}{5}$.

• $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 - 0 + 6 = 8$. .... (1)

• $\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos(\theta) = \sqrt{22 + m^2}\sqrt{3}\left(\frac{2}{5}\right)$. .... (2)

• Equate equations (1) and (2):

$8 = \sqrt{m^2 + 22}\sqrt{3}\left(\frac{2}{5}\right)$ **[H1]**

Consequential on $\left|\underset{\sim}{a}\right|$.

$\Rightarrow \frac{20}{\sqrt{3}} = \sqrt{m^2 + 22} \qquad \Rightarrow \frac{400}{3} = m^2 + 22$

$\Rightarrow m^2 = \frac{400}{3} - 22 = \frac{334}{3} \qquad \Rightarrow m = \pm\sqrt{\frac{334}{3}}$.

Answer: $m = \pm\sqrt{\frac{334}{3}}$. **[A1]**

Question 2

a.

• It is convenient to define the following notation:

Let V be the random variable '*Volume (in ml) of milk in a 2 L carton*'.

$V \sim \text{Normal}\left(\mu_V = 2000,\ \ \sigma_V = 15\right)$.

Let W be the random variable '*Volume (in ml) of milk in a 1 L carton*'.

$W \sim \text{Normal}\left(\mu_W = 1000,\ \ \sigma_W = 4\right)$.

Let $D = V - W_1 - W_2$ where W_1 and W_2 are independent copies of W.

It is important to understand that $D \neq V - 2W$. This will most likely be the most common mistake made in this type of question.

• $\text{Var}(D) = \text{Var}(V - W_1 - W_2)$

$= \text{Var}(V) + \text{Var}(W_1) + \text{Var}(W_2)$

since V, W_1 and W_2 are independent random variables (adapted from VCAA formula sheet)

$= \text{Var}(V) + \text{Var}(W) + \text{Var}(W)$

$= \text{Var}(V) + 2\text{Var}(W)$

$= \sigma_V^2 + 2\sigma_W^2$

$= 15^2 + 2(4)^2 = 225 + 32 = 257$.

Answer: 257. **[A1]**

Note:

$\text{Var}(V - 2W) \qquad = \text{Var}(V) + 2^2\,\text{Var}(W) \qquad = \text{Var}(V) + 4\text{Var}(W)$

$= \sigma_V^2 + 4\sigma_W^2$

$= 15^2 + 4(4)^2 = 225 + 64 = 289$.

289 will most likely be the most common incorrect answer to this question.

b.

• The sample mean $\overline{X}$ is a random variable whose value varies between samples of size $n=36$ of 1 L cartons.

• The sample of size $n=36$ is collected from a normally distributed population of 1 L milk cartons with mean $\mu_W = 1000$ and standard deviation $\sigma_W = 4$.

• Therefore $\overline{X} \sim \text{Norm}\left(\mu_{\overline{X}} = \mu_W = 1000,\ \sigma_{\overline{X}} = \dfrac{\sigma_W}{\sqrt{n}} = \dfrac{4}{\sqrt{36}} = \dfrac{2}{3}\right)$.

• Let $\overline{x}$ be the mean of the sample collected by the quality control officer.

The C% confidence interval for the population mean is $\left(\overline{x} - z^* \dfrac{\sigma_W}{\sqrt{n}},\ \overline{x} + z^* \dfrac{\sigma_W}{\sqrt{n}}\right)$ (adapted from the VCAA formula sheet).

Therefore: $\left(\overline{x} - \dfrac{2}{3}z^*,\ \overline{x} + \dfrac{2}{3}z^*\right)$.

Method 1:

• Compare $\left(\overline{x} - \dfrac{2}{3}z^*,\ \overline{x} + \dfrac{2}{3}z^*\right)$ with the given C% confidence interval of (994.7, 997.3):

$\overline{x} - \dfrac{2}{3}z^* = 994.7$. .... (1)

$\overline{x} + \dfrac{2}{3}z^* = 997.3$. .... (2)

Both equations: **[M1]**

• $(2)-(1)$:

$$\frac{4}{3}z^* = 2.6 = \frac{26}{10} = \frac{13}{5}$$

$$\Rightarrow z^* = \frac{39}{20}.$$

Answer: $z^* = \dfrac{39}{20}$. **[A1]**

Accept 1.95.

Method 2:

- $\bar{x}$ is the midpoint of the confidence interval and is given by

$$\bar{x} = \frac{994.7 + 997.3}{2} = \frac{1992}{2} = 996.$$

- Substitute $\bar{x} = 996$ into $\left(\bar{x} - \frac{2}{3}z^*,\ \bar{x} + \frac{2}{3}z^*\right)$:

$$\left(996 - \frac{2}{3}z^*,\ 996 + \frac{2}{3}z^*\right).$$

- Compare $\left(996 - \frac{2}{3}z^*,\ 996 + \frac{2}{3}z^*\right)$ with the given C% confidence interval of (994.7, 997.3):

$$996 - \frac{2}{3}z^* = 994.7. \qquad \text{.... (1)}$$

$$996 + \frac{2}{3}z^* = 997.3. \qquad \text{.... (2)}$$

Either equation: **[M1]**

- From either equation (1) or equation (2):

$$\frac{2}{3}z^* = 1.3 = \frac{13}{10}$$

$$\Rightarrow z^* = \frac{39}{20}.$$

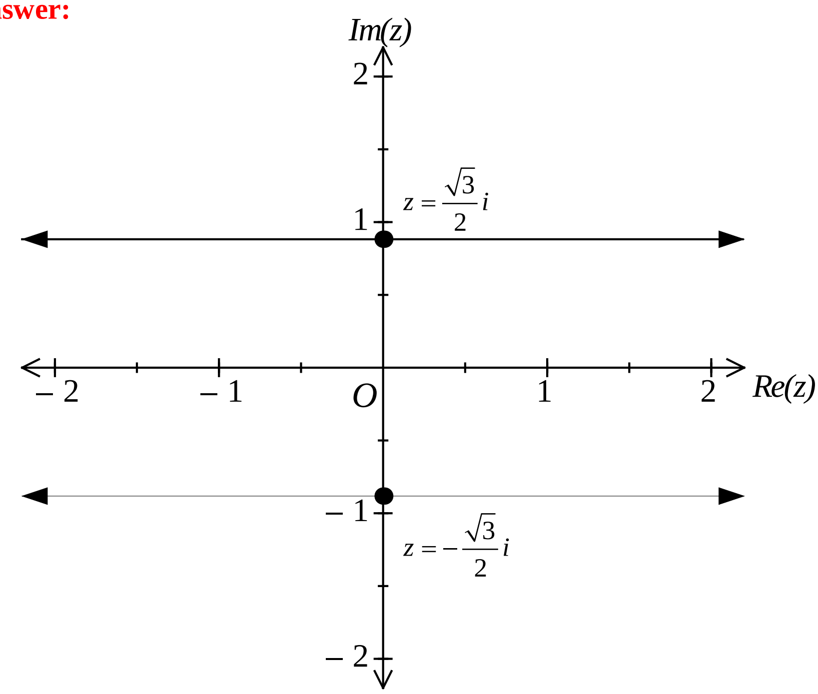
Answer: $z^* = \frac{39}{20}$. **[A1]**

Accept 1.95.

Question 3

a.

Answer:



Horizontal line and *y*-intercept labelled $z=\frac{\sqrt{3}}{2}i$ (must be in roughly correct position relative to the given scale): **[A1]**

Horizontal line and *y*-intercept labelled $z=-\frac{\sqrt{3}}{2}i$ (must be in roughly correct position relative to the given scale): **[A1]**

Recognition that $\frac{1}{2}<\frac{\sqrt{3}}{2}<1$ is required.

Note: $\frac{\sqrt{1}}{2}=\frac{1}{2}<\frac{\sqrt{3}}{2}<\frac{\sqrt{4}}{2}=1$.

Working:

Substitute $z=x+iy$ into $\left|z-\bar{z}+1\right|=2$:

$\left|(x+iy)-(x-iy)+1\right|=2 \quad \Rightarrow \left|1+2yi\right|=2$

$\Rightarrow \sqrt{1^2+(2y)^2}=2 \quad \Rightarrow \sqrt{1+4y^2}=2$

$\Rightarrow 1+4y^2=4 \quad \Rightarrow y^2=\frac{3}{4}$

$\Rightarrow y=\pm\frac{\sqrt{3}}{2}$.

b.

It is easiest to work in polar form.

- $1+i=\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$.

- $-\sqrt{3}+i=2\text{cis}\left(\frac{5\pi}{6}\right)$

$\Rightarrow\left(-\sqrt{3}+i\right)^3=2^3\text{cis}\left(3\times\frac{5\pi}{6}\right)$ **[A1]**

$=8\text{cis}\left(\frac{15\pi}{6}\right)=8\text{cis}\left(\frac{5\pi}{2}\right)=8\text{cis}\left(2\pi+\frac{\pi}{2}\right)=8\text{cis}\left(\frac{\pi}{2}\right)$.

- Therefore:

$$z=\frac{1+i}{(-\sqrt{3}+i)^3}$$

$$=\frac{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}{8\text{cis}\left(\frac{\pi}{2}\right)} \quad =\frac{\sqrt{2}}{8}\text{cis}\left(\frac{\pi}{4}-\frac{\pi}{2}\right) \quad =\frac{\sqrt{2}}{8}\text{cis}\left(-\frac{\pi}{4}\right).$$

Answer: $-\frac{\pi}{4}$. **[A1]**

Question 4

Method 1:

• $\operatorname{cosec}(2\theta) = -\frac{13}{12} \quad \Rightarrow \frac{1}{\sin(2\theta)} = -\frac{13}{12} \quad \Rightarrow \sin(2\theta) = -\frac{12}{13}$

$\Rightarrow \cos(2\theta) = \pm\frac{5}{13}$

(using either a 'triangle' or the Pythagorean Identity).

• But:

$\pi < 2\theta < \frac{3\pi}{2}$ (3rd quadrant).

Therefore:

$\cos(2\theta) = -\frac{5}{13}$. **[M1]**

• Substitute $\cos(2\theta) = 1 - 2\sin^2(\theta)$ (from the VCAA formula sheet):

$1 - 2\sin^2(\theta) = -\frac{5}{13}$ **[H1]**

Consequential on value of $\cos(2\theta)$.

$\Rightarrow \sin^2(\theta) = \frac{9}{13} \quad \Rightarrow \sin(\theta) = \pm\frac{3}{\sqrt{13}}$.

• But:

$\pi < 2\theta < \frac{3\pi}{2} \quad \Rightarrow \frac{\pi}{2} < \theta < \frac{3\pi}{4}$ (subset of 2nd quadrant).

Therefore:

$\sin(\theta) = \frac{3}{\sqrt{13}}$.

Answer: $\frac{3}{\sqrt{13}}$. **[A1]**

Method 2: Not recommended (inefficient and difficult to apply rigorously and correctly).

- $\operatorname{cosec}(2\theta) = -\frac{13}{12} \quad \Rightarrow \frac{1}{\sin(2\theta)} = -\frac{13}{12} \quad \Rightarrow \sin(2\theta) = -\frac{12}{13}$

$\Rightarrow 2\sin(\theta)\cos(\theta) = -\frac{12}{13} \quad \Rightarrow 4\sin^2(\theta)\cos^2(\theta) = \frac{144}{169}$

$\Rightarrow \sin^2(\theta)\left(1-\sin^2(\theta)\right) = \frac{36}{169}$

$\Rightarrow \sin^4(\theta) - \sin^2(\theta) + \frac{36}{169} = 0$. **[M1]**

- This is a quadratic in $\sin^2(\theta)$.

Use the quadratic formula:

$$\sin^2(\theta) = \frac{1 \pm \sqrt{1-4(1)\left(\frac{36}{169}\right)}}{2} = \frac{1 \pm \sqrt{1-\frac{144}{169}}}{2} = \frac{1 \pm \sqrt{\frac{25}{169}}}{2} = \frac{1 \pm \frac{5}{13}}{2} = \frac{13 \pm 5}{26}.$$

$\sin^2(\theta) = \frac{9}{13}$ or $\frac{4}{13}$. **[H1]**

Consequential on $\sin^4(\theta) - \sin^2(\theta) + \frac{36}{169} = 0$.

- But:

$\pi < 2\theta < \frac{3\pi}{2} \quad \Rightarrow \frac{\pi}{2} < \theta < \frac{3\pi}{4}$

$\Rightarrow \sin\left(\frac{3\pi}{4}\right) < \theta < \sin\left(\frac{\pi}{2}\right) \quad \Rightarrow \frac{1}{\sqrt{2}} < \sin(\theta) < 1$

$\Rightarrow \frac{1}{2} < \sin^2(\theta) < 1$

therefore $\sin^2(\theta) = \frac{4}{13}$ is rejected.

Therefore $\sin^2(\theta) = \frac{9}{13} \quad \Rightarrow \sin(\theta) = \pm\frac{3}{\sqrt{13}}$.

- But $\pi < 2\theta < \frac{3\pi}{2} \Rightarrow \frac{3\pi}{4} < \theta < \frac{\pi}{2}$ (subset of 2nd quadrant).

Therefore: $\sin(\theta) = \frac{3}{\sqrt{13}}$.

Answer: $\frac{3}{\sqrt{13}}$. **[A1]**

Question 5

• Since the degree of the numerator is not smaller than the degree of the denominator, $\dfrac{2x^2+1}{4x^2-4x+3}$ must be re-written using polynomial long division:

$$\begin{array}{r|l} & \quad\dfrac{1}{2} \\ 4x^2-4x+3 & 2x^2+\underbrace{0x}_{\text{'Ghost' term}}+1 \\ & -\left(2x^2-2x+\dfrac{3}{2}\right) \\ \hline & \qquad 2x-\dfrac{1}{2} \end{array}$$

Therefore:

$$\frac{2x^2+1}{4x^2-4x+3}=\frac{1}{2}+\frac{2x-\frac{1}{2}}{4x^2-4x+3}.$$ **[M1]**

• $\displaystyle\int \frac{2x^2+1}{4x^2-4x+3}\,dx=\int \frac{1}{2}+\frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx$

$$=\frac{1}{2}x+\int \frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx.$$

• Consider $\displaystyle\int \frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx$.

Note that the denominator is an irreducible quadratic polynomial and so partial fraction decomposition is NOT suggested.

Note that the numerator is linear and that the derivative of the denominator is $8x-4$. This motivates the following algebraic re-arrangement to get an integral of the form

$$\int \frac{f'(x)}{f(x)}\,dx$$

where $f(x)=4x^2-4x+3$ and $f'(x)=8x-4$:

$$\frac{2x-\frac{1}{2}}{4x^2-4x+3}=\frac{(2x-1)+\frac{1}{2}}{4x^2-4x+3}$$

$$=\frac{2x-1}{4x^2-4x+3}+\frac{\frac{1}{2}}{4x^2-4x+3}$$

$$=\frac{1}{4}\left(\frac{8x-4}{4x^2-4x+3}\right)+\frac{1}{2}\left(\frac{1}{4x^2-4x+3}\right).$$

Therefore:

$$\int\frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx=\frac{1}{4}\int\frac{8x-4}{4x^2-4x+3}\,dx+\frac{1}{2}\int\frac{1}{4x^2-4x+3}\,dx.$$

- $\int\frac{8x-4}{4x^2-4x+3}\,dx=\log_e\left|4x^2-4x+3\right|.$ **[H1]**

This integral is found by making the substitution $u=4x^2-4x+3$ and recognising the resulting standard form $\int\frac{1}{u}\,du$.

Consideration of any reasonable integral of the form $\int\frac{f'(x)}{f(x)}\,dx$ and correct calculation of this integral.

- $\int\frac{1}{4x^2-4x+3}\,dx=\int\frac{1}{(2x-1)^2+2}\,dx$

$$=\frac{1}{\sqrt{2}}\int\frac{\sqrt{2}}{(2x-1)^2+\left(\sqrt{2}\right)^2}\,dx\qquad=\frac{1}{2\sqrt{2}}\arctan\left(\frac{2x-1}{\sqrt{2}}\right).$$ **[H1]**

This integral is found by making the substitution $u=2x-1$ and recognising the resulting standard form $\frac{1}{2}\int\frac{a}{u^2+a^2}\,du$ (from theVCAA formula sheet) where $a=\sqrt{2}$.

Consideration of any reasonable integral that leads to an arctan form and correct calculation of this integral.

• Therefore:

$$\int \frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx = \frac{1}{4}\int \frac{8x-4}{4x^2-4x+3}\,dx + \frac{1}{2}\int \frac{1}{4x^2-4x+3}\,dx$$

$$= \frac{1}{4}\log_e\left|4x^2-4x+3\right| + \frac{1}{4\sqrt{2}}\arctan\left(\frac{2x-1}{\sqrt{2}}\right).$$

• Therefore:

$$\int \frac{2x^2+1}{4x^2-4x+3}\,dx = \frac{1}{2}x + \int \frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx$$

$$= \frac{1}{2}x + \frac{1}{4}\log_e\left|4x^2-4x+3\right| + \frac{1}{4\sqrt{2}}\arctan\left(\frac{2x-1}{\sqrt{2}}\right).$$

Answer: $\frac{1}{2}x + \frac{1}{4}\log_e\left|4x^2-4x+3\right| + \frac{1}{4\sqrt{2}}\arctan\left(\frac{2x-1}{\sqrt{2}}\right).$ **[A1]**

Acceptable alternative answers:

• Accept all equivalent answers such as $\dfrac{4x+2\log_e\left|4x^2-4x+3\right|+\sqrt{2}\arctan\left(\frac{2x-1}{\sqrt{2}}\right)}{8}$.

• Answers that include a constant term are acceptable: The arbitrary constant of anti-differentiation can be taken as zero since only *an* anti-derivative (rather than *the* anti-derivative) is required.

• Answers without a modulus are acceptable: Since $4x^2-4x+3=(2x-1)^2+2>0$ for all $x \in R$ the modulus in $\log_e\left|4x^2-4x+3\right|$ is not required.

Question 6

a.

- $p = mv = 5v \qquad \Rightarrow |p| = 5|v|$

therefore the value of v when $x = 2$ is required.

Option 1: Use $a = v\dfrac{dv}{dx}$.

$v\dfrac{dv}{dx} = -e^{-2x}$.

This is a separable differential equation:

$$\int v\ dv = -\int e^{-2x} dx$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + c.$$ **[M1]**

Option 2: Use $a = \dfrac{1}{2}\dfrac{d}{dx}\left(v^2\right)$.

$\dfrac{1}{2}\dfrac{d}{dx}\left(v^2\right) = -e^{-2x}$

$$\Rightarrow \frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + c.$$ **[M1]**

- Substitute $v = \pm 1 \Rightarrow v^2 = 1$ when $x = 0$ and solve for c: $c = 0$.
- Therefore:

$$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} \qquad \Rightarrow v^2 = e^{-2x} = \left(e^{-x}\right)^2$$

$$\Rightarrow v = \pm\sqrt{\left(e^{-x}\right)^2} = \pm\left|e^{-x}\right|$$

$\Rightarrow v = \pm e^{-x}$ since $e^{-x} > 0$ for $x \in R$

$\Rightarrow |v| = e^{-x}$.

- Substitute $x = 2$: $|p| = 5|v| = 5e^{-2}$.

Answer: $5e^{-2}$ kg ms^{-1}. **[A1]**

b.

• From **part a.**: $v = \pm e^{-x}$.

Both solutions must be considered because it is the **speed** (**not** velocity) that is given to be 1 ms^{-1} when $x = 0$.

Substitute $v = \frac{dx}{dt}$:

$$\frac{dx}{dt} = \pm e^{-x}$$

$$\Rightarrow \frac{dt}{dx} = \pm e^{x}$$

$$\Rightarrow t = \int \pm e^{x}\,dx$$

$$= \pm e^{x} + k.$$ **[M1]**

Case 1: $t = e^{x} + k$.

• Substitute $x = 0$ when $t = 0$ and solve for *k*: $k = -1$.

• $t = e^{x} - 1 \quad \Rightarrow x = \log_e(1+t)$.

• Substitute $t = 3$: $x = \log_e(4) = 2\log_e(2)$.

Case 2: $t = -e^{x} + k$.

• Substitute $x = 0$ when $t = 0$ and solve for *k*: $k = 1$.

• $t = -e^{x} + 1 \quad \Rightarrow x = \log_e(1-t)$.

• Reject because *x* is not defined for $t = 3$.

Note: This solution is also rejected because $x \to +\infty$ as $t \to 1$.

Answer: $2\log_e(2)$ m. **[A1]**

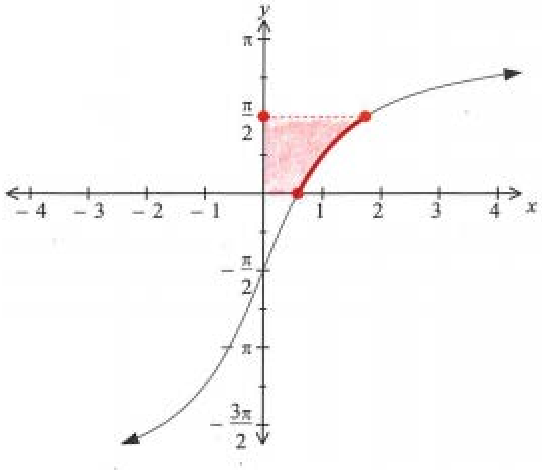
Accept $\log_e(4)$ m.

Question 7

• A rough graph of $y = 3\arctan(x) - \dfrac{\pi}{2}$ over the relevant interval $x \in \left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$ should be drawn and the relevant area shaded.

$$x = \frac{1}{\sqrt{3}}: \quad y = 3\arctan\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{2} = 3\left(\frac{\pi}{6}\right) - \frac{\pi}{2} = 0.$$

$$x = \sqrt{3}: \quad y = 3\arctan\left(\sqrt{3}\right) - \frac{\pi}{2} = 3\left(\frac{\pi}{3}\right) - \frac{\pi}{2} = \frac{\pi}{2}.$$



a.

• Required volume:

Rotation around the y-axis therefore $V = \pi \displaystyle\int_0^h x^2 \, dy$.

• $y = 3\arctan(x) - \dfrac{\pi}{2}$

$$\Rightarrow \arctan(x) = \frac{y + \frac{\pi}{2}}{3} = \frac{2y + \pi}{6}$$

$$\Rightarrow x = \tan\left(\frac{2y + \pi}{6}\right).$$

Answer: $V = \pi \displaystyle\int_0^h \tan^2\left(\frac{2y + \pi}{6}\right) dy$. **[A1]**

Accept $V = \pi \displaystyle\int_0^h \tan^2\left(\frac{y}{3} + \frac{\pi}{6}\right) dy$.

b.

• The tank is full when $h = \frac{\pi}{2}$.

• It follows from **part a.** that

$$V = \pi \int_0^{\pi/2} \tan^2\left(\frac{2y+\pi}{6}\right) dy.$$

• 'Tidy' the integrand by substituting $u = \frac{2y+\pi}{6}$.

$$u = \frac{2y+\pi}{6} \quad \Rightarrow \frac{du}{dy} = \frac{1}{3} \quad \Rightarrow dy = 3du.$$

$$y = 0: \ u = \frac{\pi}{6}. \qquad y = \frac{\pi}{2}: \ u = \frac{\pi+\pi}{6} = \frac{\pi}{3}.$$

$$V = 3\pi \int_{\pi/6}^{\pi/3} \tan^2(u)\, du.$$

• Substitute the trigonometric identity

$1 + \tan^2(u) = \sec^2(u)$ Use of trigonometric identity: **[M1]**

(from the VCAA formula sheet):

$$V = 3\pi \int_{\pi/6}^{\pi/3} \sec^2(u) - 1\, du$$

$$= 3\pi\left[\tan(u) - u\right]_{\pi/6}^{\pi/3}$$

$$= 3\pi\left(\left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right] - \left[\tan\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right]\right)$$

$$= 3\pi\left(\sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) \quad = 3\pi\left(\frac{2}{\sqrt{3}} - \frac{\pi}{6}\right) \quad = 3\pi\left(\frac{4\sqrt{3}}{6} - \frac{\pi}{6}\right) \quad = \frac{4\pi\sqrt{3} - \pi^2}{2}.$$

Answer: $V = \frac{4\pi\sqrt{3} - \pi^2}{2}$. **[A1]**

c.

- $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$.
- It is given that $\dfrac{dV}{dt} = k$ m^3 per minute.
- From **part a.**: $V = \pi \displaystyle\int_0^h \tan^2\left(\frac{2y+\pi}{6}\right) dy$.

There are two options for finding $\dfrac{dV}{dh}$.

Option 1: Use the integral solution to a differential equation.

This option is recommended for its efficiency. The number of marks the question is worth (2 marks and one of those marks is for the answer) suggests that an efficient method exists (as opposed to **option 2** below).

$V = \pi \displaystyle\int_0^h \tan^2\left(\frac{2y+\pi}{6}\right) dy$ is the integral solution to $\dfrac{dV}{dh} = \pi \tan^2\left(\dfrac{2h+\pi}{6}\right)$, $V = 0$ when $h = 0$.

Therefore $\dfrac{dV}{dh} = \pi \tan^2\left(\dfrac{2h+\pi}{6}\right)$ when $V = \pi \displaystyle\int_0^h \tan^2\left(\frac{2y+\pi}{6}\right) dy$.

$\dfrac{dV}{dh} = \pi \tan^2\left(\dfrac{2h+\pi}{6}\right)$. **[H1]**

Consequential on answer to **part a.**

Option 2: Calculate $V = \pi \displaystyle\int_0^h \tan^2\left(\frac{2y+\pi}{6}\right) dy$ and differentiate the answer with respect to h. This option is inefficient and is **not** recommended.

The calculation is done in a similar way to the calculation in **part b.**:

- $V = 3\pi \displaystyle\int_{\pi/6}^{(2h+\pi)/6} \tan^2(u)\, du$

$= 3\pi \left[\tan(u) - u\right]_{\pi/6}^{(2h+\pi)/6}$

$= 3\pi\left(\left(\tan\left(\dfrac{2h+\pi}{6}\right) - \dfrac{(2h+\pi)}{6}\right) - \left(\tan\left(\dfrac{\pi}{6}\right) - \dfrac{\pi}{6}\right)\right)$.

- Therefore:

$$\frac{dV}{dh}=3\pi\left(\frac{1}{3}\sec^2\left(\frac{2h+\pi}{6}\right)-\frac{1}{3}\right)\qquad=\pi\left(\sec^2\left(\frac{2h+\pi}{6}\right)-1\right)$$

$$=\pi\tan^2\left(\frac{2h+\pi}{6}\right).$$

Therefore $\dfrac{dh}{dV}=\dfrac{1}{\pi\tan^2\left(\dfrac{2h+\pi}{6}\right)}$.

Substitute $h=\dfrac{\pi}{4}$:

$$\frac{dh}{dV}=\frac{1}{\pi\tan^2\left(\dfrac{\frac{\pi}{2}+\pi}{6}\right)}=\frac{1}{\pi\tan^2\left(\frac{\pi}{4}\right)}=\frac{1}{\pi}.$$

- Substitute $\dfrac{dV}{dt}=k$ and $\dfrac{dh}{dV}=\dfrac{1}{\pi}$ into $\dfrac{dh}{dt}=\dfrac{dh}{dV}\times\dfrac{dV}{dt}$:

$$\frac{dh}{dt}=\frac{1}{\pi}\times k.$$

Answer: $\dfrac{dh}{dt}=\dfrac{k}{\pi}$ metres per minute. **[A1]**

Question 8

a.

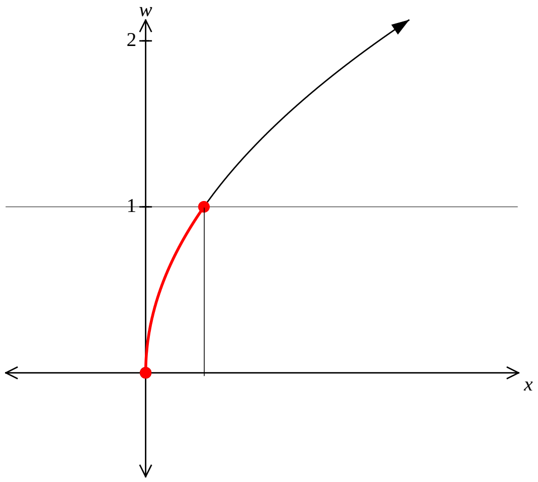
• Let $y = \arccos\left(2\sqrt{x}\right)$.

• Let $w = 2\sqrt{x}$.

• Then $y = \arccos(w)$.

Implied domain: Solve $-1 \le w \le 1$.

• Draw a graph of $w = 2\sqrt{x}$:



• Solve $w = 1$:

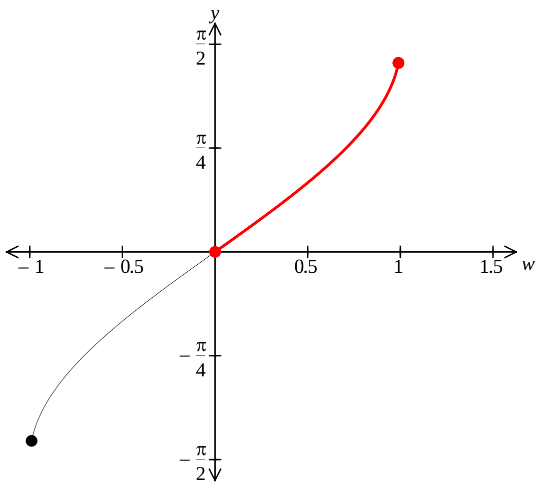
$2\sqrt{x} = 1 \qquad \Rightarrow x = \dfrac{1}{4}$.

Answer: $x \in \left[0,\ \dfrac{1}{4}\right]$. **[A1]**

Implied range:

• From the graph of $w = 2\sqrt{x}$ it can be seen that only the subset $0 \le w \le 1$ of allowed values $-1 \le w \le 1$ is used.

• Draw a graph of $y = \arccos(w)$ over the domain $0 \le w \le 1$:



• From this graph it is seen that $0 \le y \le \dfrac{\pi}{2}$.

Answer: $y \in \left[0, \dfrac{\pi}{2}\right]$. **[A1]**

b.

• Let $y = \arccos\left(2\sqrt{x}\right)$.

Use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dw} \times \frac{dw}{dx}$$

(from the VCAA formula sheet).

Note: It is important that *y* is defined before using this form of the chain rule.

Let $w = 2\sqrt{x} \qquad \Rightarrow \frac{dw}{dx} = \frac{1}{\sqrt{x}}$.

$y = \arccos(w) \qquad \Rightarrow \frac{dy}{dw} = \frac{-1}{\sqrt{1-w^2}}$ (from the VCAA formula sheet).

• Substitute $\frac{dy}{dw} = \frac{-1}{\sqrt{1-w^2}}$ and $\frac{dw}{dx} = \frac{1}{\sqrt{x}}$ into the chain rule:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-w^2}} \times \frac{1}{\sqrt{x}}.$$

Back-substitute $w = 2\sqrt{x}$:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-4x}} \times \frac{1}{\sqrt{x}}$$ **[M1]**

$$= \frac{-1}{\sqrt{x-4x^2}}.$$

• Substitute $x = \frac{1}{8}$:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\frac{1}{8} - 4\left(\frac{1}{8}\right)^2}} = \frac{-1}{\sqrt{\frac{1}{8} - \frac{1}{16}}} = \frac{-1}{\sqrt{\frac{1}{16}}} = \frac{-1}{\frac{1}{4}} = -4.$$

Answer: -4. **[A1]**

c.

- The y-coordinate of the point on the graph where $x=\frac{1}{8}$ is required.

Substitute $x=\frac{1}{8}$ into $\arccos\left(2\sqrt{x}\right)-\frac{1}{3}\arcsin(y)=\frac{\pi}{3}$ and solve for y:

$$\arccos\left(2\sqrt{\frac{1}{8}}\right)-\frac{1}{3}\arcsin(y)=\frac{\pi}{3}$$

$$\Rightarrow \arccos\left(\frac{2}{2\sqrt{2}}\right)-\frac{1}{3}\arcsin(y)=\frac{\pi}{3}$$

$$\Rightarrow \arccos\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{3}\arcsin(y)=\frac{\pi}{3}$$

$$\Rightarrow \frac{\pi}{4}-\frac{1}{3}\arcsin(y)=\frac{\pi}{3}$$

$$\Rightarrow \arcsin(y)=-\frac{\pi}{4}$$

$$\Rightarrow y=-\frac{1}{\sqrt{2}}.$$ **[A1]**

• The gradient of the tangent is given by the value of $\frac{dy}{dx}$ at the point $\left(\frac{1}{8},\ -\frac{1}{\sqrt{2}}\right)$.

Use implicit differentiation to find $\frac{dy}{dx}$.

Differentiate both sides of $\arccos\left(2\sqrt{x}\right)-\frac{1}{3}\arcsin(y)=\frac{\pi}{3}$ with respect to *x*:

$$\underbrace{\frac{-1}{\sqrt{x-4x^2}}}_{\text{From part (a)}}-\frac{1}{3}\underbrace{\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}}_{\substack{\text{From the chain rule:} \\ \frac{d}{dy}(\arcsin(y))\times\frac{dy}{dx}}}=0.$$

$$\frac{-1}{\sqrt{x-4x^2}}-\frac{1}{3}\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0.$$

[H1]

Derivative of $y=\arccos\left(2\sqrt{x}\right)$ is consequential on **part a.**

• Substitute $x=\frac{1}{8}$ and $y=-\frac{1}{\sqrt{2}}$ and then solve for $\frac{dy}{dx}$.

Note: This is more efficient and there is less risk of making a careless mistake than first solving for $\frac{dy}{dx}$.

$$\underbrace{-4}_{\text{From part (a)}}-\frac{1}{3}\frac{1}{\sqrt{1-\frac{1}{2}}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{\frac{1}{\sqrt{2}}}\frac{dy}{dx}=-12 \qquad \Rightarrow \sqrt{2}\frac{dy}{dx}=-12$$

$$\Rightarrow \frac{dy}{dx}=-\frac{12}{\sqrt{2}} \qquad =-6\sqrt{2}.$$

Answer: $-6\sqrt{2}$.

[A1]

Simplified answer is required.

Question 9

a.

• Let $I = \int \sin\left(\frac{\pi}{3}t\right)\cos\left(\frac{\pi}{6}t\right)\,dt$.

• $\sin\left(\frac{\pi}{3}t\right)\cos\left(\frac{\pi}{6}t\right) = \sin\left(2\times\frac{\pi}{6}t\right)\cos\left(\frac{\pi}{6}t\right)$.

Substitute the double angle formula $\sin\left(2\times\frac{\pi}{6}t\right) = 2\sin\left(\frac{\pi}{6}t\right)\cos\left(\frac{\pi}{6}t\right)$
(adapted from the VCAA formula sheet):

$$\sin\left(\frac{\pi}{3}t\right)\cos\left(\frac{\pi}{6}t\right) = 2\sin\left(\frac{\pi}{6}t\right)\cos^2\left(\frac{\pi}{6}t\right).$$

• Therefore:

$$I = 2\int \sin\left(\frac{\pi}{6}t\right)\cos^2\left(\frac{\pi}{6}t\right)\,dt.$$ **[M1]**

There are two methods for calculating this integral.

Method 1:

Substitute $u = \cos\left(\frac{\pi}{6}t\right)$:

$$\frac{du}{dt} = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right) \qquad \Rightarrow dt = -\frac{6}{\pi}\frac{du}{\sin\left(\frac{\pi}{6}t\right)}.$$

$$I = 2\int \sin\left(\frac{\pi}{6}t\right)u^2\left(-\frac{6}{\pi}\right)\frac{du}{\sin\left(\frac{\pi}{6}t\right)} \qquad = -\frac{12}{\pi}\int u^2\,du \qquad = -\frac{4}{\pi}u^3.$$

Note: The arbitrary constant of anti-differentiation can be taken as zero since only *an* anti-derivative (rather than *the* anti-derivative) is required.

• Back-substitute $u = \cos\left(\frac{\pi}{6}t\right)$:

$$I = -\frac{4}{\pi}\cos^3\left(\frac{\pi}{6}t\right).$$

Method 2:

'Construct' the answer 'by inspection'.

- 'By inspection' $2\int \sin\left(\frac{\pi}{6}t\right)\cos^2\left(\frac{\pi}{6}t\right)dt$ will have the form $k\cos^3\left(\frac{\pi}{6}t\right)$

where k is a constant.

- The value of k is found by differentiating $k\cos^3\left(\frac{\pi}{6}t\right)$ and 'forcing' the result

to equal the integrand $2\sin\left(\frac{\pi}{6}t\right)\cos^2\left(\frac{\pi}{6}t\right)$:

$$\frac{d}{dx}\left(k\cos^3\left(\frac{\pi}{6}t\right)\right) = -3\left(\frac{\pi}{6}\right)k\cos^2\left(\frac{\pi}{6}t\right)\sin\left(\frac{\pi}{6}t\right) = -\frac{k\pi}{2}\cos^2\left(\frac{\pi}{6}t\right)\sin\left(\frac{\pi}{6}t\right).$$

Compare the coefficients of $-\frac{k\pi}{2}\cos^2\left(\frac{\pi}{6}t\right)\sin\left(\frac{\pi}{6}t\right)$ and $2\sin\left(\frac{\pi}{6}t\right)\cos^2\left(\frac{\pi}{6}t\right)$:

$$-\frac{k\pi}{2} = 2 \qquad \Rightarrow k = -\frac{4}{\pi}.$$

Answer: $-\frac{4}{\pi}\cos^3\left(\frac{\pi}{6}t\right)$. **[A1]**

Answers that include a constant term are acceptable.

b.

• Substitute $\underset{\sim}{\mathrm{v}}(t) = \dfrac{d\underset{\sim}{\mathrm{r}}}{dt}$:

$$\frac{d\underset{\sim}{\mathrm{r}}}{dt} = \frac{3\sqrt{t}}{2+t\sqrt{t}}\underset{\sim}{\mathrm{i}} + \sin\left(\frac{\pi}{3}t\right)\cos\left(\frac{\pi}{6}t\right)\underset{\sim}{\mathrm{j}}$$

$$\Rightarrow \underset{\sim}{\mathrm{r}}(t) = \int \frac{3\sqrt{t}}{2+t\sqrt{t}}\,dt\,\underset{\sim}{\mathrm{i}} + \underbrace{\int \sin\left(\frac{\pi}{3}t\right)\cos\left(\frac{\pi}{6}t\right)dt\,\underset{\sim}{\mathrm{j}}}_{\text{Calculated in part (a)}}.$$

• Let $I = \displaystyle\int \frac{3\sqrt{t}}{2+t\sqrt{t}}\,dt$.

Method 1:

Substitute $u = 2 + t\sqrt{t} = 2 + t^{3/2}$:

$$\frac{du}{dt} = \frac{3}{2}t^{1/2} = \frac{3}{2}\sqrt{t} \qquad \Rightarrow dt = \frac{2}{3\sqrt{t}}\,du.$$

$$I = \int \frac{3\sqrt{t}}{u}\left(\frac{2}{3\sqrt{t}}\right)du$$ **[M1]**

$$= 2\int \frac{1}{u}\,du$$

$$= 2\log_e|u|$$

$$= 2\log_e|2+t\sqrt{t}|$$ **[M1]**

$$= 2\log_e(2+t\sqrt{t}) \text{ since } t \geq 0.$$

Method 2:

- Substitute $u = \sqrt{t}$:

$$\frac{du}{dt} = \frac{1}{2\sqrt{t}} \qquad \Rightarrow dt = 2\sqrt{t}du = 2udu.$$

$$I = \int \frac{3u}{2+u^3}\,(2u)\,du$$

$$= 6\int \frac{u^2}{2+u^3}\,du.$$ **[M1]**

- Substitute $w = 2 + u^3$:

$$I = \int \frac{2}{w}\,dw$$

$$= 2\log_e |w|$$

$$= 2\log_e |2+u^3|$$

$$= 2\log_e \left|2 + \left(\sqrt{t}\right)^3\right|$$

$$= 2\log_e |2 + t\sqrt{t}|$$ **[M1]**

$$= 2\log_e (2 + t\sqrt{t}) \text{ since } t \geq 0.$$

- Therefore:

$$\underset{\sim}{r}(t) = 2\log_e(2+t\sqrt{t})\,\underset{\sim}{i} \underbrace{- \frac{4}{\pi}\cos^3\left(\frac{\pi}{6}t\right)}_{\text{Answer to part (a)}}\,\underset{\sim}{j} + \underset{\sim}{c}$$

where $\underset{\sim}{c}$ is an arbitrary constant vector.

- Substitute $\underset{\sim}{r}(0) = \underset{\sim}{0}$ and solve for $\underset{\sim}{c}$:

$$\underset{\sim}{0} = 2\log_e(2)\,\underset{\sim}{i} - \frac{4}{\pi}\,\underset{\sim}{j} + \underset{\sim}{c}$$

$$\Rightarrow \underset{\sim}{c} = -2\log_e(2)\,\underset{\sim}{i} + \frac{4}{\pi}\,\underset{\sim}{j}.$$

• Therefore:

$$\underset{\sim}{r}(t) = 2\log_e(2 + t\sqrt{t})\,\underset{\sim}{i} - \frac{4}{\pi}\cos^3\left(\frac{\pi}{6}t\right)\underset{\sim}{j} - 2\log_e(2)\,\underset{\sim}{i} + \frac{4}{\pi}\underset{\sim}{j}$$

[H1]

Consequential on answer to **part a.**

$$= \left(2\log_e(2 + t\sqrt{t}) - 2\log_e(2)\right)\underset{\sim}{i} + \left(\frac{4}{\pi} - \frac{4}{\pi}\cos^3\left(\frac{\pi}{6}t\right)\right)\underset{\sim}{j}$$

$$= 2\log_e\left(\frac{2 + t\sqrt{t}}{2}\right)\underset{\sim}{i} + \frac{4}{\pi}\left(1 - \cos^3\left(\frac{\pi}{6}t\right)\right)\underset{\sim}{j}$$

• Substitute $t = 8$:

$$\underset{\sim}{r}(8) = 2\log_e\left(\frac{2 + 8\sqrt{8}}{2}\right)\underset{\sim}{i} + \frac{4}{\pi}\left(1 - \cos^3\left(\frac{8\pi}{6}\right)\right)\underset{\sim}{j}$$

$$= 2\log_e(1 + 8\sqrt{2})\,\underset{\sim}{i} + \frac{4}{\pi}\left(1 - \left(-\frac{1}{2}\right)^3\right)\underset{\sim}{j}$$

$$= 2\log_e(1 + 8\sqrt{2})\,\underset{\sim}{i} + \frac{9}{2\pi}\underset{\sim}{j}.$$

Answer: $2\log_e(1 + 8\sqrt{2})\,\underset{\sim}{i} + \dfrac{9}{2\pi}\underset{\sim}{j}$. **[A1]**

Accept $\log_e(129 + 16\sqrt{2})\,\underset{\sim}{i} + \dfrac{9}{2\pi}\underset{\sim}{j}$.

END OF SOLUTIONS