YEAR 12 *Trial Exam Paper*

2018

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- fully worked solutions
- mark allocations
- tips on how to approach the exam.

Question 1a.

Worked solution

$$\frac{v}{z}=\frac{1+3i}{2+i}\times\frac{2-i}{2-i}$$

$$=\frac{5+5i}{5}$$

$$=1+i$$

$$\left|\frac{v}{z}\right|=\sqrt{1^2+1^2}=\sqrt{2}$$

$$\text{Arg}\left(\frac{v}{z}\right)=\tan^{-1}\left(\frac{1}{1}\right)$$

$$=\tan^{-1}(1)=\frac{\pi}{4}$$

$$\Rightarrow\frac{v}{z}=\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

Mark allocation: 2 marks

- 1 mark for correctly simplifying $\frac{v}{z}$
- 1 mark for correctly showing $\left|\frac{v}{z}\right|=\sqrt{2}$ and $\text{Arg}\left(\frac{v}{z}\right)=\frac{\pi}{4}$

Tips

- *Do not attempt to convert* z *and* v *into polar form. You must simplify* $\frac{v}{z}$ *first as* $\frac{v\bar{z}}{z\bar{z}}$ *and then convert to polar form.*
- *Minimise errors by recognising that* $1+i=\sqrt{2}\text{cis}\frac{\pi}{4}$ *without having to calculate modulus and argument.*

Question 1b.

Worked solution

$$\text{Arg}\left(\frac{\overline{v}}{\overline{z}}\right) = \text{Arg}\overline{\left(\frac{v}{z}\right)}$$

$$= \frac{-\pi}{4}$$

$$\text{Arg}\left(\frac{v}{zi}\right) = \text{Arg}\left(\frac{v}{z} \div i\right)$$

$$= \text{Arg}\left(\frac{v}{z}\right) - \text{Arg}(i)$$

$$= \frac{\pi}{4} - \frac{\pi}{2}$$

$$= \frac{-\pi}{4}$$

Alternatively, solving for $\text{Arg}\left(\frac{v}{zi}\right)$:

$$\frac{v}{zi} = \frac{1+3i}{-1+2i} \times \frac{-1-2i}{-1-2i}$$

$$= \frac{5-5i}{5}$$

$$= 1-i$$

$$\text{Arg}\left(\frac{v}{zi}\right) = \tan^{-1}\left(\frac{-1}{1}\right)$$

$$= -\tan^{-1}(1) = \frac{-\pi}{4}$$

Mark allocation: 2 marks

- 1 mark for the correct answer for $\text{Arg}\left(\frac{\overline{v}}{\overline{z}}\right)$
- 1 mark for the correct answer for $\text{Arg}\left(\frac{v}{zi}\right)$

Question 2a.

Worked solution

$$\text{LHS} = \log_e(1) + 1^2 - 1 = 0 + 1 - 1$$
$$= 0 = \text{RHS}$$

Mark allocation: 1 mark

- 1 mark for the correct evaluation

Question 2b.

Worked solution

Using log laws, the relation can be written as $\log_e x - \log_e y + x^2 - y = 0$.

Now use implicit differentiation:

$$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} + 2x - \frac{dy}{dx} = 0$$

At the point (1, 1), we have

$$1 - \frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}$$

Since the tangent passes through the point (1, 1), the equation of the tangent (using the formula $y - y_1 = m(x - x_1)$) is

$$y - 1 = \frac{3}{2}(x - 1)$$
$$2y - 2 = 3x - 3$$

and so $3x - 2y - 1 = 0$.

Mark allocation: 3 marks

- 1 mark for correctly differentiating both sides of the equation with respect to x
- 1 mark for the correct evaluation of $\frac{dy}{dx}$
- 1 mark for the correct equation

Question 3a.

Worked solution

The mean $\overline{x}$ is the middle of the confidence interval:

$$\overline{x} = \frac{6.98 + 5.02}{2} = 6\,.$$

Therefore

$$6 + 1.96 \times \frac{s}{\sqrt{16}} = 6.98$$
$$\Rightarrow 1.96 \times \frac{s}{\sqrt{16}} = 0.98$$

and so

$$\begin{aligned} s &= \frac{4 \times 0.98}{1.96} \\ &= \frac{4 \times 0.98}{2 \times 0.98} \\ &= 2 \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for the correct mean
- 1 mark for the correct evaluation of s

Question 3b.

Worked solution

$H_0 : \mu = 4.5$

$H_1 : \mu < 4.5$

$$\begin{aligned} p &= \Pr\left(z < \frac{4 - 4.5}{\frac{1.5}{\sqrt{36}}} \right) = \Pr\left(\frac{-0.5}{0.25} \right) \\ &= \Pr(z < -2) \end{aligned}$$

$\Rightarrow\ p = 0.025$

$\therefore$ Reject the null hypothesis as $p = 0.025 < 0.05$.

Mark allocation: 4 marks

- 1 mark for the correct null and alternate hypothesis
- 1 mark for the correct evaluation of z
- 1 mark for the correct value of p
- 1 mark for the correct reason for rejecting the null hypothesis

Question 4

Worked solution

$$\int_0^{\sqrt{e^2-1}} \frac{4x\log_e(x^2+1)}{x^2+1}\,dx$$

$$=2\int_0^{\sqrt{e^2-1}} \frac{2x}{x^2+1}\log_e(x^2+1)dx$$

let $u=\log_e(x^2+1)$

$$\Rightarrow \frac{du}{dx}=\frac{2x}{x^2+1}$$

$$x=\sqrt{e^2-1}\Rightarrow u=\log_e(e^2)=2$$

$$x=0\Rightarrow u=\log_e(1)=0$$

$$\therefore 2\int_0^{\sqrt{e^2-1}} \frac{2x}{x^2+1}\log_e(x^2+1).dx = 2\int_0^2 u.\frac{du}{dx}.dx=\int_0^2 2u.du$$

$$=\left[u^2\right]_0^2$$

$$=4-0=4$$

Mark allocation: 3 marks

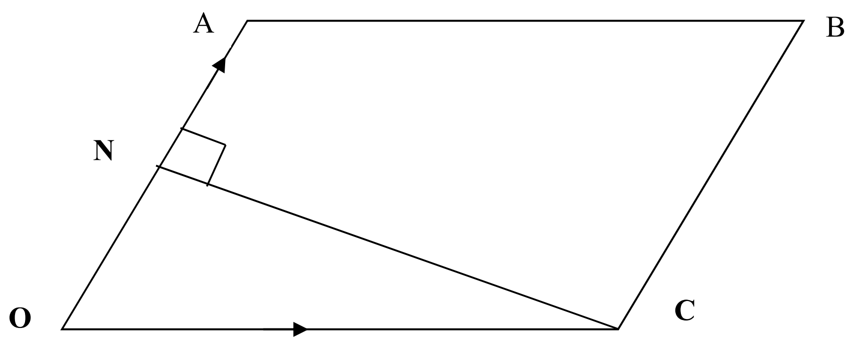
- 1 mark for correctly setting up the integrand for a substitution of $u=\log_e(x^2+1)$
- 1 mark for obtaining a correct integrand with respect to u
- 1 mark for the correct answer

Tip

- *Use the substitution* $u=\log_e(x^2+1)$ *as its derivative,* $\frac{2x}{x^2+1}$*, is a factor of the integrand.*

Question 5a.

Worked solution



For N to be closest to C, it must be perpendicular to $\overrightarrow{OA}$.

Let $\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

and $\overrightarrow{OC} = \underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$.

The scalar resolute of $\overrightarrow{OC}$ in the direction of equation of $\overrightarrow{OA} = ON$.

$$ON = \frac{\underset{\sim}{c} \cdot \underset{\sim}{a}}{a} = \frac{2-2+2}{\sqrt{1^2+2^2+(-1)^2}} = \frac{2}{\sqrt{6}}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5b.

Worked solution

$OC = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$

Using Pythagoras,

$$\begin{aligned} CN^2 &= OC^2 - ON^2 \\ &= 9 - \frac{4}{6} \\ &= 9 - \frac{2}{3} \\ &= \frac{25}{3} \end{aligned}$$

$\therefore CN = \dfrac{5}{\sqrt{3}}$

Mark allocation: 2 marks

- 1 mark for the correct evaluation of $\left|\overrightarrow{OC}\right|$
- 1 mark for the correct answer

Question 6

Worked solution

Let $s_L(t) =$ downward displacement of the lift floor, t seconds after the cup is dropped

and $s_C(t) =$ downward displacement of the cup, t seconds after the cup is dropped

$$\Rightarrow h = s_C\left(\frac{1}{2}\right) - s_L\left(\frac{1}{2}\right)$$

Using $s = ut + \frac{1}{2}at^2$

Lift floor: $u = v,\ a = 0$

$$\Rightarrow s_L(t) = vt$$

$$\Rightarrow s_L\left(\frac{1}{2}\right) = \frac{v}{2}$$

Coffee cup: $u = v,\ a = g$

$$\Rightarrow s_C(t) = vt + \frac{gt^2}{2}$$

$$\Rightarrow s_C\left(\frac{1}{2}\right) = \frac{v}{2} + \frac{g}{8}$$

Solve for h:

$$h = s_C\left(\frac{1}{2}\right) - s_L\left(\frac{1}{2}\right) = \frac{g}{8}$$

or $h = \frac{9.8}{8} = 1.225$

Mark allocation: 4 marks

- 1 mark for correctly expressing h as $s_C\left(\frac{1}{2}\right) - s_L\left(\frac{1}{2}\right)$
- 1 mark for the correct evaluation of $s_L\left(\frac{1}{2}\right)$
- 1 mark for the correct evaluation of $s_C\left(\frac{1}{2}\right)$
- 1 mark for the correct answer

Tip

- *There are several ways to do this question. The simplest method is to realise that since the initial speed of the coffee cup downwards equals the (constant) speed of the lift, we can model this situation as an object dropped from rest at a height* h *m above the ground.*

Question 7

Worked solution

$$\frac{dy}{dx} = \frac{e^{0.5x}}{3\sqrt{y}}$$

$$\Rightarrow \int 3y^{\frac{1}{2}}.dy = \int e^{0.5x}.dx$$

$$2y^{\frac{3}{2}} = 2e^{0.5x} + c$$

$y = 1$ when $x = 0$

$$\Rightarrow 2 = 2 + c$$

$$\Rightarrow c = 0$$

$$2y^{\frac{3}{2}} = 2e^{0.5x}$$

$$y^{\frac{3}{2}} = e^{0.5x}$$

$$\therefore y = e^{\frac{x}{3}}$$

Mark allocation: 3 marks

- 1 mark for correctly integrating using separation of variables
- 1 mark for correctly evaluating the constant of antidifferentiation
- 1 mark for the correct answer

Tip

- *Use the separation of variables technique to antidifferentiate.*

Question 8

Worked solution

$\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + 4t\underset{\sim}{j} + \underset{\sim}{c}$

$\dot{\underset{\sim}{r}}(0) = \underset{\sim}{c} = -2\underset{\sim}{i} - 3\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + 4t\underset{\sim}{j} - 2\underset{\sim}{i} - 3\underset{\sim}{k}$

$\Rightarrow \dot{\underset{\sim}{r}}(t) = (2t-2)\underset{\sim}{i} + 4t\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{r}(t) = (t^2-2t)\underset{\sim}{i} + 2t^2\underset{\sim}{j} - 3t\underset{\sim}{k} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = \underset{\sim}{d} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{r}(t) = (t^2-2t)\underset{\sim}{i} + 2t^2\underset{\sim}{j} - 3t\underset{\sim}{k} + \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\Rightarrow \underset{\sim}{r}(t) = (t^2-2t+1)\underset{\sim}{i} + (2t^2-1)\underset{\sim}{j} + (2-3t)\underset{\sim}{k}$

$\underset{\sim}{r}(2) = \underset{\sim}{i} + 7\underset{\sim}{j} - 4\underset{\sim}{k}$

$\underset{\sim}{r}(2) - \underset{\sim}{r}(0) = 8\underset{\sim}{j} - 6\underset{\sim}{k}$

$|\underset{\sim}{r}(2) - \underset{\sim}{r}(0)| = \sqrt{8^2 + (-6)^2} = 10$

$\therefore$ the particle is 10 metres from its initial position after 2 seconds.

Mark allocation: 4 marks

- 1 mark for the correct velocity vector $\dot{\underset{\sim}{r}}(t)$
- 1 mark for the correct position vector $\underset{\sim}{r}(t)$
- 1 mark for the correct vector $\underset{\sim}{r}(2) - \underset{\sim}{r}(0)$
- 1 mark for the correct answer

Question 9

Worked solution

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)=\dfrac{1}{2}$

Let $\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)=A$

$\Rightarrow \sin(x+y)+\sin(x-y)=2\sin(x)\cos(y)=\dfrac{1}{2}+A$

$\Rightarrow \sin(x+y)-\sin(x-y)=2\cos(x)\sin(y)=\dfrac{1}{2}-A$

$$\Rightarrow \frac{2\sin(x)\cos(y)}{2\sin(y)\cos(x)}=\tan(x)\cot(y)=\frac{\frac{1}{2}+A}{\frac{1}{2}-A}.$$

But $\tan(x)\cot(y)=\dfrac{\tan(x)}{\tan(y)}=\dfrac{\frac{1}{2}+A}{\frac{1}{2}-A}=3$

$\dfrac{1}{2}+A=\dfrac{3}{2}-3A$

$4A=1$

$A=\dfrac{1}{4}$

$\therefore A=\sin(x-y)=\dfrac{1}{4}$

Mark allocation: 4 marks

- 1 mark for correctly using the compound angle formula to correctly set up equations for $\sin(x+y)$ and $\sin(x-y)$
- 1 mark for correctly using the compound angle formula to add and subtract the first two equations
- 1 mark for obtaining the correct expression for $\dfrac{\tan(x)}{\tan(y)}$
- 1 mark for the correct answer

Alternatively,

$$\frac{\tan(x)}{\tan(y)} = \frac{\frac{\sin(x)}{\cos(x)}}{\frac{\sin(y)}{\cos(y)}} = 3$$

$$\Rightarrow \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)} = 3$$

$$\Rightarrow \sin(x)\cos(y) = 3\cos(x)\sin(y)$$

As

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) = \frac{1}{2}$$

$$\sin(x+y) = 3\cos(x)\sin(y) + \cos(x)\sin(y) = 4\cos(x)\sin(y) = \frac{1}{2}$$

So

$$\cos(x)\sin(y) = \frac{1}{8}$$

Therefore

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$= 3\cos(x)\sin(y) - \cos(x)\sin(y)$$

$$= 2\cos(x)\sin(y)$$

$$= \frac{1}{4}$$

Mark allocation: 4 marks

- 1 mark for the correct equation $\sin(x)\cos(y) = 3\cos(x)\sin(y)$
- 1 mark for the correct equations $\cos(x)\sin(y) = \frac{1}{8}$
- 1 mark for the correct obtaining $\sin(x-y) = 2\cos(x)\sin(y)$
- 1 mark for the correct answer

Question 10a.

Worked solution

$x = \arcsin(t) \Rightarrow t = \sin(x)$

$y = 2t^2\sqrt{1-t^2} \Rightarrow y = 2\sin^2(x)\sqrt{1-\sin^2(x)}$

$y = 2\sin^2(x)\cos(x)$

Mark allocation: 1 mark

- 1 mark for the correct equation

Question 10b.

Worked solution

Volume, $V = \pi\int_0^{\frac{\pi}{2}}\left(2\sin^2(x)\cos(x)\right)^2.dx \;=\pi\int_0^{\frac{\pi}{2}}\left(4\sin^4(x)\cos^2(x)\right).dx$

$$V = \pi\int_0^{\frac{\pi}{2}}\left(4\sin^2(x)\cos^2(x)\right)\sin^2(x).dx$$

$$V = \pi\int_0^{\frac{\pi}{2}}\left(2\sin(x)\cos(x)\right)^2.\sin^2(x).dx \quad = \pi\int_0^{\frac{\pi}{2}}\sin^2(2x).\frac{1-cos(2x)}{2}.dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2x).dx \text{ - } \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2x).\cos(2x).dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos(4x)}{2}.dx - \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2x).\cos(2x).dx$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\left(1-\cos(4x)\right).dx - \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2x).\cos(2x).dx$$

$$= \frac{\pi}{4}\left[x - \frac{\sin(4x)}{4}\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2x).\cos(2x).dx$$

$$= \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right] - \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2x).\cos(2x).dx$$

$$V = \frac{\pi^2}{8} - \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2x).\cos(2x).dx$$

Let $u = \sin(2x)$

$x = \frac{\pi}{2} \Rightarrow u = \sin(\pi) = 0$ and $x = 0 \Rightarrow u = \sin(0) = 0$

$\frac{du}{dx} = 2\cos 2x \;\Rightarrow \cos(2x) = \frac{1}{2}.\frac{du}{dx}$

$$\Rightarrow V = \frac{\pi^2}{8} - \frac{\pi}{2}\int_0^0 u^2.\frac{1}{2}.\frac{du}{dx}.dx = \frac{\pi^2}{8} - \frac{\pi}{4}\int_0^0 u^2.du$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4}\left[\frac{u^3}{3}\right]_0^0 = \frac{\pi^2}{8} - 0 = \frac{\pi^2}{8}$$

$\therefore$ the volume generated is $\frac{\pi^2}{8}$ cubic units.

Alternatively, the volume of the solid is

$$V = \pi\int_0^{\frac{\pi}{2}} 4\sin^4 x\cos^2 x\,dx$$

$$= 4\pi\int_0^{\frac{\pi}{2}} \frac{1}{4}(1-\cos 2x)^2 \cdot \frac{1}{2}(1+\cos 2x)dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1-2\cos 2x+\cos^2 2x)(1+\cos 2x)dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x-\cos^2 2x+\cos^3 2x)dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x-\frac{1}{2}-\frac{1}{2}\cos 4x+\cos^3 2x)dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\,dx - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos 2x\,dx - \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \cos 4x\,dx + \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos^3 2x\,dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\,dx$$

$$= \frac{\pi}{2}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi^2}{8}$$

To see the simplification that occurred before the final answer, note that

$$\int_0^{\frac{\pi}{2}} \cos 2x\,dx = \int_0^{\frac{\pi}{2}} \cos^3 2x\,dx = \int_0^{\frac{\pi}{2}} \cos 4x\,dx = 0$$

Mark allocation: 4 marks

- 1 mark for correctly setting up the integrand to represent the volume required
- 1 mark for using the double angle formulas correctly to set the integrand up for antidifferentiating
- 1 mark for correctly antidifferentiating
- 1 mark for the correct answer

Tips

- *Correct application of the double angle formulas is necessary to obtain the volume.*
- *It is useful to know the formulas:*

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

END OF WORKED SOLUTIONS