SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

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2017 Trial Examination

<u>SOLUTIONS</u>

SECTION A: Multiple-choice questions (1 mark each)

Question 1

***Answer:* D**

Explanation:

$$x = \frac{1}{3}\cos(2t) = \frac{1}{3}(1 - 2\sin^2(t)) = \frac{1}{3}(1 - 2(2y)^2) \Rightarrow$$
$$3x + 8y^2 = 1$$

Question 2

Answer: B

Explanation

$$-\frac{\pi}{2} < \arctan(3x+1) < \frac{\pi}{2}$$
$$-\frac{3\pi}{2} < -3\arctan(3x+1) < \frac{3\pi}{2}$$
$$2\arctan(\sqrt{3}) - \frac{3\pi}{2} < 2\arctan(\sqrt{3}) - 3\arctan(3x+1) < 2\arctan(\sqrt{3}) + \frac{3\pi}{2}$$
$$-\frac{5\pi}{6} = \frac{2\pi}{3} - \frac{3\pi}{2} < 2\arctan(\sqrt{3}) - 3\arctan(3x+1) < \frac{2\pi}{3} + \frac{3\pi}{2} = \frac{13\pi}{6}$$

Question 3

Answer: E

Explanation:
Simplify by CAS

$$\text{expand}\left(\frac{2\cdot x^3+5\cdot x^2+2\cdot x-1}{x^2-1}\right) \qquad \frac{4}{x-1}+2\cdot x+5$$

$f(x) = 2x + 5 + \frac{4}{x-1}$ has an oblique asymptote $y = 2x + 5$
a vertical asymptote $x = 1$

Question 4

Answer: D

Explanation:
Let $z_1 = 2 + 3i$, $z_2 = 2 - 3i$ and z_3 be the three roots of the polynomial.
Then $P(z) = (z - z_1)(z - z_2)(z - z_3)$

$$-z_1 z_2 z_3 = -39$$
$$z_3 = \frac{39}{z_1 z_2} = \frac{39}{13} = 3$$

Expand $(z - z_1)(z - z_2)(z - z_3)$ by CAS

$\text{expand}\left((z-(2+3\cdot \boldsymbol{i}))\cdot(z-(2-3\cdot \boldsymbol{i}))\cdot(z-3)\right)$	$z^3-7\cdot z^2+25\cdot z-39$

$$b = -7, \ \ c = 25$$

Question 5

Answer: A

Explanation:
Using CAS

Define $arg(a,b)=\text{angle}(a-2\cdot b\cdot \boldsymbol{i})$	*Done*
$arg(-6,-\sqrt{3})$	$\frac{5\cdot\pi}{6}$

Question 6

Answer: E

Explanation:

$$z^4 + ri = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$$
$$= z^4 - (z_1 + z_2 + z_3 + z_4)z^3 + (z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_1)z^2$$
$$-(z_1 z_2 z_3 + z_2 z_3 z_4 + z_3 z_4 z_1 + z_4 z_1 z_2)z + z_1 z_2 z_3 z_4$$

Therefore

$$z_1 + z_2 + z_3 + z_4 = 0$$
$$z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_1 = 0$$
$$z_1 z_2 z_3 + z_2 z_3 z_4 + z_3 z_4 z_1 + z_4 z_1 z_2 = 0$$

$z_1 z_2 z_3 z_4 = ri$, ***hence it's impossible to have*** $\boldsymbol{z_1 = \overline{z_2}}$, $\boldsymbol{z_3 = \overline{z_4}}$**.**

OR

$z^4 = -ri = r\ cis\left(-\frac{\pi}{2}\right) \Rightarrow$

z_1, z_2, z_3 and z_4 have principal arguments $-\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$ and $-\frac{5\pi}{8}$. They are not in conjugate pairs.

Question 7

Answer: C

Explanation:
Using CAS

Define $x(t)=\tan(t)-2\cdot t$	*Done*
Define $y(t)=2\cdot \ln(\sec(t))$	*Done*
⚠ $\frac{d}{dt}(y(t))$	$2\cdot \tan(t)$
$\frac{d}{dt}(x(t))$	$\frac{-(2\cdot (\cos(t))^2-1)}{(\cos(t))^2}$
⚠ $\frac{2\cdot \tan(t)}{\frac{-(2\cdot (\cos(t))^2-1)}{(\cos(t))^2}}$	$\frac{-2\cdot \sin(t)\cdot \cos(t)}{2\cdot (\cos(t))^2-1}$
⚠ tCollect$\left(\frac{-2\cdot \sin(t)\cdot \cos(t)}{2\cdot (\cos(t))^2-1}\right)$	$-\tan(2\cdot t)$

Question 8

Answer: D

Explanation:
Let $u = x^3 - x^2 + 19x + 8.$
Then $\frac{du}{dx} = 3x^2 - 2x + 19 \Rightarrow dx = \frac{1}{3x^2-2x+19}du$

$$\int_0^1 ((3x^2-2x+19)\sqrt[3]{x^3-x^2+19x+8})dx = \int_8^{27} u^{\frac{1}{3}}\,du$$

Question 9

Answer: C

Explanation:
From CAS we have

Define $f(x)=\sin(x)+\cos(x)$ *Done*

$\text{euler}(f(x),x,y,\{0,0.3\},1,0.1)$ $\begin{bmatrix} 0. & 0.1 & 0.2 & 0.3 \\ 1. & 1.1 & 1.20948 & 1.32736 \end{bmatrix}$

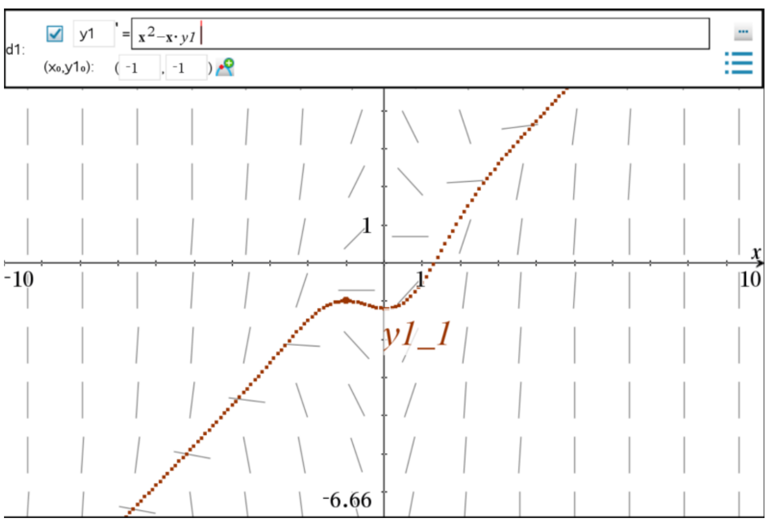
Therefore

$$y_3 = y_2 + 0.1 \times f(0.2) = 1.21 + 0.1f(0.2)$$

Question 10

Answer: E

Explanation:
From CAS we have



Question 11

Answer: D

Explanation:
A useful mathematical model for setting up differential equations of dynamic systems

$$\frac{dQ}{dt} = R_{in} \times C_{in} - R_{out} \times C_{out}$$

where R_{in} and R_{out} are the flowing in and flowing out rate; C_{in} and C_{out} are the concentrations of the solutions which are flowing in and flowing out respectively.
Therefore

$$\frac{dQ}{dt} = 8 \times 200 - 6 \times \frac{Q}{300 + (8-6)t} = 1600 - \frac{3Q}{150 + t}$$

$$\frac{dQ}{dt} + \frac{3Q}{150 + t} = 1600$$

Question 12

Answer: A

Explanation:
Solve $\left|\underset{\sim}{u} - (\underset{\sim}{u} \cdot \underset{\sim}{\hat{v}})\underset{\sim}{\hat{v}}\right| = \frac{\sqrt{10}}{2}$ for a in CAS

Define $u=[{}^{-}1 \quad a \quad 1]$	*Done*
Define $v=[1 \quad 3 \quad 2]$	*Done*
Define $vh=\text{unitV}(v)$	*Done*
$\text{solve}\left(\text{norm}(u-\text{dotP}(u,vh)\cdot vh)=\frac{\sqrt{10}}{2},a\right)$	$a=\frac{-4}{5}$ or $a=2$

Question 13

Answer: C

Explanation:
Solve $\det\left(\begin{bmatrix} 3 & 2 & 4 \\ m & 1 & -2 \\ 1 & m & 2 \end{bmatrix}\right) \neq 0$ for m in CAS, $m \neq -1$ and $m \neq \frac{1}{2}$

Question 14

Answer: E

Explanation:

$$\underset{\sim}{a} = \frac{1}{m}\left(\underset{\sim}{F_1} + \underset{\sim}{F_2}\right) = \frac{1}{12}\left(-5\underset{\sim}{i} + 2\underset{\sim}{j}\right) \Rightarrow \left|\underset{\sim}{a}\right| = \frac{1}{12}\sqrt{(-5)^2 + 2^2} = \frac{\sqrt{29}}{12}$$

Question 15

Answer: A

Explanation:
By Lammi's Theorem

$$\frac{F_1}{\sin(120°)} = \frac{F_2}{\sin(90°)} = \frac{F_3}{\sin(150°)}$$

Therefore

$$\frac{2\sqrt{3}}{3}F_1 = F_2 = 2F_3$$

Question 16

Answer: D

Explanation:
The velocity $v = 30\cos(15°)\,\underset{\sim}{i} + (30\sin(15°) + 9.8t)\underset{\sim}{j}$

The vertical height travelled $h = 30\sin(15°)\,t + \frac{1}{2} \times 9.8t^2 = 2.5 \Rightarrow t \approx 0.274\ s$

Therefore, the travelled distance

$$L = \int_0^{0.2744} \left|30\cos(15°)\,\underset{\sim}{i} + (30\sin(15°) + 9.8t)\underset{\sim}{j}\right| dt \approx 8.34\ m$$

$\text{solve}\left(30\cdot\sin(15°)\cdot t + \frac{1}{2}\cdot 9.8\cdot t^2 = 2.5, t\right)$ — $t = -1.85905$ or $t = 0.274443$

$\int_0^{0.274443} \text{norm}\left(\left[30\cdot\cos(15°) \quad 30\cdot\sin(15°) + 9.8\cdot t\right]\right)\, dt$ — 8.33891

Question 17

Answer: B

Explanation:

$$v\frac{dv}{dx} = \frac{v(2x+1)}{(x+1)(3x+2)}$$

$$v(10) - v(2) = \int_2^{10} \frac{2x+1}{(3x+2)(x+1)} dx$$

$$v(10) = \int_2^{10} \left(\frac{-1}{3x+2} - \frac{-1}{x+1}\right) dx + v(2) = \int_2^{10} \left(\frac{1}{x+1} - \frac{1}{3x+2}\right) dx + 5$$

Question 18

Answer:

Explanation:

$$mean = 5 \times 430 + 10 \times 250 = 4650$$

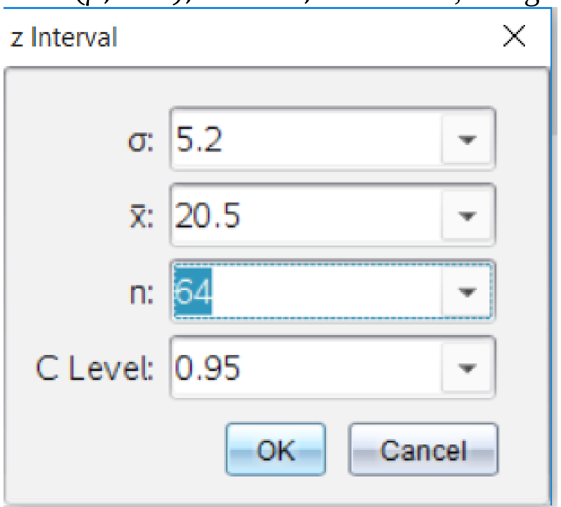
$$std = \sqrt{5 \times 25^2 + 10 \times 10^2} = 5\sqrt{165}$$

Question 19

Answer: E

Explanation:

$X \sim N(\mu, 5.2^2)$, $n = 64$, $\bar{x} = 20.5$, using CAS



zInterval 5.2,20.5,64,0.95: *stat.results*

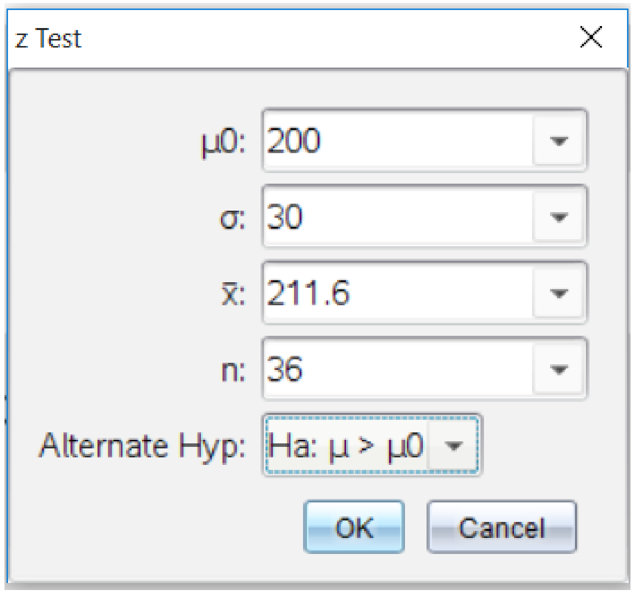
"Title"	"z Interval"
"CLower"	19.226
"CUpper"	21.774
"x̄"	20.5
"ME"	1.27398
"n"	64.
"σ"	5.2

Question 20

Answer: B

Explanation:

Enter the statistics into CAS



zTest 200,30,211.6,36,1: *stat.results*

"Title"	"z Test"
"Alternate Hyp"	"μ > μ0"
"z"	2.32
"PVal"	0.01017
"x̄"	211.6
"n"	36.
"σ"	30.

$p \approx 0.0102$ is less than 0.05, hence enough evidence to reject H_0.

SECTION B: Extended Response questions

Question 1 (10 marks)

a. $f'(x) = \frac{4x^3-16x^2+16x+3}{(x-2)^2}$

Solve $f'(x) = 0$ for x, $x \approx -0.1607$.

$f(-0.1607) \approx 2.440$

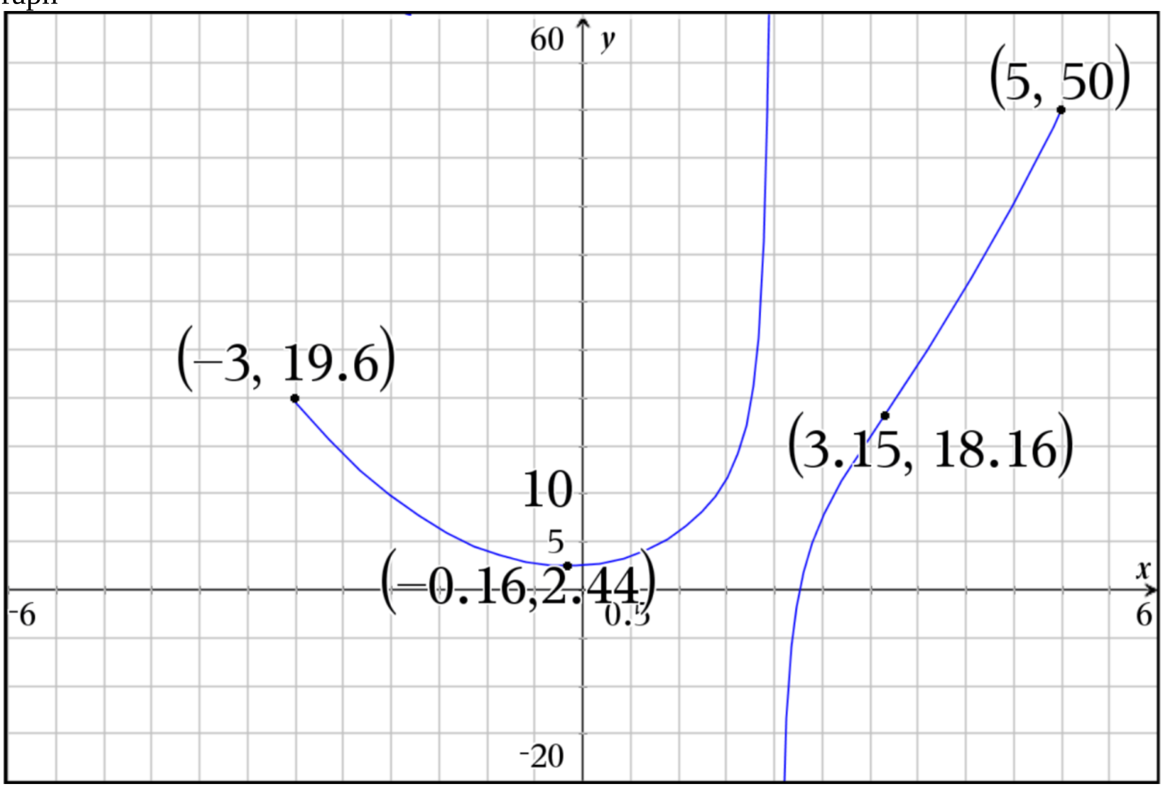
The required stationary point is $(-0.16, 2.44)$. 1 mark

b. Solve $f''(x) = 0$ for x by CAS, $x \approx 3.1447$ 1 mark

$f(3.1447) \approx 18.1577$

The required point of inflection is $(3.14, 18.16)$ 1 mark

c. Graph



Labelling the turning point and the point of inflection 1 mark

Labelling the end points 1 mark

Correct graph for $x \in [-3, 2) \cup (2, 5]$ 1 mark

d. *i.* $L = \int_{-3}^{1} \sqrt{1 + \left(f'(x)\right)^2}\, dx = \int_{-3}^{1} \sqrt{1 + \left(\frac{4x^3-16x^2+16x+3}{(x-2)^2}\right)^2}\, dx$ 1 mark

ii. $\approx 21.44\ units$ 1 mark

e. $V = \pi \int_{-3}^{1} \left(f(x)\right)^2 dx$ 1 mark

$\approx 888\ units^3$ 1 mark

Question 2 (11 marks)

a. Solve $\left|x+yi+\frac{9}{2}+\left(\frac{7\sqrt{3}}{2}-1\right)i\right| = \left|x+yi-(\sqrt{3}+1)i\right|$ for y in CAS. 1 mark

$$\text{solve}\left(\left|x+y\cdot \boldsymbol{i}+\frac{9}{2}+\left(\frac{7\cdot\sqrt{3}}{2}-1\right)\cdot \boldsymbol{i}\right|=\left|x+y\cdot \boldsymbol{i}-\left(\sqrt{3}+1\right)\cdot \boldsymbol{i}\right|,y\right) \quad y=\frac{-\sqrt{3}\cdot\left(x-\sqrt{3}+6\right)}{3}$$

$$\text{expand}\left(\frac{-\sqrt{3}\cdot\left(x-\sqrt{3}+6\right)}{3}\right) \quad \frac{-\sqrt{3}\cdot x}{3}-2\cdot\sqrt{3}+1$$

Therefore, $y=\frac{-\sqrt{3}}{3}x-2\sqrt{3}+1$ 1 mark

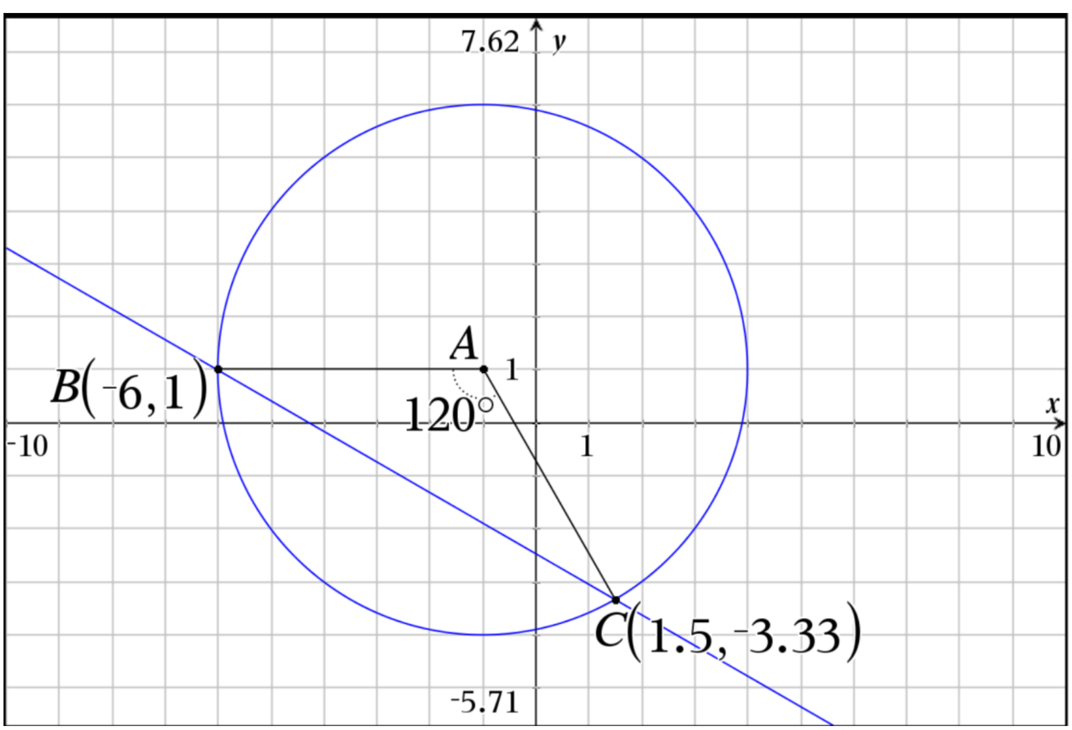
b. Solve $\left|z+\frac{9}{2}+\left(\frac{7\sqrt{3}}{2}-1\right)i\right| = \left|z-(\sqrt{3}+1)i\right|$ and $|z+1-i|=5$ simultaneously in CAS.
1 mark

$$\text{solve}\left(\begin{cases}\left|x+y\cdot \boldsymbol{i}+\frac{9}{2}+\left(\frac{7\cdot\sqrt{3}}{2}-1\right)\cdot \boldsymbol{i}\right|=\left|x+y\cdot \boldsymbol{i}-\left(\sqrt{3}+1\right)\cdot \boldsymbol{i}\right| \\ \left|x+y\cdot \boldsymbol{i}+1-\boldsymbol{i}\right|=5\end{cases},x,y\right)$$

$$x=-6 \text{ and } y=1 \text{ or } x=\frac{3}{2} \text{ and } y=\frac{-\left(5\cdot\sqrt{3}-2\right)}{2}$$

The intersections are $B(-6,1)$ and $C\left(\frac{3}{2},\ 1-\frac{5\sqrt{3}}{2}\right)$. 1 mark

c. Graph



Correct graphs for the circle and the line 1 mark
Labelling the intersections 1 mark

d. $\angle ABC = \arctan\left(\frac{\sqrt{3}}{3}\right) = 30° \Rightarrow$ 1 mark
$\angle BAC = 180° - 2 \times 30° = 120°$ 1 mark

e. The area of the minor sector $A_1 = \frac{1}{3} \times \pi r^2 = \frac{25\pi}{3}$ 1 mark

The area of the triangle ABC $A_2 = \frac{1}{2} \times (-1+6) \times \left(1 - \left(1 - \frac{5\sqrt{3}}{2}\right)\right) = \frac{25\sqrt{3}}{4}$

1 mark

Alternative: $A_2 = \frac{1}{2} AB \times AC \times \sin(120°) = \frac{25\sqrt{3}}{4}$

Therefore, the area of the minor segment

$$A = \frac{25\pi}{3} - \frac{25\sqrt{3}}{4}$$ 1 mark

Question 3 (9 marks)

a. i. Solve $\frac{dp}{dt} = r\,p(1 - \frac{p}{k})$ by CAS,
1 mark

$\text{deSolve}\left(p' = r \cdot p \cdot \left(1 - \frac{p}{k}\right), t, p\right)$ $\quad p = \frac{k \cdot \mathbf{e}^{r \cdot t}}{\mathbf{e}^{r \cdot t} + k \cdot \boldsymbol{c1}}$

Therefore $p = \frac{k\,e^{rt}}{e^{rt} + kc}$ or $\frac{k\,e^{rt}}{e^{rt} + A}$, where c and A are constants.

1 mark

ii. $5000 = \lim_{t \to \infty} \frac{k\,e^{rt}}{e^{rt} + kc} = \lim_{t \to \infty} \frac{k}{1 + kc\ e^{-rt}} = k$
$k = 5000$

1 mark

b. From Parts a i) and a ii), $P(t)=\frac{5000\times e^{rt}}{e^{rt}+5000c}$

Solve $P(0)=2500$ and $P(10)=2600$ simultaneously by CAS,

$$c=\frac{1}{5000},\quad r=\frac{1}{10}\ln\left(\frac{13}{12}\right)$$ 1 mark

Therefore

$$P(t)=\frac{5000\times e^{\ln\left(\frac{13}{12}\right)^{\frac{1}{10}}}}{e^{\ln\left(\frac{13}{12}\right)^{\frac{1}{10}}}+5000\times\frac{1}{5000}}$$ 1 mark

$$=\frac{5000\times\left(\frac{13}{12}\right)^{\frac{t}{10}}}{1+\left(\frac{13}{12}\right)^{\frac{t}{10}}}$$ 1 mark

c. $Let\ u=\frac{dp}{dt}=\frac{1}{10}\ln\left(\frac{13}{12}\right)p\left(1-\frac{p}{5000}\right)$

$\frac{d^2p}{dt^2}=\frac{du}{dt}=\frac{du}{dp}\times\frac{dp}{dt}$ 1 mark

$=\frac{1}{10}\ln\left(\frac{13}{12}\right)\left(1-\frac{p}{2500}\right)\times\frac{1}{10}\ln\left(\frac{13}{12}\right)p\left(1-\frac{p}{5000}\right)$ 1 mark

$=\left(\frac{1}{10}\ln\left(\frac{13}{12}\right)\right)^2 p\left(1-\frac{p}{2500}\right)\left(1-\frac{p}{5000}\right)$ 1 mark

Question 4 (11 marks)

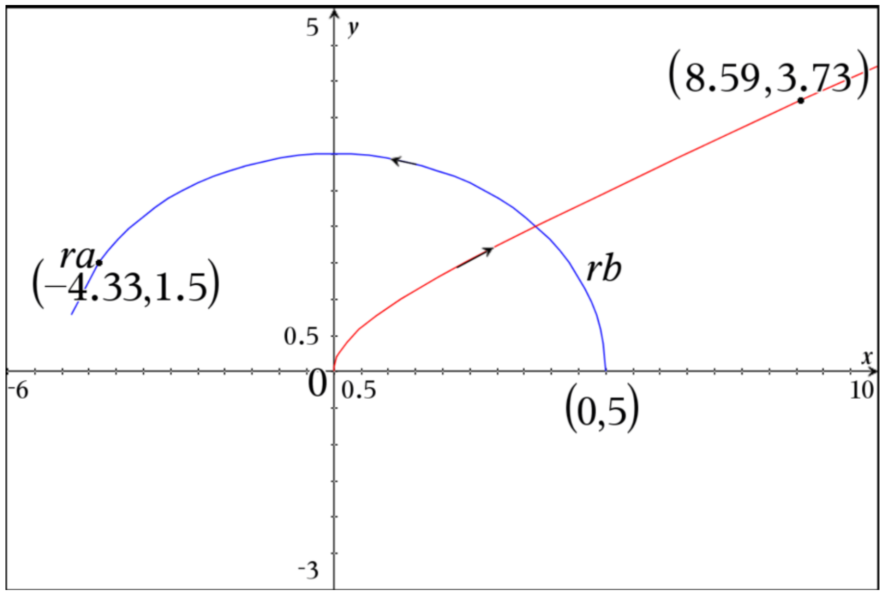
a. Solve $\underset{\sim}{r_A}(s)=\underset{\sim}{r_B}(t)$ for s,t by CAS,

At $s\approx 0.37$ Particle A passes $(3.72, 2.00)$ 2 mark

At $t=1.11$ Particle B passes $(3.72, 2.00)$ 1 mark

Define $ra(t)=\begin{bmatrix}5\cdot\cos(2\cdot t)\\3\cdot\sin(2\cdot t)\end{bmatrix}$	*Done*
Define $rb(t)=\begin{bmatrix}-3+3\cdot\sec(t)\\\tan(t)\end{bmatrix}$	*Done*
⚠ solve$(ra(s)=rb(t),s,t)\vert 0\le s\le\frac{5\cdot\pi}{12}$ and $0\le t\le\frac{5\cdot\pi}{12}$	$s=0.365859$ and $t=1.10804$
$ra(0.365859)$	$\begin{bmatrix}3.72014\\2.00445\end{bmatrix}$

b. Graph



Correct end points	1 mark
Correct directions	1 mark
Correct shapes	1 mark

c. $\dot{\underset{\sim}{r}}_A(t) = -10\sin(2t)\,\underset{\sim}{i} + 6\cos(2t)\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}_B(t) = 3\sec(t)\tan(t)\,\underset{\sim}{i} + \sec^2(t)\,\underset{\sim}{j}$ 1 mark

Let θ be the angle between the moving directions when they cross the same point.

$$\cos(\theta) = \frac{\dot{\underset{\sim}{r}}_A(0.366)\cdot\dot{\underset{\sim}{r}}_B(1.108)}{\left|\dot{\underset{\sim}{r}}_A(0.366)\right|\left|\dot{\underset{\sim}{r}}_B(1.108)\right|} = -0.5855$$

$\theta \approx 126°$ 1 mark

d. **i.** Let $h(t) = \left|\underset{\sim}{r}_A(t) - \underset{\sim}{r}_B(t)\right|$ 1 mark

Solve $\frac{d}{dt}(h(t)) = 0$ for t, $t \approx 0.703\ s$ 1 mark

ii. $h(0.7035) \approx 2.11\ m$ 1 mark

Question 5 (11marks)

a. Proof

$ma = -mg - 0.025mv^2$

$40a = -40g - v^2$ 1 mark

$40\frac{dv}{dt} = 40\,v\frac{dv}{dx} = -40g - v^2$

$\frac{dv}{dx} = -\frac{40g+v^2}{40v}$ 1 mark

b. Solve $x' = -\frac{40v}{40g+v^2}$ in CAS, $x = c - 20\ln(v^2 + 40g)$ 1 mark

$\text{deSolve}\left(x'=\frac{-40\cdot v}{40\cdot g+v^2},v,x\right)$	$x=\mathbf{c1}-20\cdot\ln\left(v^2+40\cdot g\right)$

Substitute $x = 0, v = 80$,

$0 = c - 20\ln(80^2 + 40g) \Rightarrow c = 20\ln(6400 + 40g)$

$x = 20\ln(\frac{6400+40g}{v^2+40g})$ 1 mark

c. At the highest point $v = 0$. 1 mark

$x = 20\ln\left(\frac{6400+40\times9.8}{0^2+40\times9.8}\right) \approx 57.04\,m$ 1mark

d. Solve the differential equation $y' = \frac{1000v}{1000\times9.8-v^2}$ in CAS

$y = c - 500\ln|v^2 - 9800|$ 1 mark

Substitute $v = 0,\ y = 0,$

$c = 500\ln(9800)$ 1 mark

$y = 500\ln|\frac{9800}{9800-v^2}|$

Solve $120 = 500\ln|\frac{9800}{9800-v^2}|$ for , $v = 45.728\,ms^{-1}$ 1 mark

e. $a = \frac{dv}{dt} = g - 0.001v^2$ 1 mark

$t = \int_0^{45.728}\frac{1}{9.8-0.001v^2}dt \approx 5.05\,s$ 1 mark

Question 6 (8 marks)

a. Let $Y = X_1 + X_2 + \cdots + X_{25}$, where $X_i \sim N(200, 5^2)$.
Therefore $E(Y) = 25 \times 200 = 5000$, $Var(Y) = 25 \times 5^2 = 625$ 1 mark

$\Pr(Y > 5050) = 0.02275$ 1 mark

b. Let $\bar{X}$ be the sample mean of the amount of the liquid. Then $\bar{X} \sim N(200, \left(\frac{5}{5}\right)^2)$. 1 mark

$\Pr(\bar{X} > 201) = 0.1587$ 1 mark

c. Find from CAS

zInterval 5,197.5,25,0.95: *stat.results*

"Title"	"z Interval"
"CLower"	195.54
"CUpper"	199.46
"$\bar{x}$"	197.5
"ME"	1.95996
"n"	25.
"σ"	5.

(195.54, 199.46)

d. $H_0: \mu = 200$, $H_1: \mu < 200$ 1 mark

e. $p = \Pr(\bar{X} < 197.5 | \mu = 200) = 0.00621$

zTest 200,5,197.5,25,-1: *stat.results*

"Title"	"z Test"
"Alternate Hyp"	"μ < μ0"
"z"	-2.5
"PVal"	0.00621
"$\bar{x}$"	197.5
"n"	25.
"σ"	5.

Conclusion: There is sufficient evidence to reject H_0.