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Free Exam
For 2016-18 VCE Study design

Engage
Education
Foundation

Units 3 and 4 Specialist Maths: Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

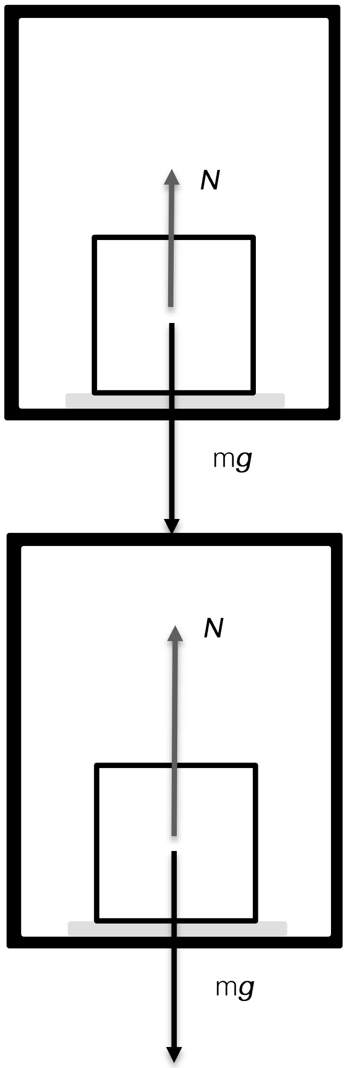
Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a

The box is accelerating downward at 2.8 m/s^2, in one dimensional motion. Taking the downward direction to be positive, the net force acting on the box can be written as:

$F = ma = mg - N$ [1]

(where mg is the force due to gravity, and N is the reaction force)



Rearranging:

$$m = \frac{N}{(g-a)}$$
$$= \frac{N}{9.8 - 2.8}$$
$$= \frac{N}{7}$$

The scale reads a value proportional to the reaction force N. Specifically, it outputs:

m': $N = m'g$ [1]

Hence:

$$m = \frac{N}{7}$$
$$= \frac{m'g}{7}$$
$$= \frac{35g}{7}$$
$$= 5g$$
$$= 49 \text{ kg}$$ [1]

Question 1b

Now taking the upwards direction to be positive:

$F = ma = N - mg$ [1]

Rearranging:

$$N = ma + mg$$
$$= m(a+g)$$

Thus the output on the scale is given:

$$m' = \frac{N}{g}$$
$$= \frac{m(a+g)}{g}$$
$$= \frac{m \times \frac{3g}{2}}{g} \text{ since } a = 4.9 = \frac{g}{2}$$
$$= \frac{3m}{2}$$
$$= 73.5 \text{ kg}$$ [1]

Question 2

Method 1:

This method is a variation on completing the square.

$z^4 + 9z^2 + 64 = 0$
$z^4 + 16z^2 + 8^2 - 7z^2 = 0$ [1]
$(z^2 + 8)^2 = 7z^2$
$z^2 + 8 = \pm z\sqrt{7}$ [1]

$\Rightarrow z^2 + z\sqrt{7} + 8 = 0$ or $z^2 - z\sqrt{7} + 8 = 0$ [1]

Each equation can then be solved by the quadratic formula, yielding:

$z = \frac{-\sqrt{7} \pm \sqrt{7 - 4 \times 1 \times 8}}{2 \times 1}$ or $z = \frac{\sqrt{7} \pm \sqrt{7 - 4 \times 1 \times 8}}{2 \times 1} = \frac{\sqrt{7} \pm 5\boldsymbol{i}}{2}$ [1]
$z = \frac{-\sqrt{7} \pm 5i}{2}$ or $z = \frac{\sqrt{7} \pm 5i}{2}$

So the solutions are:

$z_1 = \frac{-\sqrt{7} - 5i}{2}$ $z_2 = \frac{-\sqrt{7} + 5i}{2}$ $z_3 = \frac{\sqrt{7} - 5i}{2}$ $z_4 = \frac{\sqrt{7} + 5i}{2}$ [1]

Method 2:

$z^4 + 9z^2 + 64 = 0$
Let $a = z^2$
$a^2 + 9a + 64 = 0$
$\left(a + \frac{9}{2}\right)^2 - \frac{81}{4} + \frac{256}{4} = 0$
$\left(a + \frac{9}{2}\right)^2 = -\frac{175}{4}$
$\therefore a = z^2 = \frac{-9 \pm 5\sqrt{-7}}{2}$ [1]
$\Rightarrow i^2 z^2 = \frac{9 \pm 5\sqrt{-7}}{2}$
$\Rightarrow iz = \pm\sqrt{\frac{9 \pm 5\sqrt{-7}}{2}}$
$= \pm\frac{\sqrt{18 \pm 10\sqrt{-7}}}{2}$ [1]

The problem now is that the real and imaginary terms under the square root cannot be separated.

Suppose:

$18 \pm 10\sqrt{-7} = (A \pm B)^2$
$9 + X \pm 10\sqrt{-7} + 9 - X = (A \pm B)^2$
$\left(\sqrt{9 + X}\right)^2 \pm 10\sqrt{-7} + \left(\sqrt{9 - X}\right)^2 = A^2 \pm 2AB + B^2$

Now take:

$A = \sqrt{9+X}$
$B = \sqrt{9-X}$

And choose X such that:

$2AB = 10\sqrt{-7}$
$\sqrt{(9+X)(9-X)} = 5\sqrt{-7}$
$\Rightarrow 81 - X^2 = 25(-7)$
$X^2 = 256$
$\Rightarrow X = 16$ (sign is irrelevant because of symmetry in A, B) [1]

Hence:

$18 \pm 10\sqrt{-7} = \left(\sqrt{9+16} \pm \sqrt{9-16}\right)^2$
$= \left(5 \pm \sqrt{7}i\right)^2$ [1]

And:

$$iz = \pm\frac{\sqrt{18 \pm 10\sqrt{-7}}}{2}$$
$$= \pm\frac{\sqrt{\left(5 \pm \sqrt{7}i\right)^2}}{2}$$
$$= \pm\frac{\left(5 \pm \sqrt{7}i\right)}{2}$$
$$\Rightarrow z = -i \cdot iz$$
$$= \pm\frac{\left(\sqrt{7} \pm 5i\right)}{2}$$

Giving the solutions:

$$z_1 = \frac{-\sqrt{7} - 5\boldsymbol{i}}{2} \quad z_2 = \frac{-\sqrt{7} + 5\boldsymbol{i}}{2} \quad z_3 = \frac{\sqrt{7} - 5\boldsymbol{i}}{2} \quad z_4 = \frac{\sqrt{7} + 5\boldsymbol{i}}{2} \quad [1]$$

Question 3a
Method 1: By implicit differentiation:

Take the natural logarithm of both sides to eliminate the power.

$y = x^x$

$\Rightarrow \ln(y) = x\ln(x) \quad \left[\frac{1}{2}\right]$

$\frac{d}{dy}\ln(y) \times \frac{dy}{dx} = \frac{d}{dx}(x\ln(x)) \quad \left[\frac{1}{2}\right]$

$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln(x) = 1 + \ln(x)$

$\frac{dy}{dx} = y + y \times \ln(x) = x^x + x^x.\ln(x)$ [1]

Method 2:

$y = x^x$
$= e^{\ln(x^x)}$ $\left[\frac{1}{2}\right]$
$= e^u$ where $u = x\ln x$
$\Rightarrow \frac{dy}{dx} = \frac{d(e^u)}{du} \times \frac{d(x\ln x)}{dx}$ $\left[\frac{1}{2}\right]$
$= e^u\left(x\left(\frac{1}{x}\right) + (1)\ln(x)\right)$
$= x^x(1 + \ln(x))$ [1]

Question 3b i

$f(x) \in R^- \cup \{0\}$ [1]

This can be worked out from a rough sketch of $y = -x$ and $y = arctan(x)$. They have opposite signs regardless of x.

Question 3b ii

By the product rule:

$\frac{d}{dx}(x\cos^{-1}(x)) = \cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$ [1]

Question 3b iii

This is done through anti-differentiation by recognition. The use of the above answer is required for 3 marks. Any other method will be awarded 2 marks.

$\text{Area} = \int_{\frac{1}{2}}^{1} \cos^{-1} x\, dx$ $\left[\frac{1}{2}\right]$

To evaluate the integral, utilising the answer to part (ii), one may integrate by recognition:

$\frac{d}{dx}(x\cos^{-1}(x)) = \cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \cos^{-1}(x) = \frac{d}{dx}(x\cos^{-1}(x)) + \frac{x}{\sqrt{1-x^2}}$

$\therefore \int \cos^{-1}(x)\, dx = x\cos^{-1}(x) + \int \frac{x}{\sqrt{1-x^2}} dx$ [1]

So:

$\text{Area} = \int_{\frac{1}{2}}^{1} \cos^{-1} x\, dx$
$= [\, x\cos^{-1}(x)]_{\left(\frac{1}{2}\right)}^{1} + \int_{\left(\frac{1}{2}\right)}^{1} \frac{x}{\sqrt{1-x^2}} dx$
$= -\frac{\pi}{6} + \int_{\left(\frac{1}{2}\right)}^{1} \frac{x}{\sqrt{1-x^2}} dx$

The integral $\int \frac{x}{\sqrt{1-x^2}} dx$ can be evaluated by substitution:

Let $u = 1 - x^2 \Rightarrow \frac{du}{dx} = -2x$.

Therefore, $x = -\frac{1}{2}.\frac{du}{dx}$.

Hence, we have:

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int -\frac{1}{2\sqrt{u}}du = -\sqrt{u} \quad \left[\frac{1}{2}\right]$$

And so:

$$\text{Area} = -\frac{\pi}{6} + \int_{\left(\frac{1}{2}\right)}^{1} \frac{x}{\sqrt{1-x^2}}dx$$
$$= -\frac{\pi}{6} - \int_{\left(\frac{3}{4}\right)}^{0} \frac{1}{2\sqrt{u}}du$$
$$= -\frac{\pi}{6} - \left[\sqrt{u}\right]_{\left(\frac{3}{4}\right)}^{0}$$
$$= -\frac{\pi}{6} + \frac{\sqrt{3}}{2} \text{ square units} \qquad [1]$$

Question 4a

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= \overrightarrow{AB} + \overrightarrow{AD}$ since $\overrightarrow{AD} = \overrightarrow{BC}$
$= \boldsymbol{a} + \boldsymbol{d} \qquad [1]$

Question 4b

Since M is midpoint of AC, we have:

$\overrightarrow{AM} = \overrightarrow{MC} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{d}$

To show that B, M, and D are collinear, it must be shown that $\overrightarrow{DM} = k\overrightarrow{DB}$, that is, that these vectors are parallel.

$\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB}$
$= \boldsymbol{a} - \boldsymbol{d}$

$\overrightarrow{DM} = \overrightarrow{DA} + \overrightarrow{AM}$
$$= -\boldsymbol{d} + \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{d}$$
$$= \frac{1}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{d}$$
$$= \frac{1}{2}\overrightarrow{DB}$$

(Completion up to this stage is awarded **[1].** Note that the expression for $\overrightarrow{DM}$ cannot be obtained by claiming $\overrightarrow{DM}$ equal $\frac{1}{2}\overrightarrow{DB}$, since it has not been proven that M is midpoint of $\overrightarrow{DB}$.)

Hence, we have shown $\overrightarrow{DM} = \frac{1}{2}\overrightarrow{DB}$, as required [1].

Question 5

$3\sin^2(\theta) + \cos^2(\theta) + \sqrt{3}\sin(2\theta) = 4$

$3\sin^2(\theta) + \cos^2(\theta) + 2\sqrt{3}\sin(\theta)\cos(\theta) = 4$ [1]

$\left(\sqrt{3}\sin(\theta) + \cos(\theta)\right)^2 = 4$

$\sqrt{3}\sin(\theta) + \cos(\theta) = 2$ (1) or $\sqrt{3}\sin(\theta) + \cos(\theta) = -2$ (2) [1]

Solving (1):

$\sqrt{3}\sin(\theta) + \cos(\theta) = 2$

$\frac{\sqrt{3}}{2}\sin(\theta) + \frac{1}{2}\cos(\theta) = 1$

$\cos\left(\frac{\pi}{6}\right)\sin(\theta) + \sin\left(\frac{\pi}{6}\right)\cos(\theta) = 1$

$\Rightarrow \sin\left(\frac{\pi}{6} + \theta\right) = 1$ by the compound angle formula [1]

$\Rightarrow \theta = \frac{\pi}{3}$ [1]

Solving (2):

By the same method, obtain:

$\sin\left(\frac{\pi}{6} + \theta\right) = -1$

$\Rightarrow \theta = \frac{-2\pi}{3}$ [1]

Deduct [1] if only (I) or (II) is solved, failing to acknowledge ± 2.

Alternatively:

$3\sin^2(\theta) + \cos^2(\theta) + \sqrt{3}\sin(2\theta) = 4$
$\Rightarrow 2\sin^2(\theta) + \sqrt{3}\sin(2\theta) = 3$ since $\sin^2(\theta) + \cos^2(\theta) = 1$ [1]
$-(1 - 2\sin^2(\theta)) + \sqrt{3}\sin(2\theta) = 2$
$\Rightarrow -\cos(2\theta) + \sqrt{3}\sin(2\theta) = 2$ since $1 - 2\sin^2(\theta) = \cos(2\theta)$ [1]
$\left(-\frac{1}{2}\right)\cos(2\theta) + \left(\frac{\sqrt{3}}{2}\right)\sin(2\theta) = 1$
$\sin\left(\frac{-\pi}{6}\right)\cos(2\theta) + \cos\left(-\frac{\pi}{6}\right)\sin(2\theta) = 1$
$\Rightarrow \sin\left(2\theta - \frac{\pi}{6}\right) = 1$ by the compound angle formula [1]

Hence:

$2\theta - \frac{\pi}{6} = -\frac{3\pi}{2}, \frac{\pi}{2}$ only, since $\theta \in [-\pi, \pi] \Rightarrow \left(2\theta - \frac{\pi}{6}\right) \in \left[-\frac{11\pi}{6}, \frac{13\pi}{6}\right]$ [1]
$\Rightarrow \theta = -\frac{2\pi}{3}, \frac{\pi}{3}$ [1]

Question 6

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^2 + 1 \quad [1]$

Hence, using the initial condition:

$$\int_2^v d\left(\frac{v^2}{2}\right) = \int_1^x (2\chi^2 + 1)\, d\chi$$

$$\frac{v^2}{2} - 2 = \left[\frac{2\chi^3}{3} + \chi\right]_1^x = \left[\frac{2\chi^3}{3} + \chi\right]_1^0 \text{ at } x = 0$$

$$\frac{v^2}{2} = 2 - \frac{2}{3} - 1$$

$$= \frac{1}{3}$$

$$\therefore v = \pm\frac{\sqrt{6}}{3} \text{ ms}^{-1}$$

Question 7a i

$E(4X - 2Y + 6) = 4E(X) - 2E(Y) + 6 = 44 \ [1]$

Question 7a ii

$Var(4X - 2Y + 6) = 4^2Var(X) + 2^2Var(Y)$ since the distributions are independent

$= 100 \ [1]$.

Hence, $sd = 10$.

Question 7b

The standard error for 95% confidence interval is $z\frac{s}{\sqrt{n}} = 1.96 \times \frac{\sqrt{256}}{\sqrt{64}} = 3.92$ [1]

The 95% CI is [173-3.92 , 173+3.92] = [169.08 , 176.92].

176 ∈ 95% CI. Hence, the prediction is **not rejected. [1]** (saying the prediction is **accepted** is not acceptable).

Let H be the height of a given individual.

$H \sim N(\mu, 256)$ where μ is the true mean of the population

$\bar{H} = \frac{1}{n}\sum H_i \sim N\left(\mu, \frac{256}{n}\right)$ since it is a some of identical independent normal random variables

Thus,

$$Z = \frac{\bar{H} - \mu}{\left(\frac{16}{\sqrt{n}}\right)} \sim N(0,1)$$

$\Pr(-1.96 < Z < 1.96) = 0.95$

$\Rightarrow Z \in (-1.96, 1.96)$ with 95% confidence $\quad \left[\frac{1}{2}\right]$

$$\Rightarrow \frac{\bar{H} - \mu}{\left(\frac{16}{\sqrt{n}}\right)} \in (-1.96, 1.96)$$

$$\Rightarrow \mu \in \left(\bar{H} - 1.96\left(\frac{16}{\sqrt{n}}\right), \bar{H} + 1.96\left(\frac{16}{\sqrt{n}}\right)\right)$$

Substituting the given numbers:

$\mu \in \left(173 - 1.96\left(\frac{16}{8}\right), 173 + 1.96\left(\frac{16}{8}\right)\right)$
$\mu \in (169.08, 176.92) \text{ with } 95\% \text{ confidence} \quad [1/2]$

Since the prediction, 176cm is an element of the 95% confidence interval, the prediction cannot be rejected at the 0.05 significance level [1].

Question 8a
$x = k\tan(t) - 1$

$\therefore \tan(t) = \frac{x+1}{k}$

$y = 3\sec(t) - 8$

$\therefore \sec(t) = \frac{y+8}{3}$

From the trigonometric identity:

$\sec^2(t) - \tan^2(t) = 1$

$\Rightarrow \frac{(y+8)^2}{9} - \frac{(x+1)^2}{k^2} = 1$

Question 8b
Implicit differentiation is used. The use of chain rule for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ is also accepted. Working needed.

$\frac{2(y+8)}{9} \times \frac{dy}{dx} - \frac{2(x+1)}{k^2} = 0$ [1]

$\therefore \frac{dy}{dx} = \frac{9(x+1)}{(y+8)k^2}$ [1]

Question 8c
The normal has a gradient of $\frac{-2}{3}$. Hence, the gradient of the path at point x = 2 equals $\frac{3}{2}$, so we have at, $x = 2$:

$$\frac{dy}{dx} = \frac{9(x+1)}{(y+8)k^2} = \frac{9(2+1)}{(y+8)k^2} = \frac{3}{2} \ (1)$$

The point at $x = 2$ also satisfies the equation of the normal:

$3y + 2(2) = -2 \Rightarrow y = -2.$

Sub $y = -2$ into (1), we have:

$\frac{27}{6k^2} = \frac{3}{2}$
$\therefore k^2 = 3$

$\therefore k = \sqrt{3} \text{ (only, since } k > 0)$