SPECIALIST MATHEMATICS

Units 3 and 4 – Written Examination 1

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2016 Trial Examination

SOLUTIONS

Question 1 (4 marks)

Denote the mass and speed of a toy car by m and v separately. Then

$$m \sim N(200, 3^2), \quad v \sim N(80, 4^2)$$

and
Exerted force $P = 4m$
Resistant force $R = 3v$
The resultant force $F = 4m - 3v$

a.

$E(F) = 4E(m) - 3E(R) = 4 \times 200 - 3 \times 80 = 560$ N 1 mark

$Var(F) = 4^2 Var(m) + (-3)^2 Var(v) = 288$ 1 mark
$\sigma(F) = \sqrt{288} = 12\sqrt{2}$

b.

The standard deviation of the sample mean $= \frac{\sigma}{\sqrt{n}} = \frac{12\sqrt{2}}{11}$. 1 mark

Question 2 (3 marks)

P is the intersection of the lines AD and CB .⇒

$\overrightarrow{CP} = x\overrightarrow{CB}$ and $\overrightarrow{DP} = y\overrightarrow{DA}$

$\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$

$= \overrightarrow{OC} + x\overrightarrow{CB}$

$= \overrightarrow{OC} + x\overrightarrow{CO} + x\overrightarrow{OB}$

$= -3\underset{\sim}{a} + 3x\underset{\sim}{a} + x\underset{\sim}{b}$

$= (3x - 3)\underset{\sim}{a} + x\underset{\sim}{b}$

Also

$\overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP}$

$= \overrightarrow{OD} + y\overrightarrow{DA}$

$= \overrightarrow{OD} + y\overrightarrow{DO} + y\overrightarrow{OA}$

$= 4\underset{\sim}{b} - 4y\underset{\sim}{b} + y\underset{\sim}{a}$ 1 mark

Therefore

$3x - 3 = y$ and $x = 4 - 4y$ 1 mark

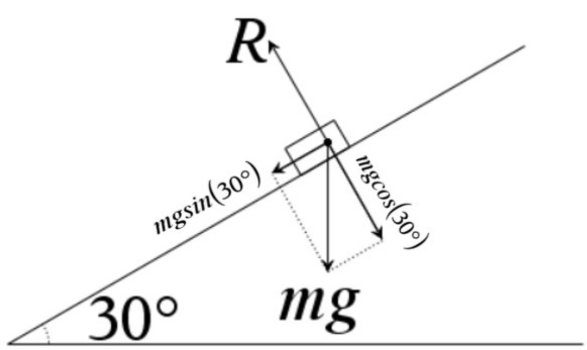
Solve them simultaneously, $x = \frac{16}{13},\ y = \frac{9}{13}$.

Hence

$\overrightarrow{OP} = \frac{9}{13}\underset{\sim}{a} + \frac{16}{13}\underset{\sim}{b}$ 1 mark

Question 3 (3marks)

All forces are labelled in the diagram below.



The equations of the motion are

$$R = mgcos(30°), \qquad mgsin(30°) = ma$$ 1 mark

Therefore the acceleration

$$a = \frac{1}{2}g$$ 1 mark

The travelled distance in $\frac{10}{\sqrt{g}}$ seconds is

$s = ut + \frac{1}{2}at^2 = \frac{1}{2} \times \frac{1}{2}g \times \left(\frac{10}{\sqrt{g}}\right)^2 = 25$ m 1 mark

Question 4 (3 marks)

The velocity

$\underset{\sim}{v}(t) = \frac{\sqrt{3}}{4} t \underset{\sim}{i} + \left(t^2 - \frac{7}{2}\right) \underset{\sim}{j} + \frac{\sqrt{3}}{2} t \underset{\sim}{k}$

$\underset{\sim}{v}(2) = \frac{\sqrt{3}}{2} \underset{\sim}{i} + \frac{1}{2} \underset{\sim}{j} + \sqrt{3} \underset{\sim}{k}$ 1 mark

Then

$\underset{\sim}{v}(2) \cdot \underset{\sim}{k} = \sqrt{3}$ and $\left|\underset{\sim}{v}(2)\right| = 2$ 1 mark

If θ is the angle between $\underset{\sim}{v}(2)$ and $\underset{\sim}{k}$ then

$\cos(\theta) = \frac{\underset{\sim}{v}(2) \cdot \underset{\sim}{k}}{\left|\underset{\sim}{v}(2)\right|} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30°$

Therefore the angle between $\underset{\sim}{v}(2)$ and the horizontal direction is $90° - 30° = 60°$.

1 mark

Question 5 (6 marks)

a. $z = -2 + i$ is a solution $\Rightarrow z = -2 - i$ is also a solution. 1 mark

Hence

$z^4 + 5z^3 + az^2 + bz + c = (z + 3)(z + 2 - i)(z + 2 + i)(z - r)$ 1 mark

for a real number $r \in R$.

When expanding the right hand side the coefficient of z^3 is

$(3 + 2 - i + 2 + i - r)$ or *(7 – r)* 1 mark

Hence

$3 + 2 - i + 2 + i - r = 5$

Therefore

$r = 2$ 1 mark

b. $z^4 + 5z^3 + az^2 + bz + c$

$= (z + 3)(z + 2 - i)(z + 2 + i)(z - 2)$ 1 mark

$= (z + 3)(z^2 + 4z + 5)(z - 2)$

$= (z^2 + z - 6)(z^2 + 4z + 5)$

$= z^4 + 5z^3 + 3z^2 - 19z - 30$

Therefore

$a = 3, \quad b = -19, \quad c = -30$ 1 mark

Question 6 (4 marks)

Let V be the volume of the water.

$V=\frac{1}{3}\pi r^2 h$ where r is the radius of water surface.

$\frac{r}{4}=\frac{h}{8}\Rightarrow r=\frac{1}{2}h$

$\therefore\ V=\frac{1}{12}\pi h^3\Rightarrow\frac{dV}{dh}=\frac{1}{4}\pi h^2$ 1 mark

$\Rightarrow\frac{dh}{dt}=\frac{dV}{dt}\times\frac{dh}{dV}=-\frac{\pi(h^2-1)}{8}\times\frac{4}{\pi h^2}=-\frac{h^2-1}{2h^2}$ 1 mark

$\Rightarrow t=-\int_8^2\frac{2h^2}{h^2-1}dh$ 1 mark

$=-\int_8^2(2+\frac{1}{h-1}-\frac{1}{h+1}dh$

$=-[2h+\ln(\frac{h-1}{h+1})]_8^2$

$=12+\ln(\frac{7}{3})$ 1 mark

Question 7 (3+3=6 marks)

a. $\frac{dy}{dx}=\frac{e^x-e^{-x}}{2}$ 1 mark

The required arc length

$L=\int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\int_{-1}^{1}\sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\,dx$ 1 mark

$=\int_{-1}^{1}\sqrt{1+\frac{e^{2x}-2+e^{-2x}}{4}}\,dx$

$=\int_{-1}^{1}\sqrt{\frac{e^{2x}+2+e^{-2x}}{4}}\,dx$

$=\int_{-1}^{1}\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2}\,dx$

$=\int_{-1}^{1}\frac{e^x+e^{-x}}{2}\,dx$

$=\left[\frac{e^x-e^{-x}}{2}\right]_{-1}^{1}=e-\frac{1}{e}$ 1 mark

b. The required volume

$V=\pi\int_{-1}^{1}\left(f(x)\right)^2dx=\pi\int_{-1}^{1}\left(\frac{e^x+e^{-x}}{2}\right)^2dx=\pi\int_{-1}^{1}\frac{e^{2x}+2+e^{-2x}}{4}dx$ 1 mark

$=\pi\left[\frac{e^{2x}+4x-e^{-2x}}{8}\right]_{-1}^{1}$ 1 mark

$=\frac{\pi(e^2+4-e^{-2})}{4}$ 1 mark

Question 8 (6 marks)

a. $\arctan(x)$ is an increasing only function$\Rightarrow$

$x^2 < 2x + 15 \Rightarrow$ 1 mark

$(x+3)(x-5) < 0 \Rightarrow$ 1 mark

$-3 < x < 5$ 1 mark

b. $\sqrt{3}\cos(2x) + \sin(2x) = \sqrt{2}, 0 \le x \le \pi \Rightarrow$

$2(\frac{\sqrt{3}}{2}\cos(2x) + \frac{1}{2}\sin(2x)) = \sqrt{2}, 0 \le x \le \pi \Rightarrow$ 1 mark

$2\left(\cos\left(\frac{\pi}{6}\right)\cos(2x) + \sin\left(\frac{\pi}{6}\right)\sin(2x)\right) = \sqrt{2}\,, 0 \le x \le \pi \Rightarrow$

$\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2},\ -\frac{\pi}{6} \le 2x - \frac{\pi}{6} \le 2\pi - \frac{\pi}{6} \Rightarrow$ 1 mark

$2x - \frac{\pi}{6} = \frac{\pi}{4},\ 2\pi - \frac{\pi}{4} \Rightarrow$

$x = \frac{5\pi}{12}, \frac{23\pi}{12}$ 1 mark

Question 9 (5 marks)

$\frac{dy}{dx} = \frac{1+y^2}{(2+e^x)y} \Rightarrow \frac{y}{1+y^2}dy = \frac{1}{2+e^x}dx \Rightarrow$

$\int \frac{y}{1+y^2}dy = \int \frac{1}{2+e^x}dx \Rightarrow$ 1 mark

$\frac{1}{2}\int \frac{1}{1+y^2}d(1+y^2) = \int \frac{e^{-x}}{2e^{-x}+1}dx \Rightarrow$

$\frac{1}{2}\ln(1+y^2) = -\frac{1}{2}\int \frac{1}{2e^{-x}+1}d(2e^{-x}+1) \Rightarrow$ 1 mark

$\frac{1}{2}\ln(1+y^2) = -\frac{1}{2}\ln(2e^{-x}+1) + c$ 1 mark

Substitute $x = 0, y = 0$ and solve for c, $c = \frac{1}{2}\ln(3)$

Therefore

$\ln(1+y^2) = \ln\left(\frac{3}{2e^{-x}+1}\right) = \ln\left(\frac{3e^x}{2+e^x}\right)$ 1 mark

Hence

$$y = \sqrt{\frac{2e^x - 2}{e^x + 2}}$$ 1 mark