Neap:

Trial Examination 2016

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

Question	Answer
1	C
2	E
3	E
4	B
5	A
6	C
7	C
8	B
9	B
10	C
11	D
12	E
13	D
14	A
15	D
16	B
17	A
18	E
19	A
20	B

Question 1 C

From $y = \frac{x}{2} - \frac{3}{x}$, the graph has two straight-line asymptotes, namely $x = 0$ and $y = \frac{x}{2}$.

$$\frac{dy}{dx} = \frac{x^2 + 6}{2x^2} \left(\frac{dy}{dx} = \frac{1}{2} + \frac{3}{x^2}\right)$$

The equation $\frac{dy}{dx} = 0$ has no real solutions and so the graph has no stationary points.

Question 2 E

Consider the graphs of $y_1 = |4x - 1|$ and $y_2 = \left|4\left(\frac{x}{4}\right) - 1\right|$.

The graph of y_1 has been dilated by a factor of 4 from the y-axis to form the graph of y_2.

For example, the point (1, 3) is transformed to (4, 3).

Question 3 E

The turning points of the graph of $y = \sec(x)$ occur at $x = 0$, π and 2π for $0 \le x \le 2\pi$.

Hence the turning points for the graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ occur at $x = \pi - \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3}$.

That is, $x = \frac{2\pi}{3}, \frac{5\pi}{3}$.

Question 4 B

The repeated factor $(x - 1)^2$ must be re-expressed as the sum of two fractions. Hence we can disregard options **C**, **D** and **E**.

The quadratic factor $(x^2 + 16)$ must be re-expressed as a fraction with a linear factor in the numerator. Hence we can disregard option **A**.

Question 5 A

$$uv = 6a\text{cis}\left(\frac{\pi}{5} + b\right)$$

Equating the two expressions for uv we obtain $6a\text{cis}\left(\frac{\pi}{5} + b\right) = 48\text{cis}\left(\frac{\pi}{12}\right)$.

$$6a = 48 \Rightarrow a = 8 \text{ and } \frac{\pi}{5} + b = \frac{\pi}{12} \Rightarrow -\frac{7\pi}{60}$$

Question 6 C

$$i\bar{z} - iz = 2$$

Substituting for $\bar{z}$ and z we obtain $i(x - yi) - i(x + yi) = 2$.

$$-2i^2y = 2 \Rightarrow y = 1$$

Question 7 C

Options **A**, **B**, **D** and **E** all represent circles. Option **C** represents an ellipse with foci (2, 0) and (–2, 0).

Question 8 B

$y^2 - xy = -4$

$2y\frac{dy}{dx} - \left(y + x\frac{dy}{dx}\right) = 0$

$(2y - x)\frac{dy}{dx} = y$

$\frac{dy}{dx} = \frac{y}{2y - x}$ (this can be obtained directly with CAS)

$\frac{dy}{dx} = 0 \Rightarrow y = 0$

If $y^2 - xy = -4,\ y = 0 \Rightarrow \text{LHS} = 0$.

So LHS ≠ RHS and there are no stationary points.

For a tangent parallel to the y-axis to exist, we require $2y - x = 0$.

Substituting $y = \frac{x}{2}$ into $y^2 - xy = -4$ gives $\left(\frac{x}{2}\right)^2 - x\left(\frac{x}{2}\right) = -4$.

Solving $\left(\frac{x}{2}\right)^2 - x\left(\frac{x}{2}\right) = -4$ for x gives $x = \pm 4$; that is, 2 tangents.

Question 9 B

$f(x) = 3x^5 - 5x^3$

So $f'(x) = 15x^4 - 15x^2$ and $f''(x) = 60x^3 - 30x$.

$f''(x) = 30x(2x^2 - 1)$

The graph of f is concave up for values of x such that $f''(x) > 0$.

Solving $f''(x) > 0$ for x gives $-\frac{\sqrt{2}}{2} < x < 0$ or $x > \frac{\sqrt{2}}{2}$.

Question 10 C

Solving $\sqrt{6x + 4} = 2x$ for x with $x > 0$ gives $x = 2$.

$$V = \pi\int_0^2 (\sqrt{6x + 4})^2 - (2x)^2 dx$$

$$= \pi\int_0^2 (6x + 4 - 4x^2)dx$$

Question 11 D

$$\frac{dV}{dt} = -2$$

$$V = \frac{1}{2} \times 2 \times 3 \times 5$$

$$= 15 \text{ (cubic metres)}$$

So when the trough is one-quarter full, $V = 3.75$ (cubic metres).

$$\frac{b}{h} = \frac{2}{3} \Rightarrow b = \frac{2h}{3}$$

Substituting $b = \frac{2h}{3}$ into $V = \frac{5bh}{2}$ gives $V = \frac{5h^2}{3}$.

Solving $\frac{15}{4} = \frac{5h^2}{3}$ for h with $h > 0$ gives $h = \frac{3}{2}$.

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$-2 = \frac{10}{3} \times \frac{3}{2} \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{2}{5} \text{ (metres/minute)}$$

Question 12 E

Let the arc length be L.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_a^b \sqrt{1 + (\sec^2(x))^2}\,dx \text{ as } \frac{dy}{dx} = \sec^2(x)$$

$$= \int_a^b \sqrt{1 + \sec^4(x)}\,dx$$

Question 13 D

When $t = 0$, $T = 15$.

The differential equation is $\frac{dT}{dt} = -k(T - 3)$.

Question 14 A

Let d be the distance travelled by the particle.

$$d = \frac{1}{2}(2 + 3)(10) + \frac{1}{2}(1)(10)$$

So the distance travelled is 30 metres.

Let s be the displacement of the particle from its starting point.

$$s = \frac{1}{2}(2 + 3)(10) - \frac{1}{2}(1)(10)$$

So the displacement from the starting position is 20 metres.

Question 15 D

A unit vector in the direction of $-\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$ is $\frac{1}{\sqrt{42}}(-\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k})$.

Hence a vector with a magnitude of 7 parallel to $-\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$ is $\frac{7}{\sqrt{42}}(-\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k})$.

As $\frac{7}{\sqrt{42}} = \frac{\sqrt{42}}{6}$, the required vector is $\frac{\sqrt{42}}{6}(-\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k})$.

Question 16 B

$\overrightarrow{AM} = \underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$ and $\overrightarrow{AN} = \frac{1}{3}\underset{\sim}{i} + \underset{\sim}{j}$

Let θ be the required angle.

$$\theta = \cos^{-1}\left(\frac{\frac{1}{3} + \frac{1}{2}}{\frac{\sqrt{5}}{2} \times \frac{\sqrt{10}}{3}}\right)$$

So the angle between $\overrightarrow{AM}$ and $\overrightarrow{AN}$ is 45°.

Question 17 A

Resolving forces parallel and perpendicular to the plane we obtain $R = 40\cos(20°)$ and $F = 40\sin(20°)$.

Question 18 E

Let R be the resistance to the motion of the trailer.

$1600 - 400 - R = 1200 \times 8$

So $R = 240$ (newtons).

Question 19 A

Let W be the total weight of the 12 raspberries.

$E(W) = 12 \times 10 = 120$ and $\text{var}(W) = 12 \times 1.5^2$

So $W \sim N(120, 12 \times 1.5^2)$ and $\Pr(W > 130) = 0.0271$.

Question 20 B

The p-value for a two-sided H_1 is twice the value for a one-sided H_1.

So the new p-value is $2 \times 0.007 = 0.014$.

As $0.014 > 0.01(\alpha)$, Emily should not reject H_0.

SECTION B

Question 1 (10 marks)

a. $v\dfrac{dv}{dx} = \dfrac{1}{2500}(10\ 000 + v^2)$ A1

Attempting to solve this differential equation either by CAS or by hand with $v(0) = 0$. M1

$v^2 = 10\ 000e^{\frac{x}{1250}} - 10\ 000$ A1

$v = 100\sqrt{e^{\frac{x}{1250}} - 1}$ (since $v > 0$) A1

b. When $x = 900$ (m), $v = 102.7$ (m/s). A1

As $102.7 > 80$, the aeroplane will take off successfully. A1

c. i. $v = 100\tan\left(\dfrac{t}{25}\right) \Rightarrow \dfrac{dv}{dt} = 4\sec^2\left(\dfrac{t}{25}\right)$ A1

$\dfrac{dv}{dt} = \dfrac{1}{2500}\left(10\ 000 + 10\ 000\tan^2\left(\dfrac{t}{25}\right)\right)$ M1

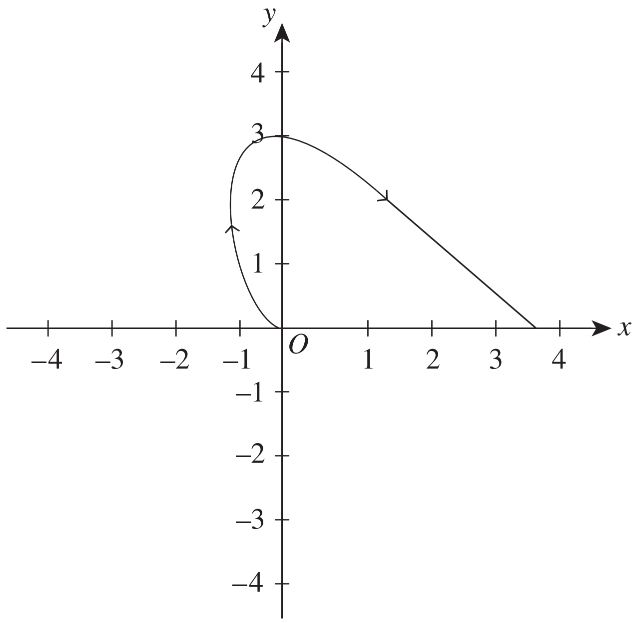
$= 4\left(1 + \tan^2\left(\dfrac{t}{25}\right)\right)$

$= 4\sec^2\left(\dfrac{t}{25}\right)$, so $v = 100\tan\left(\dfrac{t}{25}\right)$ is a solution to the differential equation. A1

ii. This model suggests that $v \to \infty$ for a finite time value $t \to \dfrac{25\pi}{2}$. A1

Question 2 (10 marks)

a.



correct shape A1

correct direction of motion indicated A1

b. **i.** Solving $\frac{t^2}{2} - \log_e(1 + t) = 0$ for t M1

we obtain $t = 1.29$ (s) (correct to two decimal places). A1

ii. $|\underset{\sim}{r}'(1.285\ldots)| = \sqrt{(x'(1.285\ldots))^2 + (y'(1.285\ldots))^2}$ A1

Attempting to evaluate. M1

speed = 1.2 (m/s) (correct to one decimal place) A1

c. Let the distance travelled be d metres.

$$d = \int_0^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$ A1

Attempting to evaluate. M1

total distance = 7.8 (m) (correct to one decimal place) A1

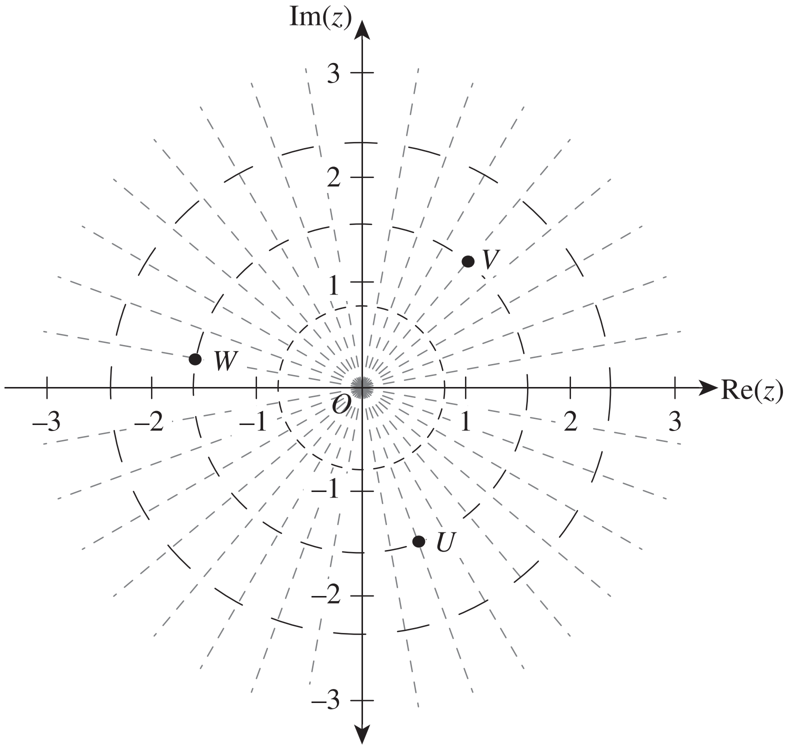
Question 3 (12 marks)

a. $z = \sqrt{12}\operatorname{cis}\left(\frac{5\pi}{6}\right)$ $\left(\text{or } z = 2\sqrt{3}\operatorname{cis}\left(\frac{5\pi}{6}\right)\right)$ A1 A1

b. $v = 12^{\frac{1}{6}}\operatorname{cis}\left(\frac{5\pi}{18}\right)$ M1 A1

$u = 12^{\frac{1}{6}}\operatorname{cis}\left(-\frac{7\pi}{18}\right)$ and $w = 12^{\frac{1}{6}}\operatorname{cis}\left(\frac{17\pi}{18}\right)$ A1 A1

c.



V plotted correctly A1
U and W plotted correctly A1

d. *UOV*, *VOW* and *UOW* are three congruent triangles with $|u| = |v| = |w| = 12^{\frac{1}{6}}$ and $\angle UOV = \angle VOW = \angle UOW = \frac{2\pi}{3}$.

Let the area be A.

$$A = 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\left(\frac{2\pi}{3}\right)$$ M1 A1 A1

$$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$$ A1

Question 4 (11 marks)

a. Let X be the total weight of the five tangerines.

$E(X) = 5 \times 200 = 1000$ and $\text{var}(X) = 5 \times 100 = 500$. A1

$\Pr(875 < X < 975) = 0.1318$ (correct to four decimal places) A1

b. Let $Y = T - 3M$, where T is the weight of a random tangerine and M is the weight of a random mandarin. M1

$E(Y) = 200 - 3 \times 75 = -25$ and $\text{var}(Y) = 10^2 + 9 \times 3^2 = 181$. A1

$\Pr(Y > 0) = 0.0316$ (correct to four decimal places) A1

c. $H_0: \mu = 100$ versus $H_1: \mu \neq 100$ A1

d. If H_0 is true, then $\bar{X} \sim \text{N}\left(100, \frac{5^2}{15}\right)$. A1

$p\text{-value} = 2\Pr(\bar{X} \le 97 \mid \mu = 100)$ M1

So p-value = 0.0201 (correct to four decimal places). A1

Note: Award full marks if the correct p-value is stated.

e. As $0.0201 < 0.05(\alpha)$, we should reject H_0 in favour of H_1. A1

We have enough evidence to conclude that the mean weights are not 100 grams. A1

Question 5 (9 marks)

a. Use of Euler's method (formula or program). M1

$f(6.1) \approx f(6) + (0.1)f'(6)$

$= 12.0586\ldots$

$f(6.2) \approx f(6.1) + (0.1)f'(6.1)$

$= 12.1180$ (correct to four decimal places) A1

b. i. $$f''(x) = \frac{(x-4)e^{\frac{x}{2}} - 4}{2x^3}$$ A1

ii. Since $f''(x)$ is positive on the interval [6, 6.2], A1
the graph of f is concave up on the interval [6, 6.2]. A1

Hence the tangent lines (Euler approximation lines) are below the actual graph of f. A1

c. $f(6.2) - f(6) = \int_{6}^{6.2} f'(x)dx$ M1 A1

$f(6.2) = 12.1189$ (correct to four decimal places) A1

Question 6 (8 marks)

a. $(8 - 6)g = (8 + 6)a$ and so $a = 1.4$ (m/s^2). M1 A1

b. Let x be the distance travelled by the particle at A.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1.4 \Rightarrow \frac{1}{2}v^2 = 1.4x + c$ M1 A1

When $x = 0$, $v = 0$ and so $c = 0$; that is, $v^2 = 2.8x$. A1

When $x = 3$, $v = \sqrt{8.4}$ (m/s). A1

c. $\frac{dv}{dt} = 1.4 \Rightarrow v = 1.4t + c$

When $t = 0$, $v = 0$ and so $c = 0$; that is $v = 1.4t$. M1

When $v = \sqrt{8.4}$ we have $t = \frac{\sqrt{8.4}}{1.4} = 2.07$ (s) (correct to two decimal places). A1

Note: Constant acceleration formulae are no longer part of the syllabus.