**The Mathematical Association of Victoria**

# Trial Examination 2016

# SPECIALIST MATHEMATICS

## Written Examination 2 - SOLUTIONS

**SECTION A**

| Question | Answer | Question | Answer |
|---|---|---|---|
| **1** | A | **11** | B |
| **2** | D | **12** | C |
| **3** | B | **13** | E |
| **4** | A | **14** | E |
| **5** | E | **15** | D |
| **6** | E | **16** | B |
| **7** | E | **17** | D |
| **8** | C | **18** | C |
| **9** | C | **19** | B |
| **10** | A | **20** | A |

**Question 1**

$$(2i)^3 - 5(2i)^2 + 4(2i) - mi = 0$$

$$20 - mi = 0$$

$$m = \frac{20}{i} = -20i$$

**Answer is A**

**Question 2**

Can begin by looking for a term which involves either $\bar{z} + z$ or $z\bar{z}$

$$\frac{1}{z} + \frac{1}{\bar{z}} = \frac{\bar{z} + z}{z\bar{z}} = \frac{x - iy + x + iy}{(x^2 + y^2)} = \frac{2x}{(x^2 + y^2)}$$

All other terms involve $i$

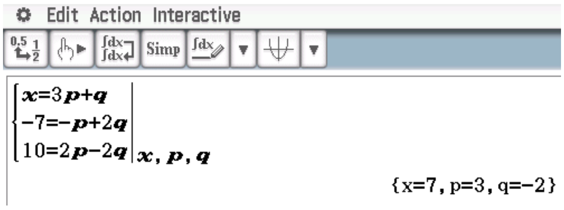
**Answer is D**

**Question 3**

To be linearly dependent can be expressed as $\underset{\sim}{c} = q\underset{\sim}{a} + p\underset{\sim}{b}$

Therefore using CAS solve

$$\begin{aligned} x &= 3p + q \\ -7 &= -p + 2q \\ 10 &= 2p - 2q \end{aligned}$$




this will give $p = 3, q = -2$ and $x = 7$

**Answer is B**

**Question 4**

$|\underset{\sim}{p}| = 4$ therefore $4 + x^2 + 9 = 16,\ x = \pm\sqrt{3}$

$\underset{\sim}{p}.\underset{\sim}{q} = -8 + \sqrt{3}x - 3y = 0,\ y = \dfrac{\sqrt{3}x - 8}{3}$

$x = \sqrt{3}, y = -\dfrac{5}{3}$ and $x = -\sqrt{3}, y = -\dfrac{11}{3}$

**Answer is A**

**Question 5**

The particles meet if the $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ components are all exactly the same at the same time.

Need to solve

$3 = t + 1 \Rightarrow t = 2$

and $2t - 6 = 4 \Rightarrow t = 5$

and $t^2 - 7t = -10 \Rightarrow t = 2, 5$

Since there is no value of $t$ in common for all of the components, they will never collide.

**Answer is E**

**Question 6**

$$a = v\frac{dv}{dx} \text{ and } \frac{dv}{dx} = \frac{1}{2}(1-2x^2)^{\frac{1}{2}} \times (-4x)$$

$$a = \frac{1}{3}(1-2x^2)^{\frac{3}{2}} \times \frac{1}{2}(1-2x^2)^{\frac{1}{2}} \times (-4x) = -\frac{2x}{3}(1-2x^2)^2$$

**Answer is E**

**Question 7**

$$\dot{\underset{\sim}{r}}(t) = 2(1+3t)\times 3\underset{\sim}{i} - \frac{18}{2}t^{-\frac{1}{2}}\underset{\sim}{k}$$

When $t = 3$ $\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{\left(60^2 + \left(\frac{9}{\sqrt{3}}\right)^2\right)} = 60.22$

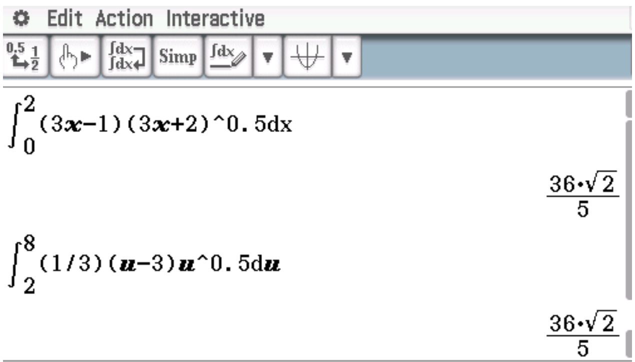
**Answer is E**

**Question 8**

$$u = 3x+2, \quad u-3 = 3x-1, \quad \frac{du}{dx} = 3$$

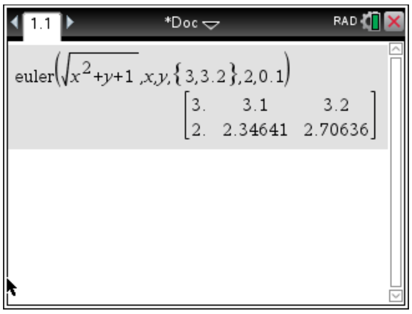
When $x = 0, u = 2$ and $x = 2, u = 8$

Therefore $\int_0^2 (3x-1)\sqrt{(3x+2)}\,dx = \frac{1}{3}\int_2^8 (u-3)\sqrt{u}\,du$




**Answer is C**

**Question 9**




$x_0 = 3, y_0 = 2$

$y_1 = 2 + 0.1\sqrt{(3^2 + 2 + 1)}$

$y_2 = y_1 + 0.1\sqrt{(3.1^2 + y_1 + 1)}$

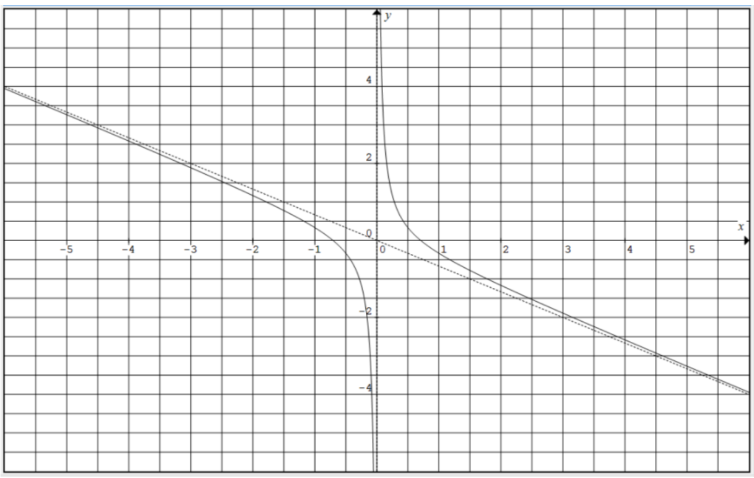
$= 2.7064$

**Answer is C**

**Question 10**

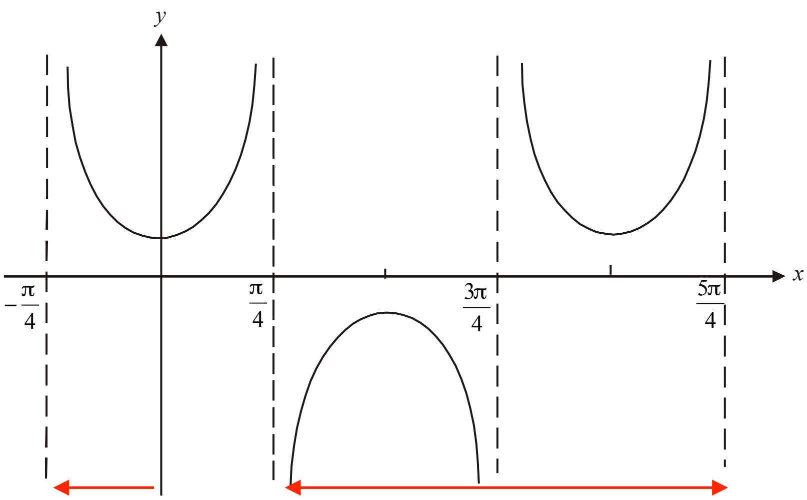
$$y = \frac{1-2x^2}{3x} = -\frac{2}{3}x + \frac{1}{3x}$$

Therefore asymptotes are $y = -\frac{2}{3}x$ and $x = 0$



**Answer is A**

**Question 11**



period $= \pi$ therefore $\pi = \frac{2\pi}{a}$, $a = 2$

Translation of $\frac{\pi}{4}$ to the left, so function is $y = \text{cosec}(2(x + \frac{\pi}{4}))$ therefore $b = -\frac{\pi}{4}$

**Answer is B**

**Question 12**

$$x^2 + 2px + y^2 + 1 = 0$$
$$(x+p)^2 - p^2 + y^2 + 1 = 0$$
$$(x+p)^2 + y^2 = p^2 - 1$$

To be a circle we require $p^2 - 1 > 0$, $p^2 > 1$ so the only correct option $|p| > 1$

**Answer is C**

**Question 13**

The graph has the form $y = \frac{k}{(x^2 + a)}$ where $k, a \in R$

The graph goes through the points $(0,4)$ and $(1,2)$, therefore

$\frac{k}{a} = 4$ and $2 = \frac{k}{a+1}$ solving these gives $a = 1, k = 4$

Thus $y = \frac{4}{(x^2 + 1)}$ and $\int y\,dx = 4\tan^{-1}(x) + c$

Alternatively, this could be done by inputting each function and plotting the gradient graph to see which matches.

**Answer is E**

**Question 14**

Range of $\cos^{-1}\theta$ is $[0,\pi]$, therefore the range of $f(x) = 4\cos^{-1}(3x+1) + \frac{\pi}{2}$ is

$$[4 \times 0 + \frac{\pi}{2}, 4 \times \pi + \frac{\pi}{2}] = [\frac{\pi}{2}, \frac{9\pi}{2}]$$

**Answer is E**

**Question 15**

When $t = 3$, $\underset{\sim}{F}_1 = \underset{\sim}{i} - 3\underset{\sim}{j}$, $\underset{\sim}{F}_2 = 2\underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{F}_3 = -7\underset{\sim}{i} + 5\underset{\sim}{j}$ therefore the resultant force $\underset{\sim}{F} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$

$$a = \frac{|\underset{\sim}{F}|}{m} = \frac{\sqrt{(4^2 + 3^2)}}{5} = 1$$

**Answer is D**

**Question 16**

Equations of motion:

$m_1 g - T = \dfrac{m_1 g}{2}$ equation 1 $\qquad T - m_2 g = \dfrac{m_2 g}{2}$ equation 2

Adding the equations gives:-

$$g(m_1 - m_2) = \frac{g}{2}(m_1 + m_2)$$

$$\frac{g}{2}m_1 = \frac{3g}{2}m_2 \quad \text{therefore } \frac{m_1}{m_2} = 3$$

**Answer is B**

**Question 17**

Vertical component of motion has $u = 30\sin(60^o) = 15\sqrt{3}\text{ms}^{-1}$, $a = -g\text{ms}^{-2}$

Solve $v = at + u$ when $v = 0$ to find the time to reach its maximum height

$0 = -gt + 15\sqrt{3}$, $t = \dfrac{15\sqrt{3}}{g}$ therefore the time to reach the ground $t = 2 \times \dfrac{15\sqrt{3}}{g} = \dfrac{30\sqrt{3}}{g}$ s

**Answer is D**

(Note, this is equivalent to using the formulae for time to travel $T = \dfrac{2V\sin(\alpha)}{g}$,

$T = \dfrac{2 \times 30 \times \sin(60)}{g} = \dfrac{30\sqrt{3}}{g}$)

An alternative method

$$\underset{\sim}{a} = -g\,\underset{\sim}{j}$$

$$\therefore \underset{\sim}{v} = 30\cos\left(60^0\right)\underset{\sim}{i} + \left(30\cos\left(60^0\right) - gt\right)\underset{\sim}{j} = 15\,\underset{\sim}{i} + \left(15\sqrt{3} - gt\right)\underset{\sim}{j}$$

$$\therefore \underset{\sim}{r} = 15t\,\underset{\sim}{i} + \left(15\sqrt{3}t - \tfrac{1}{2}gt^2\right)\underset{\sim}{j}$$

Ball at ground level when $15\sqrt{3}t - \frac{1}{2}gt^2 = 0$

Solving $t = 0, \dfrac{30\sqrt{3}}{g}$ So returns to ground when $t = \dfrac{30\sqrt{3}}{g}$ **Answer is D**

**Question 18**

$$\text{E}(T) = 3\text{E}(X) - \text{E}(Y) = 3(3.6) - 12.3 = -1.5$$
$$\text{VAR}(T) = 3^2\,\text{VAR}(X) + \text{VAR}(Y) = 3^2 \times 0.68^2 + 5.1^2$$
$$\text{Sd(T)} = \sqrt{\text{VAR}(T)} = \sqrt{(3^2 \times 0.68^2 + 5.1^2)} = 5.23$$

**Answer isA**

**Question 19**

The standard deviation of the sample mean will be $\dfrac{\mu}{2\sqrt{n}}$

To achieve a width of $0.2\mu$ for 95% confidence interval it is necessary to solve

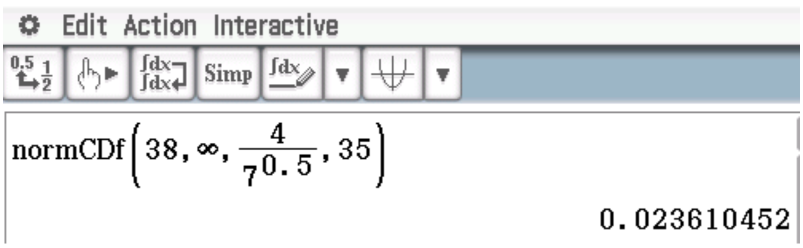
$$1.96\frac{\mu}{2\sqrt{n}} \le 0.1\mu$$
$$\sqrt{n} \ge \frac{1.96}{0.2}, \quad n \ge 96.04$$

**Answer is B**

**Question 20**

The probability required is $\Pr(\bar{X} \ge 38)$ where $\bar{X} : \text{N}\left(35, \dfrac{4^2}{7}\right)$ as there are 7 days in the week.

$$\Pr(\bar{X} \ge 38) = 0.0236$$




(Note as 38 is larger than the mean of 35, you can eliminate answers with probabilities greater than 0.5 straight away)

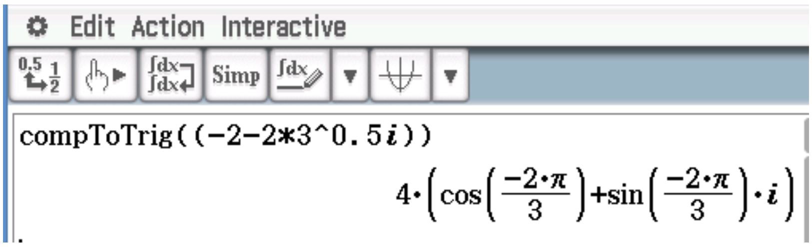
**Answer is A**

## SECTION B

### Question 1 (12 marks)

**a. i.** $r = \sqrt{((-2)^2 + (-2\sqrt{3})^2)} = \sqrt{16}, \quad \theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) = \tan^{-1}(\sqrt{3})$ [A1]

$$w^4 = 4cis\left(-\frac{2\pi}{3}\right)$$

Or using CAS 

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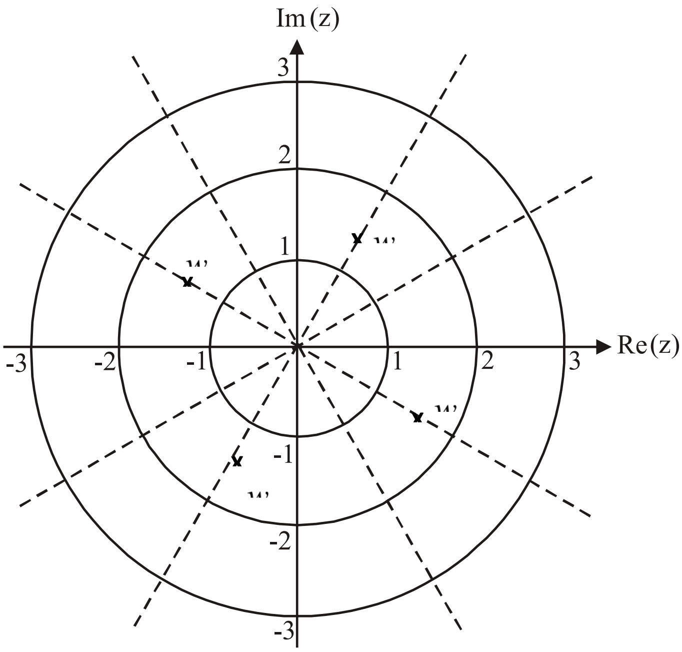
**ii.** $w = 4^{\frac{1}{4}}\text{cis}\left(-\frac{2\pi}{(3\times 4)} + \frac{2\pi k}{4}\right), \quad k = 0,1,2,3$

$$w_1 = \sqrt{2}\text{cis}\left(-\frac{2\pi}{12}\right), w_2 = \sqrt{2}\text{cis}\left(\frac{4\pi}{12}\right), w_3 = \sqrt{2}\text{cis}\left(\frac{10\pi}{12}\right), w_4 = \sqrt{2}\text{cis}\left(\frac{16\pi}{12}\right)$$

$$w_1 = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right), w_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{3}\right), w_3 = \sqrt{2}\text{cis}\left(\frac{5\pi}{6}\right), w_4 = \sqrt{2}\text{cis}\left(-\frac{2\pi}{3}\right)$$

½ mark each correct root rounded down [A2]




All roots correctly plotted and labelled. Equally spaced by $\frac{\pi}{2}$ around a circle radius $\sqrt{2}$ [A1]

**b.** $w^4 = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b\right)}$ and $w^4 = -2 - 2\sqrt{3}i$

Therefore $-\frac{a}{2} = -2$ and $\frac{a^2}{4} - b = -12$ [M1]

$a = 4$, b $= 16$ [A1]

Alternate method

Substituting $w = -2 - 2\sqrt{3}i$ into $w^8 + aw^4 + b = 0$ gives $-8 + 8\sqrt{3}i + a\left(-2 - 2\sqrt{3}i\right) + b = 0$

Equating real and imaginary parts gives $-8 - 2a + b = 0$ and $8\sqrt{3} - 2\sqrt{3}a = 0$ [M1]

Solving simultaneously to get $a$=4 and $b$=16 [A1]

**c.** **i.** $(\overline{w})^4 = (\overline{w^4}) = -2 + 2\sqrt{3}i$ and $v^2 = -2i$

therefore $\frac{v^2}{(\overline{w})^4} = \frac{-2i}{(-2 + 2\sqrt{3}i)} = \frac{1}{4}(-\sqrt{3} + i)$ [A1]

**ii**.

$\text{Arg}(\overline{u}) = \frac{\pi}{6}$ therefore $\text{Arg}(u) = -\frac{\pi}{6}$ and $\text{Arg}(u^3) = -\frac{\pi}{2}$

$\text{Arg}\left(\frac{v^2}{(\overline{w})^4}\right) = \frac{5\pi}{6}$ therefore $\text{Arg}\left(\overline{\frac{v^2}{(\overline{w})^4}}\right) = -\frac{5\pi}{6}$

$\text{Arg}\left(\overline{\left(\frac{v^2}{(\overline{w})^4}\right)}.u^3\right) = \text{Arg}\overline{\left(\frac{v^2}{(\overline{w})^4}\right)} + \text{Arg}\left(u^3\right) = -\frac{5\pi}{6} - \frac{\pi}{2} = -\frac{8\pi}{6}$ [H1]

$= \frac{2\pi}{3}$ [A1]

**d.** $\sqrt{((x+2)^2 + (y + 2\sqrt{3})^2)} = 2\sqrt{((x+2)^2 + (y - 2\sqrt{3})^2)}$ [M1]

$(x+2)^2 + \left(y - \frac{10\sqrt{3}}{3}\right)^2 = \frac{64}{3}$ [A1]

Centre $(-2, \frac{10\sqrt{3}}{3})$ and radius $\frac{8\sqrt{3}}{3}$ [A1]

**Question 2** (11 marks)

**a.** $\frac{dx}{dt} = -2\sin(t)$ and $\frac{dy}{dt} = 6\cos(2t)$ [M1]

$\frac{dy}{dx} = \frac{6(1-2\sin^2(t))}{-2\sin(t)} = -\frac{3}{\sin(t)} + \frac{6\sin^2(t)}{\sin(t)} = -3\text{cosec}(t) + 6\sin(t)$ [M1]

$p = -3, q = 6$

**b.** When $t = \frac{\pi}{6}$ $x = \sqrt{3}, y = \frac{3\sqrt{3}}{2}$ and $\frac{dy}{dx} = -3$ [A1]

Equation of the normal to the curve is $y - \frac{3\sqrt{3}}{2} = \frac{1}{3}(x - \sqrt{3})$

$$y = \frac{1}{3}x + \frac{7\sqrt{3}}{6}$$ [A1]

**c.** $y = 6\sin(t)\cos(t)$

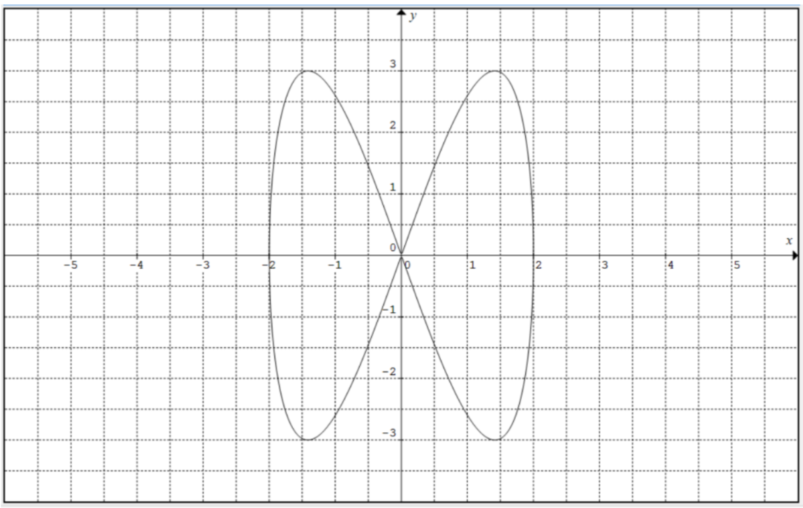
$$y^2 = 36\sin^2(t)\cos^2(t)$$
$$= \frac{9}{4}\times(4-4\cos^2(t))\times 4\cos^2(t)$$ [M1]
$$= \frac{9}{4}\times(4-x^2)x^2$$

$k = 9/4$ [A1]

**d.** $2y\frac{dy}{dx} = \frac{9}{4}(8x - 4x^3)$ [M1]

$\frac{dy}{dx} = \frac{9x\left(2-x^2\right)}{2y}$ [A1]

**e.**




[A1]

**f.** **i**.

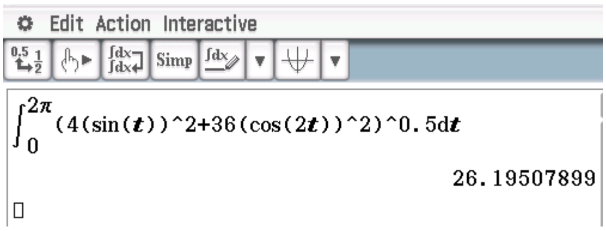
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_0^{2\pi} \sqrt{((-2\sin(t))^2 + (6\cos(2t))^2)}\, dt$$

[A1]

$$= \int_0^{2\pi} \sqrt{(4\sin^2(t) + 36\cos^2(2t))}\, dt$$

**ii**.




L = 26.20 [A1]

**Question 3** (8 marks)

**a.** Solve

$$0 = 3.5m + c$$
$$8 = 5.5m + c$$

$m = 4, c = -14$ [A1]

**b.**

$$V=\pi\int_0^h x^2\,dy$$

$$=\pi\int_0^h \frac{(y+14)^2}{16}\,dy$$ [M1]

$$V=\frac{\pi(h+14)^3}{48}-\frac{343\pi}{6}$$ [A1]

**c.** Let $h=8$, $V=517.32\text{cm}^3$ [A1]

**d.** (i) $\frac{dV}{dt}=-3\sqrt{h}$ , $\frac{dV}{dh}=\frac{\pi(h+14)^2}{16}$ and $\frac{dh}{dt}=\frac{dh}{dV}.\frac{dV}{dt}$ [A1]

therefore $\frac{dh}{dt}=-\frac{48\sqrt{h}}{\pi(h+14)^2}$ [A1]

(ii) Find the time taken for *h* to decrease from 8 to 0.

$$t=-\pi\int_8^0 \frac{(h+14)^2}{48\sqrt{h}}dh$$ [M1]

$t=104.95\text{s}$ [A1]

**Question 4** (12 marks)

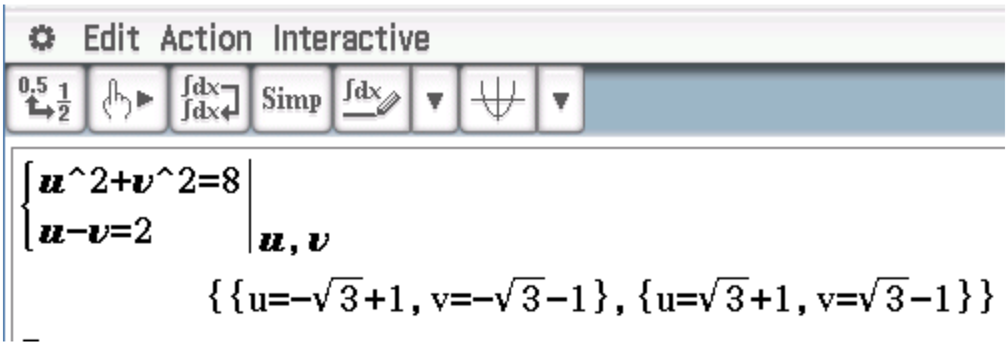
**a.** $\overrightarrow{OA}=2\underset{\sim}{i}-2\underset{\sim}{j}$ [A1]

$\overrightarrow{OB}=u\underset{\sim}{i}+v\underset{\sim}{j}$ [A1]

**b.** $|\overrightarrow{OA}|=|\overrightarrow{OB}|$ therefore $\sqrt{u^2+v^2}=2\sqrt{2}$ so $u^2+v^2=8$ [A1]

$\overrightarrow{OA}.\overrightarrow{OB}=\sqrt{8}\sqrt{8}\cos(60^o)=4$ and $\overrightarrow{OA}.\overrightarrow{OB}=2u-2v$ therefore $u-v=2$ [A1]

**c.** Solving $u^2+v^2=8$ and $u-v=2$ gives




but $u>0$ and $v>0$ therefore the solution is $u=1+\sqrt{3}$ and $v=-1+\sqrt{3}$ [A1]

**d.** $\overrightarrow{OC}=\overrightarrow{AB}=(\sqrt{3}-1)\underset{\sim}{i}+(\sqrt{3}+1)\underset{\sim}{j}$

Therefore C is the point $((\sqrt{3}-1),(\sqrt{3}+1))$ [A1]

**e.** $\overrightarrow{OB}=(1+\sqrt{3})\underset{\sim}{i}+(-1+\sqrt{3})\underset{\sim}{j}$ and $\overrightarrow{AC}=(-3+\sqrt{3})\underset{\sim}{i}+(3+\sqrt{3})\underset{\sim}{j}$ [A1]

$\overrightarrow{OB}.\overrightarrow{AC}=(1+\sqrt{3})(-3+\sqrt{3})+(-1+\sqrt{3})(3+\sqrt{3})=0$ [A1]

**f.**

$$\text{Area}=\frac{1}{2}\left|\overrightarrow{OA}\right|\,\left|\overrightarrow{OP}\right|\sin(60^o)$$

$$=\frac{1}{2}\sqrt{8}\times\frac{1}{3}\sqrt{8}\times\frac{\sqrt{3}}{2}=\frac{2\sqrt{3}}{3}$$ [A1]

**g.** $\overrightarrow{BX}=(\overrightarrow{BP}.\widehat{BA})\widehat{BA}$ [M1]

$\overrightarrow{BP}=-\frac{2}{3}(\sqrt{3}+1)\underset{\sim}{i}-\frac{2}{3}(\sqrt{3}-1)\underset{\sim}{j}$ and $\widehat{BA}=\frac{1}{\sqrt{8}}((1-\sqrt{3})\underset{\sim}{i}-(1+\sqrt{3})\underset{\sim}{j}$

$\widehat{BX}=\frac{1}{3}((1-\sqrt{3})\underset{\sim}{i}-(1+\sqrt{3})\underset{\sim}{j}$ [A1]

$\overrightarrow{OX}=\overrightarrow{OB}+\overrightarrow{BX}=\frac{2}{3}((\sqrt{3}+2)\underset{\sim}{i}+(\sqrt{3}-2)\underset{\sim}{j}$

X is the point $(\frac{2}{3}(\sqrt{3}+2),\frac{2}{3}(\sqrt{3}-2))$ [A1]

**Question 5** (9 marks)

**a.** For a sample size of 20 $\bar{X} \sim \mathrm{N}(175.5, \frac{90.57^2}{20})$

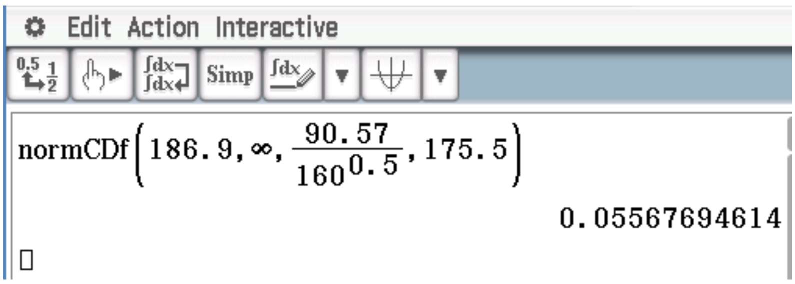
Therefore $\Pr(\bar{X} \le 150) = 0.103992... = 0.1040$ correct to 4 decimal places [A1]

**b.**

**i.** $H_0 : \mu = 175.5$
$H_1 : \mu > 175.5$ [A1]

**ii.** For a sample size of 160 $\bar{X} \sim \mathrm{N}(175.5, \frac{90.57^2}{160})$

Therefore $\mathrm{p} = \Pr(\bar{X} \ge 186.9) = 0.0557$ [A1]




**iii.** $\mathrm{p} = 0.0557 > 0.05$ therefore the null hypothesis should not be rejected at the 5% level [A1]

**c.** $T \sim \mathrm{N}(71, 4.7^2)$, for a random person $\Pr(T > 76) = 0.1437$ [A1]

Whereas, for a sample size of 4, $\bar{T} \sim \mathrm{N}(71, \frac{4.7^2}{4})$, and $\Pr(\bar{T} > 74) = 0.1009$. Therefore you would be more likely to find a random person who spends more than 76 mins on Facebook [A1]

**d.** probability required is $\Pr(2W - Y > 0)$ [M1]

$$\mathrm{E}(2W - Y) = 2\mathrm{E}(W) - \mathrm{E}(Y) = 2 \times 7.1 - 5.8 = 8.4$$
$$\mathrm{VAR}(2W - Y) = 2^2\,\mathrm{VAR}(W) + \mathrm{VAR}(Y) = 4 \times 4.7^2 + 3.5^2 = 100.61$$
[A1]

therefore $2W - Y \sim \mathrm{N}(8.4, 100.61)$

and the probability is $\Pr(2W - Y > 0) = 0.7988$ [A1]

**Question 6** (8 marks)

**a.** $F = 80g - 320v$

$= 80(g - 4v)$ newtons [A1]

**b.** $\dfrac{dv}{dt} = g - 4v$

$t = \int \dfrac{1}{g - 4v} dv, \qquad t = -\dfrac{1}{4}\ln|g - 4v| + c$

$v = \dfrac{1}{4}(Ae^{-4t} + g)$ [M1]

When $t = 0$, $v = 58.8$ therefore $v = \dfrac{1}{4}\left(g + \dfrac{1127}{5}e^{-4t}\right)$ [A1]

**c.** $t = 300$s

distance travelled $= \displaystyle\int_0^{300} v\,dt$ [M1]

$= 749.09$m [A1]

**d.** $v\dfrac{dv}{dx} = g - \dfrac{v^2}{2}$ [M1]

$x = \displaystyle\int_{58.8}^{v} \dfrac{2v}{2g - v^2} dv, \qquad v = \sqrt{(19.6 + 3437.84e^{-x})}$ [A1]

He reaches the ground when $x$ =749.09 therefore his speed $v$ = 4.43ms$^{-1}$ (correct to 2 decimal places) [A1]