Year 2016

VCE

Specialist Mathematics

Trial Examination 1

Solutions

KILBAHA MULTIMEDIA PUBLISHING	TEL: (03) 9018 5376
PO BOX 2227	FAX: (03) 9817 4334
KEW VIC 3101	kilbaha@gmail.com
AUSTRALIA	http://kilbaha.com.au

Question 1

Let $\underset{\sim}{a} = 4\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$

$|\underset{\sim}{b}| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$, $\hat{\underset{\sim}{b}} = \frac{1}{5\sqrt{2}}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$

$\underset{\sim}{a}.\underset{\sim}{b} = 12 + 8 - 10 = 10$ A1

resolving $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$

$$\left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \left(\frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{b}|}\right)\hat{\underset{\sim}{b}} = \frac{10}{5\sqrt{2}}\left(\frac{1}{5\sqrt{2}}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)\right)$$

$$\left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \frac{1}{5}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$$ A1

resolving $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$

$$\underset{\sim}{a} - \left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \left(4\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\right) - \frac{1}{5}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$$

$$= \frac{1}{5}\left(5\left(4\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\right) - \left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)\right)$$

$$= \frac{1}{5}\left(17\underset{\sim}{i} - 6\underset{\sim}{j} - 15\underset{\sim}{k}\right)$$ A1

Question 2

a. $z = \sqrt{5}\operatorname{cis}\left(-\frac{\pi}{2}\right) = -\sqrt{5}\,i$ by the conjugate root theorem, $\bar{z} = \sqrt{5}\,i$ is also a root

$$\left(z - \sqrt{5}\,i\right)\left(z + \sqrt{5}\,i\right) = \left(z^2 - 5i^2\right)$$

$= \left(z^2 + 5\right)$ is the quadratic factor A1

b.
$$f(z) = z^4 - 6z^3 + 17z^2 - 30z + 60$$
$$= \left(z^2 + 5\right)\left(z^2 - 6z + 12\right) = 0$$
$$= \left(z - \sqrt{5}\,i\right)\left(z + \sqrt{5}\,i\right)\left(z^2 - 6z + 9 + 3\right) = 0$$
$$= \left(z - \sqrt{5}\,i\right)\left(z + \sqrt{5}\,i\right)\left((z-3)^2 - 3i^2\right) = 0$$ M1
$$= \left(z - \sqrt{5}\,i\right)\left(z + \sqrt{5}\,i\right)\left(z - 3 - \sqrt{3}\,i\right)\left(z - 3 + \sqrt{3}\,i\right) = 0$$

all the roots are $z = \pm\sqrt{5}\,i\ ,\ 3 \pm \sqrt{3}\,i$ A1

Question 3

$\frac{dy}{dx}=\frac{24y^2}{9+4x^2}$ and $y\left(\frac{3}{2}\right)=1$

using variables separable $\int \frac{1}{y^2}\,dy=\int \frac{24}{9+4x^2}\,dx$, integrating M1

$-\frac{1}{y}=4\tan^{-1}\left(\frac{2x}{3}\right)+c$ using $x=\frac{3}{2}$ when $y=1$ A1

$-1=4\tan^{-1}(1)+c \quad \Rightarrow -1=\pi+c \quad \Rightarrow c=-1-\pi$

$-\frac{1}{y}=4\tan^{-1}\left(\frac{2x}{3}\right)-1-\pi$

$y=\frac{1}{\pi+1-4\tan^{-1}\left(\frac{2x}{3}\right)}$ A1

Question 4

a. $y=x^2-16$

$x^2=y+16$

$V_y=\pi\int_a^b x^2\,dy$

$V=\pi\int_0^{20}(y+16)\,dy$ M1

$=\pi\left[\frac{1}{2}y^2+16y\right]_0^{20}=\pi\left[\left(\frac{1}{2}\times 400+16\times 20\right)-0\right]$

$=520\pi \text{ cm}^3$ A1

b. $V(h)=\pi\int_0^h(y+16)\,dy$

$\frac{dV}{dh}=\pi(h+16)$, $\frac{dV}{dt}=2$ A1

$\frac{dh}{dt}=\frac{dh}{dV}\frac{dV}{dt}=\frac{2}{\pi(h+16)}$ M1

$\left.\frac{dh}{dt}\right|_{h=14}=\frac{2}{\pi(14+16)}$

$=\frac{1}{15\pi}$ cm/sec A1

Question 5

a. $\int \frac{\sin(2x)}{\cos^2(2x)}dx$ let $u=\cos(2x)$ $\frac{du}{dx}=-2\sin(2x)$, $dx=\frac{-1}{2\sin(2x)}\,du$

$$\int \frac{\sin(2x)}{\cos^2(2x)}dx=\int \frac{\sin(2x)}{u^2}\times\frac{-1}{2\sin(2x)}\,du$$

$$=-\frac{1}{2}\int\frac{1}{u^2}du=-\frac{1}{2}\int u^{-2}\,du$$ M1

$$=\frac{1}{2}u^{-1}+c=\frac{1}{2u}+c$$

$$=\frac{1}{2\cos(2x)}+c=\frac{1}{2}\sec(2x)+c$$

b.i. $\underset{\sim}{v}(t)=4\tan(2t)\sec(2t)\underset{\sim}{i}+6\sec^2(2t)\underset{\sim}{j}$

$$\underset{\sim}{v}(t)=\frac{4\sin(2t)}{\cos(2t)}\times\frac{1}{\cos(2t)}\underset{\sim}{i}+6\sec^2(2t)\underset{\sim}{j}$$

$$=\frac{4\sin(2t)}{\cos^2(2t)}\underset{\sim}{i}+6\sec^2(2t)\underset{\sim}{j}$$

$$\underset{\sim}{r}(t)=\int\frac{4\sin(2t)}{\cos^2(2t)}dt\ \underset{\sim}{i}+\int 6\sec^2(2t)\,dt\,\underset{\sim}{j}$$ M1

$$\underset{\sim}{r}(t)=2\sec(2t)\underset{\sim}{i}+3\tan(2t)\underset{\sim}{j}+\underset{\sim}{c}$$

$$\underset{\sim}{r}(0)=3\underset{\sim}{i}+2\underset{\sim}{j}=2\underset{\sim}{i}+\underset{\sim}{c}\quad\Rightarrow\quad \underset{\sim}{c}=\underset{\sim}{i}+2\underset{\sim}{j}$$

$$\underset{\sim}{r}(t)=\left(2\sec(2t)+1\right)\underset{\sim}{i}+\left(3\tan(2t)+2\right)\underset{\sim}{j}$$ A1

ii. The distance is $s=\int_a^b\sqrt{\dot{x}^2+\dot{y}^2}\,dt$

$a=0$, $b=\frac{\pi}{5}$, $\dot{x}=4\tan(2t)\sec(2t)$, $\dot{y}=6\sec^2(2t)$

$$s=\int_0^{\frac{\pi}{5}}\sqrt{16\tan^2(2t)\sec^2(2t)+36\sec^4(2t)}\,dt$$

$$s=\int_0^{\frac{\pi}{5}}\sqrt{4\sec^2(2t)\left(4\tan^2(2t)+9\sec^2(2t)\right)}\,dt$$ M1

$$s=\int_0^{\frac{\pi}{5}}\sqrt{4\sec^2(2t)\left(4\tan^2(2t)+9\left(1+\tan^2(2t)\right)\right)}\,dt$$

$$s=\int_0^{\frac{\pi}{5}}2\sec(2t)\sqrt{13\tan^2(2t)+9}\,dt$$ A1

$p=2$, $q=13$, $r=9$

Question 6

$y^2(4-x)=x^3$ expanding $4y^2-xy^2-x^3=0$

using implicit differentiation and the product rule on the second term.

$$\frac{d}{dx}\left(4y^2\right)-\frac{d}{dx}\left(xy^2\right)-\frac{d}{dx}\left(x^3\right)=0$$

$$8y\frac{dy}{dx}-y^2-2xy\frac{dy}{dx}-3x^2=0$$ M1

$$(8y-2xy)\frac{dy}{dx}=y^2+3x^2$$

$$\frac{dy}{dx}=\frac{y^2+3x^2}{8y-2xy}$$ A1

Now if $x=2 \Rightarrow 2y^2=8$, $y^2=4$

but in the fourth quadrant $y<0$ so $y=-2$, $(2,-2)$ A1

$$m_T=\left.\frac{dy}{dx}\right|_{(2,-2)}=\frac{4+12}{-16+8}=-2$$

gradient of the normal $m_N=\frac{1}{2}$ A1

Question 7

$H_0:\ \mu=100$

$H_A:\ \mu\neq100$ 2 sided test A1

$\bar{x}=103$, $\mu=100$, $\sigma=15$, $n=36$

$$Z=\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}=\frac{103-100}{\frac{15}{\sqrt{36}}}=\frac{3}{\frac{15}{6}}=\frac{18}{15}=\frac{6}{5}=1.2$$ A1

At the 95% level $z=\pm1.96$, the calculated value $z=1.2$ is in the M1

acceptable region, there is no evidence to support the alternative

hypothesis, accept the null hypothesis, students are average IQ A1

Question 8

$F = ma$

$-12x = 3a$

$a = \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$

$\frac{1}{2}v^2 = \int -4x\,dx = -2x^2 + c_1$ M1

when $t = 0$, $v = 0$, $x = 4$ $0 = -32 + c_1 \Rightarrow c_1 = 32$

$v^2 = 64 - 4x^2 = 4\left(16 - x^2\right)$ A1

$v = \frac{dx}{dt} = \pm 2\sqrt{16 - x^2}$

take the positive and separate the variables, note that if we take the negative, we obtain the same final answer.

$2t = \int \frac{1}{\sqrt{16 - x^2}}\,dx$

$2t = \sin^{-1}\left(\frac{x}{4}\right) + c_2$

when $t = 0$, $x = 4$ M1

$0 = \sin^{-1}(1) + c_2 \Rightarrow c_2 = -\frac{\pi}{2}$

M1

$2t = \sin^{-1}\left(\frac{x}{4}\right) - \frac{\pi}{2}$

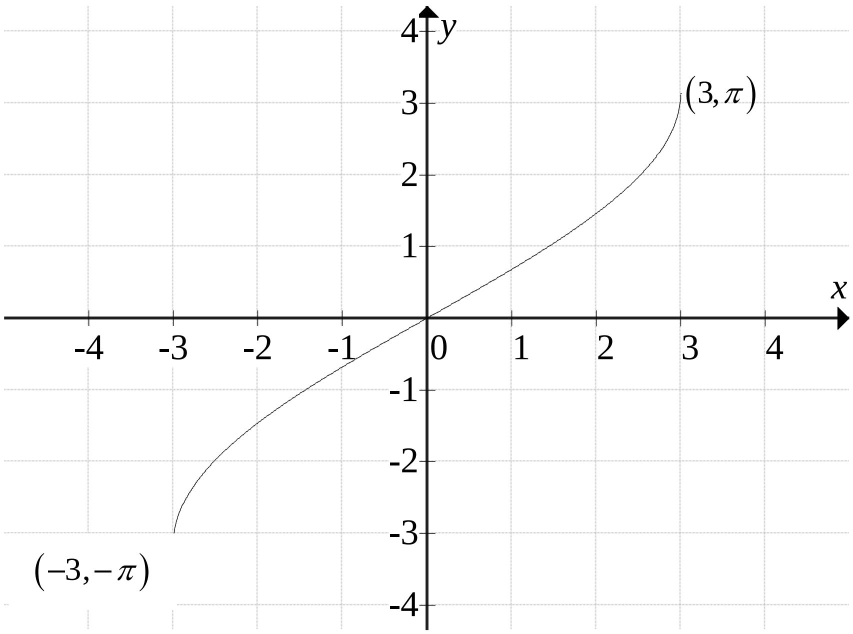
$2t + \frac{\pi}{2} = \sin^{-1}\left(\frac{x}{4}\right)$

$\frac{x}{4} = \sin\left(2t + \frac{\pi}{2}\right) = \sin(2t)\cos\left(\frac{\pi}{2}\right) + \cos(2t)\sin\left(\frac{\pi}{2}\right)$

$x = 4\cos(2t)$ A1

Question 9 $f(x) = 2\arcsin\left(\frac{x}{3}\right) = 2\sin^{-1}\left(\frac{x}{3}\right)$

a. domain $[-3,3]$ range $[-\pi,\pi]$, passes through the origin $(0,0)$
endpoints $(-3,-\pi)$,$(3,\pi)$, correct graph shape A1



b. $f: \quad y = 2\sin^{-1}\left(\frac{x}{3}\right)$

$f^{-1}: \; x = 2\sin^{-1}\left(\frac{y}{3}\right) \quad \Rightarrow \quad \frac{x}{2} = \sin^{-1}\left(\frac{y}{3}\right) \quad \Rightarrow \quad \frac{y}{3} = \sin\left(\frac{x}{2}\right) \quad \Rightarrow \quad y = 3\sin\left(\frac{x}{2}\right)$

to state the function, we must state its domain.

$f^{-1}: [-\pi,\pi] \to R \,,\; f^{-1}(x) = 3\sin\left(\frac{x}{2}\right)$ A1

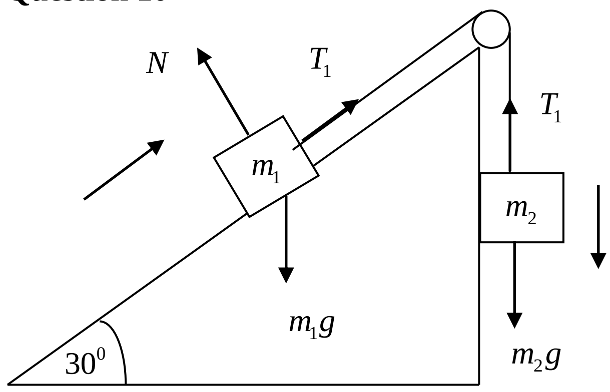
c. Let $A_1 = \int_0^3 2\sin^{-1}\left(\frac{x}{3}\right)dx$, $A_2 = \int_0^{\pi} 3\sin\left(\frac{x}{2}\right)dx$

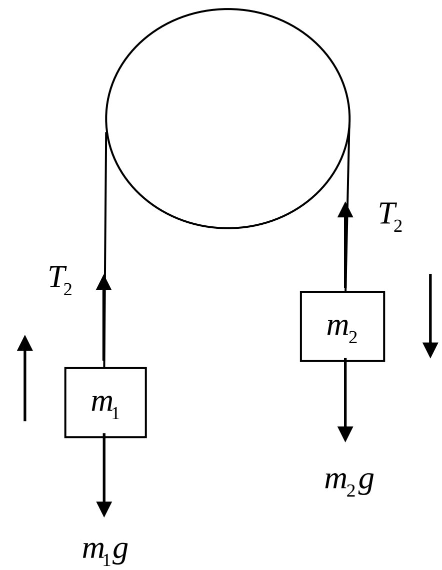
$A_1 + A_2 = 3\pi$ area of the rectangle

$A_2 = \int_0^{\pi} 3\sin\left(\frac{x}{2}\right)dx = \left[-6\cos\left(\frac{x}{2}\right)\right]_0^{\pi} = -6\cos\left(\frac{\pi}{2}\right) + 6\cos(0) = 6$ A1

$A_1 = \int_0^3 2\sin^{-1}\left(\frac{x}{3}\right)dx = 3\pi - 6 = 3(\pi - 2)$ A1

Question 10





resolving up parallel to plane around the m_1 kg mass

$$(1)\quad T_1 - m_1 g\sin\left(30^0\right) = m_1 a \quad \Rightarrow \quad T_1 - \frac{m_1 g}{2} = m_1 a$$

resolving downwards around the m_2 kg mass

$$(2)\quad m_2 g - T_1 = m_2 a$$

adding to eliminate the tension in the string,

to find the acceleration a, of the system

$$(1)+(2)\quad m_2 g - \frac{m_1 g}{2} = m_1 a + m_2 a$$

$$\frac{g}{2}\left(2m_2 - m_1\right) = a\left(m_1 + m_2\right)$$

$$a = \frac{g\left(2m_2 - m_1\right)}{2\left(m_1 + m_2\right)}$$

around m_2 kg mass A1

$$(3)\quad m_2 g - T_2 = \frac{m_2 a}{2}$$

around m_1 kg mass

$$(4)\quad T_2 - m_1 g = \frac{m_1 a}{2}$$

$$(3)+(4)\quad m_2 g - m_1 g = \frac{a}{2}\left(m_1 + m_2\right)$$

$$a = \frac{2g\left(m_2 - m_1\right)}{m_1 + m_2}$$ A1

equating the accelerations M1

$$4\left(m_2 - m_1\right) = \left(2m_2 - m_1\right)$$

$$4m_2 - 4m_1 = 2m_2 - m_1$$

$$2m_2 = 3m_1$$

$$\frac{m_2}{m_1} = \frac{3}{2}$$ A1

END OF SUGGESTED SOLUTIONS