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2016 VCAA Specialist Mathematics Sample (v2 April) Exam 2 Solutions

CAS should be used whenever possible to speed up the solution process.

SECTION A

1	2	3	4	5	6	7	8	9	10
B	B	D	D	D	B	B	B	E	C

11	12	13	14	15	16	17	18	19	20
C	E	B	C	A	D	D	E	A	C

Q1 $x^2-6x+y^2+4y=b$, $(x-3)^2+(y+2)^2=b+9+4$

.: $a=3$, $b+13=5^2$, $b=12$ B

Q2 Domain: $-1\le 4x-1\le 1$, $0\le 4x\le 2$, $0\le x\le \frac{1}{2}$

Range: $-\frac{\pi}{2}\le \sin^{-1}(4x-1)\le \frac{\pi}{2}$, $-\frac{3\pi}{2}\le 3\sin^{-1}(4x-1)\le \frac{3\pi}{2}$

$-\frac{3\pi}{2}+\frac{\pi}{2}\le 3\sin^{-1}(4x-1)+\frac{\pi}{2}\le \frac{3\pi}{2}+\frac{\pi}{2}$,

$-\pi\le 3\sin^{-1}(4x-1)+\frac{\pi}{2}\le 2\pi$ B

Q3 $f(x)=\frac{(x-3)(x-1)}{(x-3)(x+2)}$, $f(x)=\frac{x-1}{x+2}$ and $x\ne 3$ and -2 D

Q4 D

Q5 $(z+\bar{z})^2-(z-\bar{z})^2=16$, $4x^2+4y^2=16$, $x^2+y^2=2^2$ D

Q6 $z=-3i$ is also a root (conjugate root theorem) B

Q7 B

Q8 $3\times\int_0^{\pi}\sin^3 x\,dx=3\times\int_0^{\pi}(1-\cos^2 x)\sin x\,dx$

$=-3\times\int_1^{-1}(1-u^2)\,du=3\int_{-1}^{1}(1-u^2)\,du$ B

Q9 $x_0=0$, $y_0=2$, $\frac{dy}{dx}=2$

$x_1=0.10$, $y_1\approx 2+0.10\times 2=2.20$ E

Q10 $x=\frac{1}{2}\sin y$, when $x=\frac{1}{2}$, $y=\frac{\pi}{2}$

$V=\int_0^{\frac{\pi}{2}}\pi x^2\,dy=\int_0^{\frac{\pi}{2}}\pi\left(\frac{1}{2}\sin y\right)^2 dy$

$=\frac{\pi}{8}\int_0^{\frac{\pi}{2}}(1-\cos(2y))\,dy$ C

Q11 $\cos\theta°=\frac{(3\tilde{i}+6\tilde{j}-2\tilde{k}).(2\tilde{i}-2\tilde{j}+\tilde{k})}{|3\tilde{i}+6\tilde{j}-2\tilde{k}||2\tilde{i}-2\tilde{j}+\tilde{k}|}=\frac{-8}{7\times 3}$

.: $\theta=112.4$ C

Q12 $\hat{b}=\frac{1}{3}(2\tilde{i}-\tilde{j}-2\tilde{k})$, $\tilde{a}.\hat{b}=\frac{1}{3}(6+2)=\frac{8}{3}$ E

Q13 $\tilde{r}(t)=-\tilde{i}-\frac{6}{2\sqrt{t}}\tilde{j}$, $\dot{r}(9)=-\tilde{i}-\tilde{j}$ B

Q14 The diagonals of a rhombus are perpendicular. C

Q15 $a=\frac{1}{2}\frac{dv^2}{dx}=\frac{1}{2}\frac{d}{dx}(3x^2-x^3+16)=3x-\frac{3x^2}{2}$

$F=ma=12\left(3x-\frac{3x^2}{2}\right)$ A

Q16 $\frac{dv}{dt}=\frac{v}{\log_e v}$, $\frac{dt}{dv}=\frac{\log_e v}{v}$

$t=\int\frac{\log_e v}{v}\,dv=\int u\,du=\frac{u^2}{2}+c=\frac{1}{2}(\log_e v)^2+c$

$v=5$ when $t=0$, .: $t=\frac{1}{2}(\log_e v)^2-\frac{1}{2}(\log_e 5)^2$

$(\log_e v)^2=2t+(\log_e 5)^2$, $\log_e v=\sqrt{2t+(\log_e 5)^2}$

$v=e^{\sqrt{2t+(\log_e 5)^2}}$ D

Q17 $2T\sin 60°-12g=0$, $T=\frac{6g}{\sin 60°}=4\sqrt{3}\,g$ D

Q18 $\mu_Z=\bar{X}-3\bar{Y}=10-3\times 3=1$

$\sigma_Z=\sqrt{8^2+(-3)^2(2^2)}=10$ E

Q19 $\mathrm{sd}(\bar{X})=\frac{\sigma}{\sqrt{n}}=\frac{7}{\sqrt{20}}\approx 1.56525$, $\mu=30$

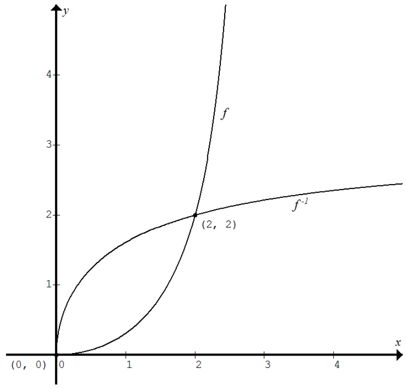
Normal: $\Pr(\bar{X}>32)\approx 0.1007$ A

Q20 C

SECTION B

Q1a $f(2) = -2 + 2\sec\left(\frac{2\pi}{6}\right) = -2 + \frac{2}{\cos\frac{\pi}{3}} = -2 + 4 = 2$

Q1b



Q1c Equation of f : $y = -2 + 2\sec\left(\frac{\pi x}{6}\right)$

Equation of f^{-1}: $x = -2 + 2\sec\left(\frac{\pi y}{6}\right)$

$\frac{x+2}{2} = \sec\left(\frac{\pi y}{6}\right)$, $\frac{2}{x+2} = \cos\left(\frac{\pi y}{6}\right)$

$y = \frac{6}{\pi}\arccos\left(\frac{2}{x+2}\right)$, $\therefore\ k = \frac{6}{\pi}$

Q1d $A = 2 \times \int_0^2 \left(\frac{6}{\pi}\arccos\left(\frac{2}{x+2}\right) - x\right) dx \approx 1.939$

Q1ei $f(x) = -2 + 2\left(\cos\left(\frac{\pi x}{6}\right)\right)^{-1}$,

$f'(x) = -2\left(\cos\left(\frac{\pi x}{6}\right)\right)^{-2}\left(-\sin\left(\frac{\pi x}{6}\right)\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{3}\frac{\sin\left(\frac{\pi x}{6}\right)}{\cos^2\left(\frac{\pi x}{6}\right)}$

Arc length $= \int_0^2 \sqrt{1 + (f'(x))^2}\ dx$

$= \int_0^2 \sqrt{1 + \frac{\pi^2 \sin^2\left(\frac{\pi x}{6}\right)}{9\cos^4\left(\frac{\pi x}{6}\right)}}\ dx$

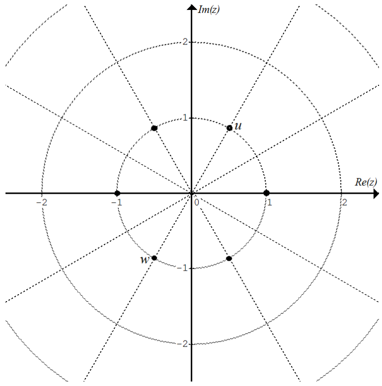
Q1eii 3.067 by CAS

Q2ai $u = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\left(\frac{\pi}{3}\right) + i\ \sin\left(\frac{\pi}{3}\right)$

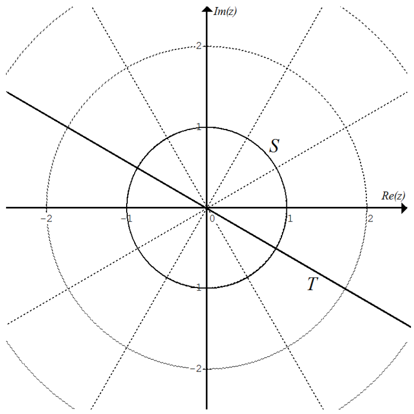
Q2aii $u^6 = \left(\cos\left(\frac{\pi}{3}\right) + i\ \sin\left(\frac{\pi}{3}\right)\right)^6 = \cos\left(\frac{6\pi}{3}\right) + i\ \sin\left(\frac{6\pi}{3}\right)$

$= \cos(2\pi) + i\ \sin(2\pi) = 1$

Q2aiii



Q2bi and ii



Q2biii $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Alternatively, $S \cap T = \left\{\frac{\sqrt{3}}{2} - \frac{1}{2}i\ ,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i\right\}$

Q3a $\log_e N = 6 - 3e^{-0.4t}$, $\frac{d}{dt}\log_e N = \frac{d}{dt}\left(6 - 3e^{-0.4t}\right)$

$\frac{1}{N}\frac{dN}{dt} = 1.2e^{-0.4t}$, $\frac{1}{N}\frac{dN}{dt} = 0.4(6 - \log_e N)$

$\therefore\ \frac{1}{N}\frac{dN}{dt} + 0.4\log_e N - 2.4 = 0$

Q3b $t = 0$, $\log_e N = 6 - 3e^{-0.4t} = 3$, $N = e^3 \approx 20$

Q3c As $t \to \infty$, $\log_e N \to 6$, $N \to 403$ for integer N

Q3di $\frac{dN}{dt} = 0.4N(6 - \log_e N)$

$\frac{d}{dt}\left(\frac{dN}{dt}\right) = \frac{d}{dt}(0.4N(6 - \log_e N))$, $\frac{d^2N}{dt^2} = 0.4(5 - \log_e N)\frac{dN}{dt}$

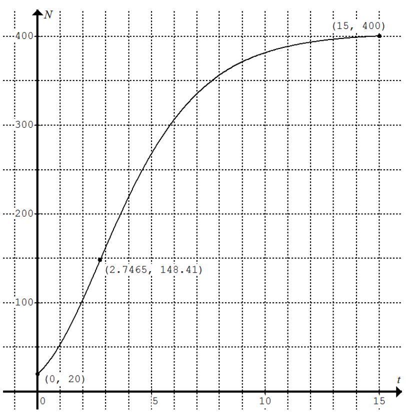
$\frac{d^2N}{dt^2} = 0.16N(5 - \log_e N)(6 - \log_e N)$

Q3dii $\frac{d^2N}{dt^2} = 0$ and $\frac{dN}{dt} \neq 0$ and $N(t) > 0$

.: $5 - \log_e N = 0$, .: $5 = 6 - 3e^{-0.4t}$

.: $t \approx 2.7$ and $N \approx 148$

Q3e



Q4a $\tilde{r}(0) = 12\cos 60° \, \tilde{i} + 12\sin 60° \, \tilde{j} = 6\,\tilde{i} + 6\sqrt{3}\,\tilde{j}$

Q4b $\dot{\tilde{r}}(t) = \int \left(-0.1t\,\tilde{i} - (g - 0.1t)\,\tilde{j}\right) dt$

$\dot{\tilde{r}}(t) = -0.05t^2\,\tilde{i} - \left(gt - 0.05t^2\right)\tilde{j} + \tilde{c}$, $\dot{\tilde{r}}(0) = \tilde{c} = 6\,\tilde{i} + 6\sqrt{3}\,\tilde{j}$

.: $\dot{\tilde{r}}(t) = \left(6 - 0.05t^2\right)\tilde{i} + \left(6\sqrt{3} - gt + 0.05t^2\right)\tilde{j}$

$\tilde{r}(t) = \int \left(\left(6 - 0.05t^2\right)\tilde{i} + \left(6\sqrt{3} - gt + 0.05t^2\right)\tilde{j}\right) dt$

$\tilde{r}(t) = \left(6t - \frac{1}{60}t^3\right)\tilde{i} + \left(6\sqrt{3}\,t - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right)\tilde{j}$, given $\tilde{r}(0) = \tilde{0}$

Q4c At $t = T$, $y = -x$, $6\sqrt{3}\,t - \frac{1}{2}gt^2 + \frac{1}{60}t^3 = -6t + \frac{1}{60}t^3$

.: $6\sqrt{3}\,T + 6T - \frac{g}{2}T^2 = 0$, $6T\left(\sqrt{3} + 1 - \frac{g}{12}T\right) = 0$

Since $T > 0$, .: $T = \frac{12}{g}\left(\sqrt{3} + 1\right)$

Q4d $T = \frac{12}{g}\left(\sqrt{3} + 1\right) = 3.3454$, $T^2 = 11.1915$

$\dot{\tilde{r}}(T)$

$= (6 - 0.05 \times 11.1915)\,\tilde{i} + \left(6\sqrt{3} - g \times 3.3454 + 0.05 \times 11.1915\right)\tilde{j}$

$= 5.4404\,\tilde{i} - 21.8330\,\tilde{j}$

Speed $= \left|\dot{\tilde{r}}(T)\right| = \sqrt{5.4404^2 + 21.8330^2} \approx 22.5$ m s^{-1}

Q5a Consider the two particles together: $3g - g = (3 + 1)a$, $a = \frac{g}{2}$

Q5b Consider the 1 kg particle: $T_1 - g = 1 \times \frac{g}{2}$, $T_1 = \frac{3g}{2}$

Q5c Consider the two particles together:

$3g\sin 30° - g = (3 + 1)b$, $b = \frac{g}{8}$

Q5d Consider the two particles together: $3g\sin\theta° - g = 0$

.: $\sin\theta° = \frac{1}{3}$, $\theta \approx 19.5$

Q5e Consider the two particles together:

$g - 3g\sin\theta° = (3 + 1)\frac{g}{4}\left(1 - \frac{3}{\sqrt{2}}\right)$, $1 - 3\sin\theta° = 1 - \frac{3}{\sqrt{2}}$

.: $\sin\theta° = \frac{1}{\sqrt{2}}$, $\theta = 45$

Q6a H_0: The mean lifetime is that claimed by the manufacturer. ($\mu = 10$)

H_1: The mean lifetime is less than that claimed by the manufacturer. ($\mu < 10$)

Q6b $\sigma \approx s = 1$, $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} \approx \frac{1}{\sqrt{25}} = 0.2$, $\mathrm{E}(\bar{X}) = \mu = 10$

p - value $= \Pr(\bar{X} \le 9.7 \mid \mu = 10) = \Pr\left(Z \le \frac{9.7 - 10}{0.2}\right) \approx 0.067$

Q6c Since p - value > 0.05, .: H_0 should not be rejected at the 5% level of significance.

Q6d $\Pr(\bar{X} < C^* \mid \mu = 10) = 0.05$, $\Pr\left(Z < \frac{C^* - 10}{0.2}\right) = 0.05$

.: $\frac{C^* - 10}{0.2} \approx -1.64485$, $C^* \approx 9.671$

Q6ei $\Pr(\bar{X} > 9.671 \mid \mu = 9.5) = \Pr\left(Z > \frac{9.671 - 9.5}{0.2}\right)$

$\approx \Pr(Z > 0.855) \approx 0.196$

Q6eii Type II error:

p - value $\approx 0.196 > 0.05$, it supports that the actual mean lifetime is $\mu = 9.5$, .: H_0 is not true but it is not rejected.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors