2016 VCAA Specialist Mathematics Sample Exam 1 (v2 April) Solutions

Q1a Let $z=\sqrt{5}-i$,

$\left(\sqrt{5}-i\right)^3-\left(\sqrt{5}-i\right)\left(\sqrt{5}-i\right)^2+4\left(\sqrt{5}-i\right)-4\left(\sqrt{5}-i\right)=0$

.: $\sqrt{5}-i$ is a solution.

Q1b $z^3-\left(\sqrt{5}-i\right)z^2+4z-4\left(\sqrt{5}-i\right)=\left(z-\left(\sqrt{5}-i\right)\right)\left(z^2+4\right)=0$

.: $z^2+4=0$, .: $z=\pm 2i$ are the other solutions.

Q2 $3x^2+2xy+y^2=11$ and $y>0$ (in the first quadrant)

At $x=1$, $-8+2y+y^2=0$, .: $y=2$

Implicit differentiation: $\frac{d}{dx}\left(3x^2+2xy+y^2\right)=0$,

$6x+2x\frac{dy}{dx}+2y+2y\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{3x+y}{x+y}$

At $(1,2)$, $\frac{dy}{dx}=-\frac{5}{3}$, .: gradient of the normal $=\frac{3}{5}$

.: equation of the normal: $y-2=\frac{3}{5}(x-1)$, $3x-5y+7=0$

Q3a $\overline{X+Y}=\overline{X}+\overline{Y}=240+10=250$ mL

$\text{Var}(X+Y)=\text{Var}(X)+\text{Var}(Y)=8^2+2^2=68$ (mL)^2

Q3bi Null hypothesis: The second machine is, on average, dispensing **not** less coffee than the first.

Alternative hypothesis: The second machine is, on average, dispensing less coffee than the first.

Q3bii $a=\frac{235-240}{8}=-0.625$, $p=\Pr(Z\le -0.625)\approx 0.266$

Since $p>0.05$, the null hypothesis should not be rejected at the 0.05 level of significance.

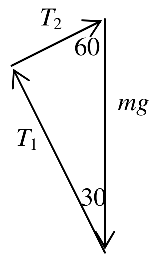
Q4a $V=\int_0^a \pi\left(e^{-x}\right)^2dx=\int_0^a \pi\, e^{-2x}dx$

Q4b $V(a)=\pi\left[\frac{e^{-2x}}{-2}\right]_0^a=\pi\left(\frac{1-e^{-2a}}{2}\right)$

Q4c $\pi\left(\frac{1-e^{-2a}}{2}\right)=\frac{5\pi}{18}$, $9-9e^{-2a}=5$, $e^{-2a}=\frac{4}{9}$, $e^{2a}=\frac{9}{4}$,

$e^a=\frac{3}{2}$, $a=\log_e\left(\frac{3}{2}\right)$

Q5a



$\frac{T_2}{T_1}=\tan 30$, .: $T_2=\frac{T_1}{\sqrt{3}}$

Q5b $\frac{T_2}{mg}=\sin 30$, let $T_2=98$

.: $m=\frac{2\times 98}{9.8}=20$ is the maximum value.

Q6

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\cos^2(2x)\sin(2x)dx=\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}-\frac{1}{2}u^2\frac{du}{dx}dx$

$=\int_{-1}^{0}-\frac{1}{2}u^2du=\left[-\frac{u^3}{6}\right]_{-1}^{0}=-\frac{1}{6}$

$u=\cos(2x)$

$\frac{du}{dx}=-2\sin(2x)$

$-\frac{1}{2}\times\frac{du}{dx}=\sin(2x)$

Q7 $\frac{dy}{dx}=\frac{y}{x^2}$, $\int\frac{1}{y}dy=\int\frac{1}{x^2}dx$, $\log_e|y|=-\frac{1}{x}+c$

Given $x=1$, $y=-1$, $\log_e|-1|=-\frac{1}{1}+c$, $c=1$

.: $\log_e|y|=1-\frac{1}{x}$, $|y|=e^{\left(1-\frac{1}{x}\right)}$, $y=\pm e^{\left(1-\frac{1}{x}\right)}$

Q8a Arc length

$=\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_0^{\pi}\sqrt{(-2\sin(2\theta))^2+(2\cos(2\theta))^2}\,d\theta$

Q8b Arc length $=2\int_0^{\pi}\sqrt{\sin^2(2\theta)+\cos^2(2\theta)}\,d\theta=2\int_0^{\pi}d\theta=2\pi$

Q9a $\underset{\sim}{b}=\underset{\sim}{i}+2\underset{\sim}{j}+m\underset{\sim}{k}$, $\left|\underset{\sim}{b}\right|=\sqrt{1^2+2^2+m^2}=2\sqrt{3}$

.: $m^2+5=12$, $m=\pm\sqrt{7}$

Q9b $\underset{\sim}{a}.\underset{\sim}{b}=0$, $1-2+2m=0$, $m=\frac{1}{2}$

Q9ci $3\underset{\sim}{c}-\underset{\sim}{a}=2\underset{\sim}{i}+4\underset{\sim}{j}-5\underset{\sim}{k}$

Q9cii Since $3\underset{\sim}{c}-\underset{\sim}{a}=2\underset{\sim}{i}+4\underset{\sim}{j}-5\underset{\sim}{k}$.: $3\underset{\sim}{c}-\underset{\sim}{a}=2\underset{\sim}{b}$ if $m=-\frac{5}{2}$

.: $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent if $m=-\frac{5}{2}$

Q10a $\frac{1}{x^2}+\frac{3}{x}+\frac{2x-1}{x^2+4}=\frac{\left(x^2+4\right)+3x\left(x^2+4\right)+x^2(2x-1)}{x^2\left(x^2+4\right)}$

$=\frac{5x^3+12x+4}{x^2\left(x^2+4\right)}$

Q10b $\int\frac{5x^3+12x+4}{x^2\left(x^2+4\right)}dx=\int\frac{1}{x^2}+\frac{3}{x}+\frac{2x-1}{x^2+4}dx$

$=\int\frac{1}{x^2}+\frac{3}{x}+\frac{2x}{x^2+4}-\frac{1}{x^2+4}dx$

$=-\frac{1}{x}+3\log_e|x|+\log_e\left(x^2+4\right)-\tan^{-1}\left(\frac{x}{2}\right)$

$=-\frac{1}{x}+\log_e\left|x^3\right|\left(x^2+4\right)-\tan^{-1}\left(\frac{x}{2}\right)$

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