YEAR 12 *Trial Exam Paper*

2016

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- fully worked solutions
- mark allocations
- tips on how to approach the exam

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SECTION A – Multiple-choice questions

Question 1

Answer: D

Worked solution

The improper rational expression $y = \dfrac{5-3x^2}{2x}$ can be written as $y = \dfrac{5}{2x} - \dfrac{3x}{2},\ x \neq 0.$

When $x \to 0^+,\ y \to +\infty$

$x \to 0^-,\ y \to -\infty$

$\therefore\ x = 0$ is a vertical asymptote.

When $x \to -\infty,\ y \to \dfrac{-3x}{2}$ from below.

When $x \to +\infty,\ y \to \dfrac{-3x}{2}$ from above.

$\therefore\ y = \dfrac{-3x}{2}$ is an oblique asymptote.

Tip

- *Division of improper rational expressions readily reveals horizontal and oblique asymptotes.*

Question 2

Answer: C

Worked solution

Use trigonometric identities:

$1+\cot^2(x)=\operatorname{cosec}^2(x)$.

$\sin(x)=-\dfrac{1}{4} \quad \Rightarrow \operatorname{cosec}(x)=-4 \quad \Rightarrow \operatorname{cosec}^2(x)=16$.

Therefore:

$1+\cot^2(x)=16$

$\Rightarrow \cot^2(x)=15$

$\Rightarrow \cot(x)=\pm\sqrt{15}$.

But $\pi \le x \le \dfrac{3\pi}{2}$ (3rd quadrant) therefore $\cot(x)>0$.

Therefore $\cot(x)=\sqrt{15}$.

Question 3

Answer: A

Worked solution

$$\frac{dy}{dx} = \frac{-1}{a\sqrt{b^2 - x^2}}$$

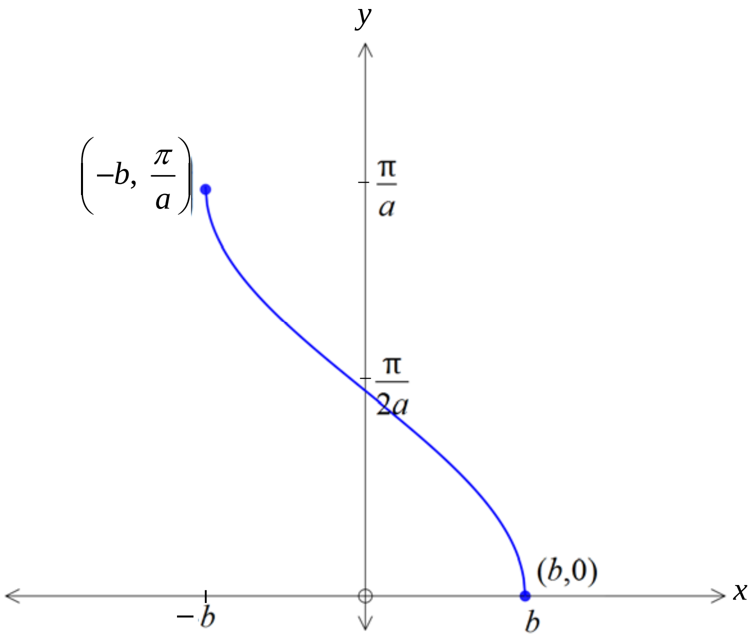
At $x = 0$:

$$\frac{dy}{dx} = \frac{-1}{a\sqrt{b^2}}$$

$$\frac{dy}{dx} = \pm\frac{1}{ab}$$

Since $a > 0$, $b > 0$, the gradient of the tangent must be negative at $x = 0$, as seen from the graph of $y = \frac{1}{a}\cos^{-1}\left(\frac{x}{b}\right)$, where $a, b > 0$.

Hence, $\frac{dy}{dx} = \frac{-1}{ab}$



Tip

- *A quick graph of the function on a CAS calculator using an appropriate substitution for a and b clearly shows a negative gradient for the tangent at x = 0.*

Question 4

Answer: D

Worked solution

$$\frac{\overline{z}}{1-z}=\frac{5+i}{1-(5-i)}$$

$$=\frac{5+i}{-4+i}\times\frac{-4-i}{-4-i}$$

$$=\frac{-20-5i-4i-i^2}{(-4)^2+(1)^2}$$

$$=\frac{-19-9i}{17}$$

$$=\frac{-19}{17}-\frac{9}{17}i$$

Question 5

Answer: E

Worked solution

$$\frac{u}{v}=-2$$

$$v=\frac{u}{-2}$$

$$v=\frac{2\operatorname{cis}\left(\frac{-3\pi}{4}\right)}{2\operatorname{cis}(\pi)}$$

$$v=\operatorname{cis}\left(\frac{-7\pi}{4}\right)$$

$$v=\operatorname{cis}\left(\frac{\pi}{4}\right)$$

Question 6

Answer: C

Worked solution

$$z^3 + z^2 - qz - p = 0$$
$$(-2i)^3 + (-2i)^2 - q(-2i) - p = 0$$
$$8i - 4 + 2qi - p = 0$$
$$(-4 - p) + i(8 + 2q) = 0 + 0i$$

Equate real and imaginary parts:

Real parts: $-4 - p = 0 \Rightarrow p = -4$
Imaginary parts: $8 + 2q = 0 \Rightarrow q = -4$

Question 7

Answer: D

Worked solution

From the slope field, it is evident that the gradient $\left(\dfrac{dy}{dx}\right)$ is zero when $x = 0$. Therefore, possible solutions are options B, C and D. Also, it appears that $\dfrac{dy}{dx} = 1$ at $\left(1, \dfrac{-1}{2}\right)$ and $\left(-1, \dfrac{1}{2}\right)$, so the differential equation could be $\dfrac{dy}{dx} = \dfrac{-x}{2y}$.

Question 8

Answer: C

Worked solution

$$y = \tan^{-1}\left(\frac{2}{3x}\right)$$

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{2}{3x}\right)^2} \times \frac{d}{dx}\left(\frac{2}{3x}\right)$$

$$= \frac{1}{1 + \frac{4}{9x^2}} \times \frac{d}{dx}\left(\frac{2}{3x}\right)$$

$$= \frac{1}{\frac{9x^2 + 4}{9x^2}} \times \frac{d}{dx}\left(\frac{2}{3x}\right)$$

$$= \frac{9x^2}{9x^2 + 4} \times \frac{d}{dx}\left(\frac{2}{3x}\right)$$

Question 9

Answer: B

Worked solution

Substitute $u = 3 - x$

$$\Rightarrow \frac{du}{dx} = -1 \text{ and } x = 3 - u.$$

$$\int \frac{x}{\sqrt{3-x}}\,dx$$

$$= \int \frac{-x}{\sqrt{3-x}}\frac{du}{dx}\,dx$$

$$= \int \frac{u-3}{\sqrt{u}}\,du$$

$$= \int \left(\sqrt{u} - \frac{3}{\sqrt{u}}\right) du$$

Question 10

Answer: E

Worked solution

$$\frac{dx}{dt} = 10t \text{ and } \frac{dy}{dt} = 7.$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 7 \times \frac{1}{10t}$$

$$\frac{dy}{dx} = \frac{7}{10t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{7}{10t}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{7}{10t}\right) \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-7}{10t^2} \times \frac{1}{10t}$$

$$\frac{d^2y}{dx^2} = \frac{-7}{100t^3}$$

Question 11

Answer: E

Worked solution

$$V = \frac{4}{3}\pi R^3$$

$$\frac{dV}{dR} = 4\pi R^2$$

$$\frac{dR}{dt} = \frac{dV}{dt} \times \frac{dR}{dV}$$

$$\frac{dR}{dt} = \frac{-3}{2} \times \frac{1}{4\pi R^2}$$

$$\frac{dR}{dt} = \frac{-3}{8\pi R^2}$$

When volume is $\frac{32\pi}{3}$ cm^3, $R = 2$ cm.

$$\frac{dR}{dt} = \frac{-3}{32\pi} \text{ cm/min}$$

The radius is decreasing at $\frac{3}{32\pi}$ cm/min.

Tip

- *Students should be aware that the negative sign is incorporated in the word ‘decreasing’. If the question asked ‘At what rate is the radius of the snowball changing when the volume is* $\frac{32\pi}{3}$ cm^3 *?’, the answer would then be option A.*

Question 12

Answer: C

Worked solution

$y = \sqrt{x} - 1$

$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

Arc length = $\displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ with a = 1 and b = 4.

Arc length = $\displaystyle\int_1^4 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$

Question 13

Answer: D

Worked solution

$$\begin{aligned}\left|\underset{\sim}{a} - 2\underset{\sim}{b}\right| &= \left|\left(5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right) - \left(6\underset{\sim}{i} - 12\underset{\sim}{j} + 12\underset{\sim}{k}\right)\right| \\ &= \left|-\underset{\sim}{i} + 11\underset{\sim}{j} - 10\underset{\sim}{k}\right| \\ &= \sqrt{(-1)^2 + (11)^2 + (-10)^2} \\ &= \sqrt{222}\end{aligned}$$

Question 14

Answer: A

Worked solution

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent, then

$\underset{\sim}{a} = \lambda \underset{\sim}{b}$ where λ is a constant.

$$2\underset{\sim}{j} - \underset{\sim}{k} = \lambda(-\underset{\sim}{j} + n\underset{\sim}{k})$$

$$2\underset{\sim}{j} - \underset{\sim}{k} = -\lambda\underset{\sim}{j} + \lambda n\underset{\sim}{k}$$

$$\therefore \lambda = -2$$

and

$$-2n = -1$$

$$n = \frac{1}{2}$$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent, then $n = \frac{1}{2}$. So, if they are linearly independent, $n \neq \frac{1}{2}$.

Question 15

Answer: C

Worked solution

If these vectors are perpendicular, then

$$\underset{\sim}{a}.\underset{\sim}{b} = 0$$

$$3 + 3 + 2p = 0$$

$$6 + 2p = 0$$

$$p = -3$$

Question 16

Answer: E

Worked solution

$$\underset{\sim}{\mathrm{r}}(t) = 2\cos(3t)\,\underset{\sim}{\mathrm{i}} - 3\sin(3t)\,\underset{\sim}{\mathrm{j}}$$

$$\dot{\underset{\sim}{\mathrm{r}}}(t) = -6\sin(3t)\,\underset{\sim}{\mathrm{i}} - 9\cos(3t)\,\underset{\sim}{\mathrm{j}}$$

$$\begin{aligned}
\text{Speed} &= \left|\dot{\underset{\sim}{\mathrm{r}}}(t)\right| \\
&= \sqrt{(-6\sin(3t))^2 + \left(-9\cos\left(3t\right)^2\right)} \\
&= \sqrt{36\sin^2(3t) + 81\cos^2(3t)} \\
&= \sqrt{36(\sin^2(3t) + \cos^2(3t)) + 45\cos^2(3t)} \\
&= \sqrt{36 + 45\cos^2(3t)}
\end{aligned}$$

$$\text{Speed}_{\text{max}} = \sqrt{36+45}$$

Speed has maximum value when $\cos^2(3t) = 1$:

$\text{Speed}_{\text{max}}$ = 9 ms^{-1}

Question 17

Answer: D

Worked solution

Option 1

$$a = v\frac{dv}{dx}$$

$$v = \sqrt{5-x}$$

$$\frac{dv}{dx} = \frac{-1}{2\sqrt{5-x}}$$

$$a = \sqrt{5-x} \times \frac{-1}{2\sqrt{5-x}}$$

$$a = \frac{-1}{2}$$

Option 2

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{(5-x)}{2}\right) = \frac{-1}{2}$$

Question 18

Answer: B

Worked solution

Let $2M$ = mass of smaller object

then $5M$ = mass of larger object.

The direction of the larger object's acceleration is down. Equations of motion on each mass are:

Larger mass: $5Mg - T = 5Ma$

Smaller mass: $T - 2Mg = 2Ma$

Where T is the magnitude of the tension force in the string, $5Mg$ is the magnitude of the gravitational force acting on the larger block and $2Mg$ is the magnitude of the gravitational force acting on the larger block.

Larger mass + Smaller mass gives:

$$3Mg = 7Ma$$
$$a = \frac{3g}{7}\ \text{ms}^{-2}$$

Tips

- *For connected particles, the equation of motion needs to be written separately for each object, knowing that the acceleration is the same in both equations.*
- *It is helpful to annotate any diagrams provided in the Q&A book with all forces.*

Question 19

Answer: E

Worked solution

E(X) = a and Var(X) = b, then

E($2X$ + 1) = 2 × E(X) + 1 = $2a$ + 1

Var(X) = 2^2 × Var(X) = $4b$

Question 20

Answer: A

Worked solution

Take the upwards direction (the direction of motion) as the positive direction.

Equation 1: $F_{net} = ma = (15)(4) = 60$.

Equation 2: $F_{net} = R - 15g$, where R is the size of the normal reaction force and $15g$ is the size of the weight force in newtons.

Equate equations 1 and 2:

$60 = R - 15g$

$\Rightarrow R = 60 + 15g$.

SECTION B

Question 1a.

Worked solution

$\bar{x} = 375$

$s = 17$

$n = 50$

To calculate a 95% confidence limit, z = 1.9600.

$$z\frac{s}{\sqrt{n}} = 1.9600 \times \frac{17}{\sqrt{50}} = 4.7122$$

$$\bar{x} - z\frac{s}{\sqrt{n}} = 375 - 4.7122 = 370.2878$$

$$\bar{x} + z\frac{s}{\sqrt{n}} = 375 + 4.7122 = 379.7122$$

Calculations need to be made to four decimal places to ensure accuracy to two decimal places.

We can be 95% confident that, on average, the service cost is between \$370.29 and \$379.71.

Mark allocation: 3 marks

- 1 mark for using z = 1.9600
- 1 mark for calculating $z\frac{s}{\sqrt{n}}$
- 1 mark for calculating the 95% confidence limits \$370.29 and \$379.71

Question 1b.

Worked solution

$\bar{x} = 375$

$s = 17$

$n = 50$

To calculate a 99% confidence limit, z = 2.5758.

$$z\frac{s}{\sqrt{n}} = 2.5758 \times \frac{17}{\sqrt{50}} = 6.1926$$

$$\bar{x} - z\frac{s}{\sqrt{n}} = 375 - 6.1926 = 368.8074$$

$$\bar{x} + z\frac{s}{\sqrt{n}} = 375 + 6.1926 = 381.1926$$

Calculations need to be made to four decimal places to ensure accuracy to two decimal places.

We can be 99% confident that, on average, the service cost is between $368.81 and $381.19.

Mark allocation: 2 marks

- 1 mark for using z = 2.5758 and calculating $z\frac{s}{\sqrt{n}}$
- 1 mark for calculating the 99% confidence limits $368.81 and $381.19

Question 1c.

Worked solution

Confidence interval formula is $\overline{x} \pm z\frac{s}{\sqrt{n}}$, where $z\frac{s}{\sqrt{n}} = 3$.

To calculate a 95% confidence limit, z = 1.9600.

$$z\frac{s}{\sqrt{n}} = 3$$

$$1.9600 \times \frac{17}{\sqrt{n}} = 3$$

$$\sqrt{n} = \frac{1.9600 \times 17}{3}$$

$$\sqrt{n} = 123.358$$

$$n = 124$$

Need to round n **up** to 124 to keep the cost to $\pm\$3$ at the 95% level of confidence.

Mark allocation: 3 marks

- 1 mark for writing $z\frac{s}{\sqrt{n}} = 3$
- 1 mark for correctly substituting z = 1.9600 and s = 17 into $z\frac{s}{\sqrt{n}} = 3$
- 1 mark for solving to find sample size n = 124 customers
- Students need to substitute answer to ensure cost is within $\pm\$3$

Question 2a.

Worked solution

$x = t - \frac{1}{t}$ $\qquad$ $y = t + \frac{1}{t}$

$\frac{dx}{dt} = 1 + t^{-2}$ $\qquad$ $\frac{dy}{dt} = 1 - t^{-2}$

$\frac{dx}{dt} = 1 + \frac{1}{t^2}$ $\qquad$ $\frac{dy}{dt} = 1 - \frac{1}{t^2}$

$\frac{dx}{dt} = \frac{t^2 + 1}{t^2}$ $\qquad$ $\frac{dy}{dt} = \frac{t^2 - 1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{t^2 - 1}{t^2} \times \frac{t^2}{t^2 + 1}$$

$$\frac{dy}{dx} = \frac{t^2 - 1}{t^2 + 1}$$

When $t = 2$, $x = \frac{3}{2}$ and $y = \frac{5}{2}$.

When $t = 2$, $\frac{dy}{dx} = \frac{3}{5}$.

Equation of tangent:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{5}{2} = \frac{3}{5}(x - \frac{3}{2})$$

$$y - \frac{5}{2} = \frac{3}{5}x - \frac{9}{10}$$

$$y = \frac{3}{5}x - \frac{9}{10} + \frac{5}{2}$$

$$y = \frac{3}{5}x + \frac{8}{5}$$

Mark allocation: 4 marks

- 1 mark for correctly differentiating each parametric equation
- 1 mark for using the chain rule to obtain $\frac{dy}{dx} = \frac{t^2 - 1}{t^2 + 1}$
- 1 mark for correctly obtaining point $\left(\frac{3}{2}, \frac{5}{2}\right)$ and gradient $m = \frac{3}{5}$
- 1 mark for equation of tangent $y = \frac{3}{5}x + \frac{8}{5}$ or other equivalent form

Question 2b.

Worked solution

$x = t - \frac{1}{t}$ and $y = t + \frac{1}{t}$, $t \neq 0$.

Eliminate parameter t by squaring both equations and subtracting.

$x^2 = t^2 - 2 + \frac{1}{t^2} \quad (1)$

$y^2 = t^2 + 2 + \frac{1}{t^2} \quad (2)$

Subtracting:

$(2) - (1)$ gives:

$y^2 - x^2 = 4$

Mark allocation: 2 marks

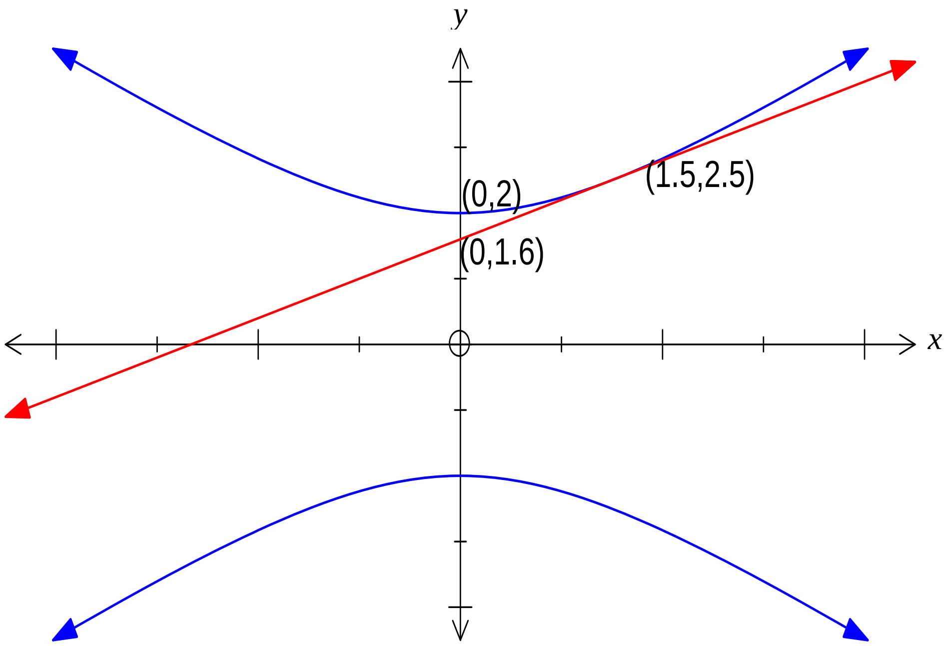
- 1 mark for correctly squaring both parametric equations
- 1 mark for subtracting to show the Cartesian equation is $y^2 - x^2 = 4$

Tip

- *Squaring parametric equations is a useful process to eliminate a parameter when other methods fail to work.*

Question 2c.

Worked solution



Mark allocation: 2 marks

- 1 mark for sketching the tangent correctly
- 1 mark for labelling tangential point and y-intercepts with their exact coordinates

Question 2d.

Worked solution

$$V = \pi \int_{1.6}^{2.5} \left(\frac{5y-8}{3} \right)^2 dy - \pi \int_{2}^{2.5} \left(y^2 - 4 \right) dy$$

$$V = \frac{2\pi}{15} \text{ cubic units}$$

Mark allocation: 2 marks

- 1 mark for $V = \pi \int_{1.6}^{2.5} \left(\frac{5y-8}{3} \right)^2 dy - \pi \int_{2}^{2.5} \left(y^2 - 4 \right) dy$
- 1 mark for $V = \frac{2\pi}{15}$ cubic units

Question 3a.

Worked solution

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

$$\cos\left(\frac{\pi}{4}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1$$

$$\frac{\sqrt{2}}{2} = 2\cos^2\left(\frac{\pi}{8}\right) - 1$$

$$2\cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2} + 1$$

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}+2}{4}$$

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}+2}{4}} \text{ since } \cos\left(\frac{\pi}{8}\right) > 0.$$

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

$$\cos\left(\frac{\pi}{4}\right) = 1 - 2\sin^2\left(\frac{\pi}{8}\right)$$

$$\frac{\sqrt{2}}{2} = 1 - 2\sin^2\left(\frac{\pi}{8}\right)$$

$$2\sin^2\left(\frac{\pi}{8}\right) = 1 - \frac{\sqrt{2}}{2}$$

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4}$$

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{2-\sqrt{2}}{4}} \text{ since } \sin\left(\frac{\pi}{8}\right) > 0.$$

Mark allocation: 4 marks

- 1 mark for using $\cos(2\theta) = 2\cos^2(\theta) - 1$
- 1 mark for working leading to $\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}+2}{4}}$
- 1 mark for using $\cos(2\theta) = 1 - 2\sin^2(\theta)$
- 1 mark for working leading to $\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{2-\sqrt{2}}{4}}$

Question 3b.

Worked solution

$$u = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)$$

$$u = \sqrt{\frac{\sqrt{2}+2}{4}} + i\sqrt{\frac{2-\sqrt{2}}{4}}$$

and

$$\bar{u} = \sqrt{\frac{\sqrt{2}+2}{4}} - i\sqrt{\frac{2-\sqrt{2}}{4}}$$

Mark allocation: 2 marks

- 1 mark for $u = \sqrt{\frac{\sqrt{2}+2}{4}} + i\sqrt{\frac{2-\sqrt{2}}{4}}$
- 1 mark for $\bar{u} = \sqrt{\frac{\sqrt{2}+2}{4}} - i\sqrt{\frac{2-\sqrt{2}}{4}}$

Question 3c.

Worked solution

$$\text{Arg}\left(\frac{u}{\bar{u}}\right) = \text{Arg}(u) - \text{Arg}(\bar{u})$$

$$\text{Arg}\left(\frac{u}{\bar{u}}\right) = \frac{\pi}{8} - \left(\frac{-\pi}{8}\right)$$

$$\text{Arg}\left(\frac{u}{\bar{u}}\right) = \frac{\pi}{4}$$

Mark allocation: 2 marks

- 1 mark for using $\text{Arg}\left(\frac{u}{\bar{u}}\right) = \text{Arg}(u) - \text{Arg}(\bar{u})$
- 1 mark for $\frac{\pi}{4}$

Question 3d.

Worked solution

$$\text{Arg}(u-\overline{u})=\left[\sqrt{\frac{\sqrt{2}+2}{4}}+i\sqrt{\frac{2-\sqrt{2}}{4}}\right]-\left[\sqrt{\frac{\sqrt{2}+2}{4}}-i\sqrt{\frac{2-\sqrt{2}}{4}}\right]$$

$$\text{Arg}(u-\overline{u})=\left(2i\sqrt{\frac{2-\sqrt{2}}{4}}\right)$$

$$\text{Arg}(u-\overline{u})=\frac{\pi}{2}$$

Explanatory notes

Since $\text{Arg}(u-\overline{u})$ is of the form $\text{Arg}\left(0+k\,i\right)$, $k > 0$, which would be the point $(0, k)$ on an Argand diagram and consequently have an Argument of $\frac{\pi}{2}$.

Mark allocation: 2 marks

- 1 mark for subtracting u and $\overline{u}$ in Cartesian form to obtain $\left(2i\sqrt{\frac{2-\sqrt{2}}{4}}\right)$
- 1 mark for $\frac{\pi}{2}$

Question 3e.

Worked solution

If u and $\overline{u}$ are the only roots of the complex equation, then

$$\left(z-u\right)\left(z-\overline{u}\right)=z^2+pz+q$$
$$z^2-uz-\overline{u}z+u\overline{u}=z^2+pz+q$$
$$z^2-(u+\overline{u})z+u\overline{u}=z^2+pz+q$$

Equating the constant terms and coefficients of z:

Constant term: $q=u\,\overline{u}$

$$q=\text{cis}\left(\frac{\pi}{8}\right)\text{cis}\left(\frac{-\pi}{8}\right)$$
$$q=\text{cis}(0)$$
$$q=\cos(0)+i\sin(0)$$
$$q=1$$

and

Coefficient of z: $p=-(u+\overline{u})$

$$p=-u-\overline{u}$$
$$p=-\left[\sqrt{\frac{\sqrt{2}+2}{4}}+i\sqrt{\frac{2-\sqrt{2}}{4}}\right]-\left[\sqrt{\frac{\sqrt{2}+2}{4}}-i\sqrt{\frac{2-\sqrt{2}}{4}}\right]$$
$$p=-\sqrt{\frac{\sqrt{2}+2}{4}}-i\sqrt{\frac{2-\sqrt{2}}{4}}-\sqrt{\frac{\sqrt{2}+2}{4}}+i\sqrt{\frac{2-\sqrt{2}}{4}}$$
$$p=-2\sqrt{\frac{\sqrt{2}+2}{4}}$$
$$p=-\sqrt{\sqrt{2}+2}$$

Mark allocation: 3 marks

- 1 mark for $z^2-(u+\overline{u})z+u\overline{u}=z^2+pz+q$
- 1 mark for $q = 1$
- 1 mark for $p=-\sqrt{\sqrt{2}+2}$

Question 4a.

Worked solution

Consider the volume of water in the tank, which is in the shape of a cylinder with fixed radius b metres and variable height h metres.

Volume $= \pi r^2 h$, where $r = b$.

$$\therefore\ V = \pi b^2 h$$

$$\frac{dV}{dh} = \pi b^2$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$\frac{dh}{dt} = -k\sqrt{h} \times \frac{1}{\pi b^2}$$

$$\frac{dh}{dt} = \frac{-k\sqrt{h}}{\pi b^2}$$

Mark allocation: 2 marks

- 1 mark for finding $\dfrac{dV}{dh} = \pi b^2$
- 1 mark for using the chain rule to find $\dfrac{dh}{dt} = \dfrac{-k\sqrt{h}}{\pi b^2}$

Question 4b.

Worked solution

$$\frac{dh}{dt} = \frac{-k\sqrt{h}}{\pi b^2}$$

$$\frac{dt}{dh} = \frac{\pi b^2}{-k\sqrt{h}}$$

$$t = \frac{-\pi b^2}{k}\int h^{-\frac{1}{2}}\,dh$$

$$t = \frac{-2\pi b^2}{k}h^{\frac{1}{2}} + c$$

When $t = 0$, $h = b$:

$$0 = \frac{-2\pi b^2}{k}\sqrt{b} + c$$

$$c = \frac{2\pi b^2}{k}\sqrt{b}$$

$$t = \frac{2\pi b^2}{k}\sqrt{b} - \frac{2\pi b^2}{k}\sqrt{h}$$

$$t = \frac{2\pi b^2}{k}\left(\sqrt{b} - \sqrt{h}\right)$$

Mark allocation: 3 marks

- 1 mark for solving the differential equation to obtain $t = \frac{-2\pi b^2}{k}h^{\frac{1}{2}} + c$
- 1 mark for finding the constant of integration $c = \frac{2\pi b^2}{k}\sqrt{b}$
- 1 mark for algebraic simplification to obtain $t = \frac{2\pi b^2}{k}\left(\sqrt{b} - \sqrt{h}\right)$

Note: Full credit should also be given for verifying the given expression for t by direct substitution of the derivative into the differential equation given in part a.

Question 4c.

Worked solution

$$t = \frac{2\pi b^2}{k}\left(\sqrt{b} - \sqrt{h}\right)$$

When $h = 0$:

$$t = T = \frac{2\pi b^2 \sqrt{b}}{k}$$

When $h = \frac{b}{2}$:

$$t = \frac{2\pi b^2}{k}\left(\sqrt{b} - \sqrt{\frac{b}{2}}\right)$$

$$t = \frac{2\pi b^2 \sqrt{b}}{k}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$t = T\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$t = \left(\frac{2 - \sqrt{2}}{2}\right)T \text{ minutes}$$

Mark allocation: 3 marks

- 1 mark for finding the time taken for the tank to empty, $T = \frac{2\pi b^2 \sqrt{b}}{k}$
- 1 method mark for manipulating the equation to obtain *t* in terms of *T*.
- 1 mark for finding the exact value for the tank to empty to half its depth, $\left(\frac{2 - \sqrt{2}}{2}\right)T$ minutes

Question 5a.

Worked solution

$$\frac{dv}{dt}=10-kt$$

$$0=10-k(20)$$

$$0=10-20k$$

$$k=\frac{1}{2}$$

Mark allocation: 1 mark

- 1 mark for showing $k=\frac{1}{2}$

Question 5b.

Worked solution

$$\frac{dv}{dt}=10-\frac{1}{2}t$$

$$v=\int\left(10-\frac{1}{2}t\right)dt$$

$$v=10t-\frac{1}{4}t^2+c$$

When $v=0,\ t=0$:

$$\Rightarrow c=0$$

$$v=10t-\frac{1}{4}t^2$$

When t = 20:

$$v=10(20)-\frac{1}{4}(20)^2$$

v = 100 ms^{-1}

Mark allocation: 3 marks

- 1 mark for antidifferentiating to find $v=10t-\frac{1}{4}t^2+c$
- 1 mark for finding c = 0
- 1 mark for showing the maximum velocity is 100 ms^{-1}

Question 5c.

Worked solution

Stage 1

$$d = \int_0^{20} \left(10t - \frac{1}{4}t^2\right) dt$$

Using CAS, $d = \frac{4000}{3}$ m

Stage 2 and stage 3

$$d = (100 \times 50) + \left(\frac{1}{2} \times 40 \times 100\right)$$

d = 7000 m

Total distance travelled = $\frac{25\,000}{3}$ m

Mark allocation: 3 marks

- 1 mark for finding distance travelled in **stage 1**
- 1 mark for finding distance travelled in **stage 2** and **stage 3**
- 1 mark for finding the total distance of $\frac{25\,000}{3}$ m

Question 5d.

Worked solution

$$\frac{dv}{dt}=\frac{-2b}{d}t+b$$

$$v=\int\left(\frac{-2b}{d}t+b\right)dt$$

$$v=\frac{-b}{d}t^2+bt+c$$

When $t=0, v=0$:

$$\Rightarrow c=0$$

$$v=\frac{-b}{d}t^2+bt$$

Maximum velocity occurs for t such that $\frac{dv}{dt}=0\Rightarrow\frac{-2b}{d}+b=0\Rightarrow t=\frac{d}{2}$.

$$v=\frac{-b}{d}\left(\frac{d}{2}\right)^2+b\left(\frac{d}{2}\right)$$

$$v=\frac{-bd}{4}+\frac{bd}{2}$$

$$v=\frac{bd}{4}\ \text{ms}^{-1}$$

Mark allocation: 2 marks

- 1 mark for finding $v=\frac{-b}{d}t^2+bt$
- 1 mark for substituting $t=\frac{d}{2}$ to obtain the maximum velocity of $\frac{bd}{4}$

Question 5e.

Worked solution

Distance travelled is the same as that of the first particle.

When d = 50, then

$$\int_0^{50}\left(\frac{-b}{50}t^2+bt\right)dt=\frac{25\,000}{3}$$

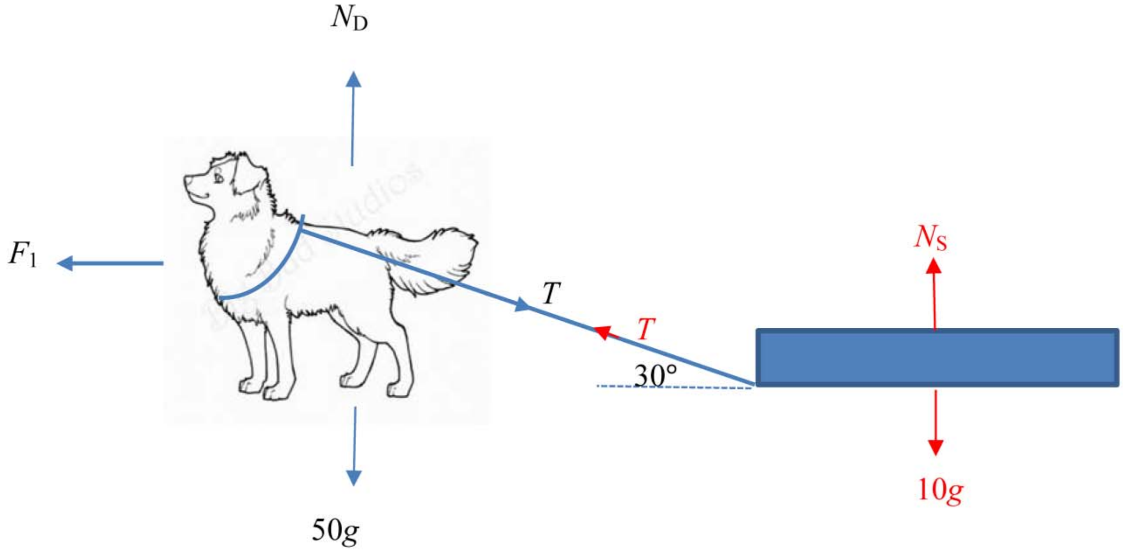
b = 20, using CAS.

Mark allocation: 2 marks

- 1 mark for $\int_0^{50}\left(\frac{-b}{50}t^2+bt\right)dt=\frac{25\,000}{3}$
- 1 mark for b = 20

Question 6a.

Worked solution



Explanatory notes

N_s is the normal reaction force and 10 g is the weight force.

Mark allocation: 1 mark

- 1 mark for marking all three forces acting on the sled. Forces need to be labelled as in the worked solution, although equivalent symbols other than those given can be marked as correct.

Question 6b.

Worked solution

If the dog and the sled are accelerating along level ground, then $\sum F = ma$ for forces resolved parallel to the ground and $\sum F = 0$ for forces resolved perpendicular to the ground.

Resolving forces on the sled parallel to the ground $\sum F = ma$:

$$T\cos(30^\circ) = 10a$$

$$T\cos(30^\circ) = 2$$

$$T\left(\frac{\sqrt{3}}{2}\right) = 2$$

$$T = \frac{4\sqrt{3}}{3}$$

$$T = \frac{4\sqrt{3}}{3} \text{ newtons}$$

Resolving forces on the sled perpendicular to the ground $\sum F = 0$:

$$T\sin(30^\circ) + N_S = 10g$$

$$\frac{4\sqrt{3}}{3}\left(\frac{1}{2}\right) + N_S = 10g$$

$$\frac{2\sqrt{3}}{3} + N_S = 10g$$

$$N_S = 10g - \frac{2\sqrt{3}}{3}$$

$$N_S = \frac{30g - 2\sqrt{3}}{3}$$

$$N_S = \frac{30g - 2\sqrt{3}}{3} \text{ newtons}$$

Mark allocation: 3 marks

- 1 mark for resolving forces on the sled parallel to and perpendicular to the ground
- 1 mark for $T = \frac{4\sqrt{3}}{3}$ newtons
- 1 mark for $N_S = \frac{30g - 2\sqrt{3}}{3}$ newtons

Question 6c.

Worked solution

Resolving forces on the dog parallel to the ground $\sum F = ma$:

$$F_1 - T\cos(30^\circ) = 50 \times 0.2$$

$$F_1 - \frac{4\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2}\right) = 10$$

$$F_1 = 12$$

$$F_1 = 12 \text{ newtons}$$

Mark allocation: 2 marks

- 1 mark for resolving forces on the dog parallel to the ground
- 1 mark for finding the exact value of F_1

Question 6d.

Worked solution

If the dog and the sled are travelling at a constant velocity up the hill, then $a = 0$ and so $\sum F = 0$ when resolving forces are parallel to the slope of the hill.

Resolving forces on the sled parallel to the hill $\sum F = 0$:

$$T\cos(30^\circ) - 10g\sin(45^\circ) = 0$$

$$T\left(\frac{\sqrt{3}}{2}\right) - 10g\left(\frac{\sqrt{2}}{2}\right) = 0$$

$$\frac{\sqrt{3}T}{2} = 5g\sqrt{2}$$

$$T = \frac{10g\sqrt{2}}{\sqrt{3}}$$

$$T = \frac{10g\sqrt{6}}{3}$$

$\therefore\ c = 10\sqrt{6}$ and $d = 3$.

Mark allocation: 2 marks

- 1 mark for resolving forces on the sled parallel to the hill
- 1 mark for solving *T*, showing $c = 10\sqrt{6}$ and $d = 3$

Question 6e.

Worked solution

Resolving forces on the dog parallel to the hill $\sum F = 0$:

$$F_2 - T\cos(30^\circ) - 50g\sin(45^\circ) = 0$$

$$F_2 - \left(\frac{10g\sqrt{6}}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - 50g\left(\frac{\sqrt{2}}{2}\right) = 0$$

$$F_2 - 5g\sqrt{2} - 25g\sqrt{2} = 0$$

$$F_2 = 30g\sqrt{2}$$

$$F_2 = 416 \text{ newtons}$$

Mark allocation: 2 marks

- 1 mark for resolving forces on the sled parallel to the hill
- 1 mark for $F_2 = 416$ newtons

END OF WORKED SOLUTIONS