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YEAR 12 *Trial Exam Paper*

2016

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- fully worked solutions
- mark allocations
- tips on how to approach the exam

Question 1a.

Worked solution

$z^2 + (a + bi)z + bi = 0$

$(2i)^2 + (a + bi)(2i) + bi = 0$

$-4 + 2ai - 2b + bi = 0$

$\Rightarrow (-4 - 2b) + (2a + b)i = 0 + 0i$

$\Rightarrow -4 - 2b = 0$

$\Rightarrow b = -2$

$2a + b = 0$

$\Rightarrow 2a - 2 = 0$

$\Rightarrow a = 1$

$\therefore a = 1$ and $b = -2$

Mark allocation: 2 marks

- 1 mark for obtaining two correct simultaneous equations in terms of a and b
- 1 mark for the correct answer

Question 1b.

Worked solution

$z^2+\left(a+bi\right)z+bi=z^2+(1-2i)z-2i=0$

$\Rightarrow z(z+1)-2i(z+1)=0$

$\Rightarrow (z+1)(z-2i)=0$

$\therefore z=-1$ is the other solution

Alternative method

Substituting the answer to part a. into $z^2+(a+bi)z-2i=0$ gives

$z^2+(1-2i)z-2i=0.$

$2i$ is known to be a solution; therefore, $z-2i$ is a factor of $z^2+(1-2i)-2i$.

$z^2+(1-2i)z-2i$ has the factorised form $(z-2i)(z+\alpha)$, $\alpha\in C$.

By comparison with $z^2+(1-2i)z-2i$, the constant term in the expansion of $(z-2i)(z+\alpha)$ must equal $-2i$, which means that $\alpha=1$.

$\therefore\ z^2+(1-2i)z-2i=(z-2i)(z+1)$

From the null factor law, the other solution to $z^2+(1-2i)z-2i=0$ is therefore $z=-1$.

Mark allocation: 2 marks

- 1 mark for correctly factorising
- 1 mark for the correct answer

Question 2a.

Worked solution

$\underset{\sim}{r}(t) = \tan^{-1}(t)\,\underset{\sim}{i} + t\,\underset{\sim}{j} + \underset{\sim}{c}$

$\underset{\sim}{r}(0) = \underset{\sim}{c} = \underset{\sim}{j}$

$\Rightarrow \underset{\sim}{r}(t) = \tan^{-1}(t)\,\underset{\sim}{i} + (t+1)\,\underset{\sim}{j}$

Mark allocation: 2 marks

- 1 mark for correctly antidifferentiating $\underset{\sim}{v}(t)$ to obtain $\underset{\sim}{r}(t)$
- 1 mark for the correct answer

Question 2b.

Worked solution

$x = \tan^{-1}(t)$
$\Rightarrow t = \tan(x)$
$y = t + 1$
$\therefore\ y = 1 + \tan(x)$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3

Worked solution

$(x-y)^2 - \log_e x = x^2 - 2xy + y^2 - \log_e x = 1$

$\Rightarrow 2x - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{1}{x} = 0$

$\Rightarrow \frac{dy}{dx}(2y-2x) = \frac{1}{x} + 2y - 2x$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{x}}{2y-2x} + 1.$

At the point $(1,\ 2)$, $\frac{dy}{dx} = \frac{1}{2} + 1 = \frac{3}{2}.$

The gradient of the tangent is $\frac{3}{2}$ or 1.5.

Alternative method 1

Another method is to avoid unnecessary algebra by substituting (1, 2) into

$2x - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{1}{x} = 0$ and then making $\frac{dy}{dx}$ the subject.

$2(1) - 2(2) - 2(1)\frac{dy}{dx} + 2(2)\frac{dy}{dx} - \frac{1}{1} = 0$

$\Rightarrow 2 - 4 - 2\frac{dy}{dx} + 4\frac{dy}{dx} - 1 = 0$

$\Rightarrow 2\frac{dy}{dx} = 3$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2}.$

Alternative method 2

Instead of expanding $(x-y)^2$ and then differentiating, this term can be directly differentiated using the chain rule.

$$\frac{d}{dx}(x-y)^2 = \underbrace{2(x-y)}_{\text{Derivative of outer function}} \times \underbrace{\left(1 - \frac{dy}{dx}\right)}_{\text{Derivative of inner fuction}}$$

This requires less work and leads to easier calculations.

Alternative method 3

$(x-y)^2-\log_e x=1$

$(x-y)^2=1+\log_e x$

$(y-x)^2=1+\log_e x$

$y-x=\pm\sqrt{1+\log_e x}$

$\Rightarrow y=x+\sqrt{1+\log_e x}$,   since $y=2$ when $x=1$

$y=x+(1+\log_e x)^{\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx}=1+\dfrac{1}{x}\times\dfrac{1}{2}\left(1+\log_e x\right)^{-\frac{1}{2}}$

$\dfrac{dy}{dx}=1+\dfrac{1}{2x\sqrt{1+\log_e x}}$

When x = 1, $\dfrac{dy}{dx}=1+\dfrac{1}{2}=\dfrac{3}{2}$.

Mark allocation: 3 marks

- 1 mark for correctly differentiating the equation
- 1 mark for obtaining a correct expression for $\dfrac{dy}{dx}$
- 1 mark for the correct answer

Tip

- *The relation can be explicitly expressed as a function of x and then differentiated to obtain* $\dfrac{dy}{dx}$*, but this would be time consuming.*

Question 4

Worked solution

$$\frac{\sin(x)}{\cos(x)} = \frac{\cos(2x)}{\sin(2x)}, \quad x \in [0, \pi]$$

$$\Rightarrow \frac{\sin(x)}{\cos(x)} = \frac{\cos(2x)}{2\sin(x)\cos(x)}, \text{ where } \cos(x) \neq 0 \text{ and } \sin(x) \neq 0$$

$$\Rightarrow 2\sin(x)\cos(x)\sin(x) = \cos(x)\cos(2x)$$

$$\Rightarrow 2\sin(x)\cos(x)\sin(x) - \cos(x)\cos(2x) = 0$$

$$\Rightarrow \cos(x)\left(2\sin^2(x) - \cos(2x)\right) = 0.$$

<u>Case 1</u>: $\cos(x) = 0$.

Reject this case because $\cos(x) \neq 0$.

<u>Case 2</u>: $2\sin^2(x) - \cos(2x) = 0$

$$\Rightarrow 2\sin^2(x) - \left(1 - 2\sin^2(x)\right) = 0$$

$$\Rightarrow 4\sin^2(x) - 1 = 0$$

$$\Rightarrow \sin^2(x) = \frac{1}{4}$$

$$\Rightarrow \sin(x) = \pm\frac{1}{2}, \quad x \in [0, \pi].$$

<u>Case 2a</u>: $\sin(x) = \frac{1}{2}$

$$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

<u>Case 2b</u>: $\sin(x) = -\frac{1}{2}$

Reject this case because $x \in [0, \pi]$

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Mark allocation: 3 marks

- 1 mark for correctly using the double angle formulas
- 1 mark for obtaining a correct equation in terms of $\sin x$ only
- 1 mark for the correct answer

Tip

- *All double angle formulas can be looked up on the formula sheet provided.*

Question 5a.

Worked solution

Let X = the time the students spend studying at home each week
and Y = the time the students spend involved in recreational activities each week.

$$\begin{aligned}\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) &= 40^2 + 30^2 \\ &= 1600 + 900 \\ &= 2500\end{aligned}$$

$\Rightarrow \text{Sd}(X + Y) = \sqrt{2500} = 50$

The standard deviation of the total time that students at Academia University spend studying at home and being involved in recreational activities each week is 50 minutes.

Mark allocation: 2 marks

- 1 mark for the correct calculation of the variance
- 1 mark for the correct answer

Question 5b.i.

Worked solution

$n = 64$, $\bar{x} = 719$, $\sigma = 40$ and $z = 1.96$

$H_0 : \mu = 730$

$H_1 : \mu \neq 730$

Mark allocation: 1 mark

- 1 mark for correctly expressing the null hypothesis and the alternative hypothesis

Question 5b.ii.

Worked solution

$n = 64, \bar{x} = 719, \sigma = 40$ and $z = 1.96$
$H_0 : \mu = 730$
$H_1 : \mu \neq 730$

Using $\alpha = 0.05$, reject H_0 if $z < -1.96$ or $z > 1.96$.

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$
$$= \frac{719 - 730}{\frac{40}{\sqrt{64}}}$$
$$\Rightarrow z = \frac{-11}{5} = -2.2$$

Because $z < -1.96$, reject H_0.

The university's claim that the mean time spent studying at home each week by its students is 730 minutes is not supported by the data at the $\alpha = 0.05$ level of significance.

Alternative method 1

Construct an approximate 95% confidence interval.

$n = 64, \bar{x} = 719, \sigma = 40$ and $z = 1.96$
$H_0 : \mu = 730$
$H_1 : \mu \neq 730$

The 95% confidence interval for μ is

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$
$$= \left(719 - 1.96 \times \frac{40}{\sqrt{64}}, 719 + 1.96 \times \frac{40}{\sqrt{64}}\right)$$
$$= (719 - 1.96 \times 5,\ 719 + 1.96 \times 5)$$
$$= (709.2,\ 728.8)$$

Because this interval does not contain the hypothesised mean of 730, reject the hypothesis that $\mu = 730$.

Alternative method 2

Calculate an approximate maximum value of the *p* value.

$H_0 : \mu = 730$
$H_1 : \mu < 730$
$E(\overline{X}) = \mu = 730$

$$SD(\overline{X}) = \frac{40}{\sqrt{64}} = 5$$

$$\begin{aligned} p \text{ value} &= 2 \times \Pr(\overline{X} \le 719 \mid \mu = 730) \\ &= 2 \times \Pr\left(Z \le \frac{719 - 730}{5} \right) \\ &= 2 \times \Pr(Z \le -2.2) \end{aligned}$$

$2 \times \Pr(Z \le -2.2) < 2 \times \Pr(Z \le -2.0) \approx 0.05$

Because the *p* value is less than the significance level of 0.05, reject the hypothesis that $\mu = 730$.

Mark allocation: 2 marks

- 1 mark for correctly evaluating *z*, the 95% confidence interval or the *p* value
- 1 mark for rejecting the claim that the mean is 730 minutes

Question 6

Worked solution

Arc length $= \int_{x_1}^{x_2} \sqrt{1+\left(f'(x)\right)^2}\, dx$.

$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$

$\Rightarrow 1+\left(f'(x)\right)^2 = 1+\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2$

$= 1+\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$

$= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}$

$= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$

Arc length $= \int_{1}^{3} \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx$

$= \int_{1}^{3} \frac{x^2}{2} + \frac{1}{2x^2}\, dx$

since $\frac{x^2}{2} + \frac{1}{2x^2} > 0$ for all $x \in R \setminus \{0\}$

$= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^3$

$= \left(\frac{27}{6} - \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$

$= \frac{26}{6} + \frac{2}{6} = \frac{28}{6} = \frac{14}{3}$

Answer: $\frac{14}{3}$ units

Mark allocation: 4 marks

- 1 mark for correctly expanding and simplifying the square root part of the integrand
- 1 mark for correctly expressing the square root part of the integrand as a perfect square
- 1 mark for simplifying the integrand
- 1 mark for the correct answer

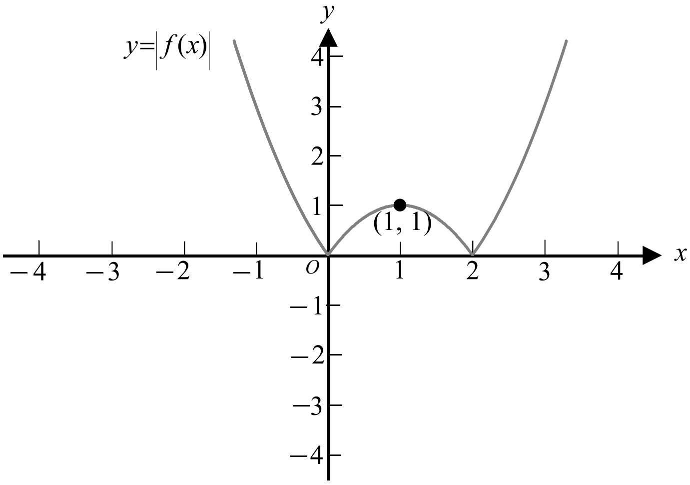
Tip

- *The VCAA requires that the answer is simplified to get the final answer mark.*

Question 7a.
Worked solution

First, graph $y = |f(x)|$.

Reflect any part of $y = f(x)$ that is below the *x*-axis through the *x*-axis.



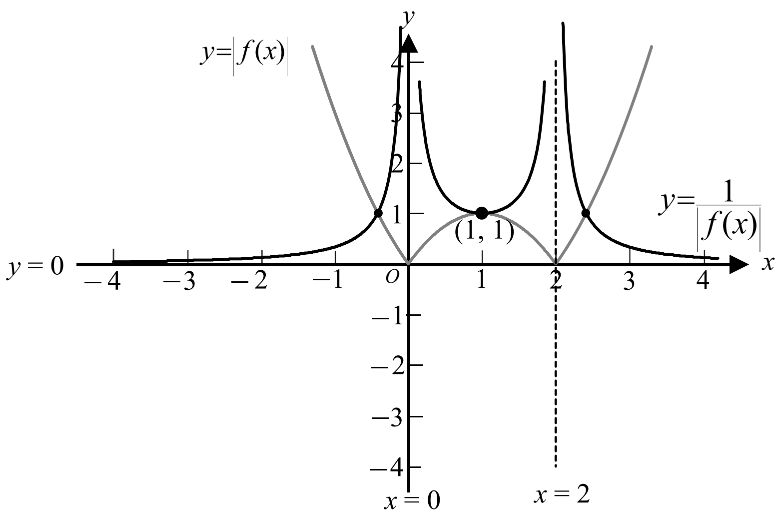
Second, graph $y = \dfrac{1}{|f(x)|}$.

When $y = |f(x)| = 0$, $y = \dfrac{1}{|f(x)|}$ will have vertical asymptotes (i.e. $x = 0$ and $x = 2$).

As $x \to \pm\infty$, $y \to 0$

Therefore, $y = 0$ is a horizontal asymptote.

When $y = |f(x)| = \pm 1$, $y = \dfrac{1}{|f(x)|} = \pm 1$.



Mark allocation: 3 marks

- 1 mark for correctly labelling the asymptotes
- 1 mark for correctly labelling the minimum stationary point of $y = \dfrac{1}{\left|x^2 - 2x\right|}$
- 1 mark for the correct graph of $y = \dfrac{1}{\left|x^2 - 2x\right|}$

Tip

- *For a correct shape, it is essential that the intersection of* $y = f(x)$ and $y = \dfrac{1}{\left|f(x)\right|}$ *aligns with* $y = 1$ *on the given vertical scale.*

Question 7b.

Worked solution

$$\frac{1}{x^2-2x}=\frac{1}{x(x-2)}$$

Therefore:

$$\frac{1}{x(x-2)}=\frac{A}{x}+\frac{B}{x-2}$$

$$=\frac{A(x-2)+Bx}{x(x-2)}$$

$\Rightarrow 1=A(x-2)+Bx$ for all $x \in R$.

Option 1: Substitute convenient values of x.

Substituting $x=2$ gives

$1=2B \Rightarrow B=\frac{1}{2}$

Substituting $x=0$ gives

$1=-2A \Rightarrow A=-\frac{1}{2}$

Option 2: Expand the right-hand side and equate coefficients.

$1=Ax-2A+Bx$.

Coefficient of x is $0=A+B$.

Constant term is $1=-2A$.

Solve simultaneously: $A=-\frac{1}{2}$, $B=\frac{1}{2}$.

Therefore, $\frac{1}{x(x-2)}=\frac{-\frac{1}{2}}{x}+\frac{\frac{1}{2}}{x-2}$

$$\Rightarrow \int \frac{1}{x(x-2)}\,dx=\frac{1}{2}\int \frac{1}{x-2}-\frac{1}{x}\,dx$$

$$=\frac{1}{2}\left(\log_e|x-2|-\log_e|x|\right)$$

$$=\frac{1}{2}\log_e\left|\frac{x-2}{x}\right|$$

Mark allocation: 4 marks

- 1 mark for setting up the partial fractions
- 1 mark for the correct value of A
- 1 mark for the correct value of B
- 1 mark for the correct answer

Question 7c.

Worked solution

$$\text{Area} = \int_3^4 \frac{1}{|x^2-2x|}\,dx = \int_3^4 \frac{1}{x^2-2x}\,dx$$

$$= \left[\frac{1}{2}\log_e\left|\frac{x-2}{x}\right|\right]_3^4$$

$$= \frac{1}{2}\log_e\left(\frac{1}{2}\right) - \frac{1}{2}\log_e\left(\frac{1}{3}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{3}{2}\right)$$

Mark allocation: 1 mark

- 1 mark for the correct answer in the form required

Question 8

Worked solution

When $t = 0$, $v = 36 \Rightarrow t(36) = 0$

Need $t = ?$, $v = 9 \Rightarrow t(9) = ?$

The time taken for the parachutist to reduce her speed from 36 ms^{-1} to 9 ms^{-1} $= t(9) - t(36)$.

$$a = \frac{dv}{dt} = -0.01v^{\frac{3}{2}}$$

$$\Rightarrow \frac{dt}{dv} = -100v^{\frac{-3}{2}}$$

$$t(9) = \int_{36}^{9} \frac{dt}{dv}\,dv + t(36)$$

$$\Rightarrow t(9) = \int_{36}^{9} -100v^{\frac{-3}{2}}\,dv + 0$$

$$= \left[\frac{-100}{-\frac{1}{2}}v^{\frac{-1}{2}}\right]_{36}^{9} = \left[\frac{200}{\sqrt{v}}\right]_{36}^{9}$$

$$= \frac{200}{3} - \frac{200}{6} = \frac{200}{3} - \frac{100}{3}$$

$$= \frac{100}{3} = 33\frac{1}{3} \quad \therefore\ t(9) - t(36) = 33\frac{1}{3} - 0 = 33\frac{1}{3}$$

It takes the parachutist $33\frac{1}{3}$ seconds to reduce her speed from 36 ms^{-1} to 9 ms^{-1}.

Mark allocation: 3 marks

- 1 mark for the correct acceleration
- 1 mark for correctly expressing the time taken as an integrand in terms of the speed
- 1 mark for the correct answer

Question 9

Worked solution

The vectors are **linearly dependent** if

$\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}=\alpha\left(2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)+\beta\left(p\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}\right)$

where $\alpha,\beta\in R$.

Equate components:

$\underset{\sim}{i}$ components: $1=2\alpha+p\beta$. (1)

$\underset{\sim}{j}$ components: $2=-\alpha+2\beta$. (2)

$\underset{\sim}{k}$ components: $-1=3\alpha+2\beta$. (3)

Solve equations (2) and (3) simultaneously:

$(3)-(2)$ $\quad -3=4\alpha$

$\Rightarrow \alpha=-\dfrac{3}{4}$

Substituting $\alpha=-\dfrac{3}{4}$ into equation (2) gives $2=\dfrac{3}{4}+2\beta$

$\Rightarrow \beta=\dfrac{5}{8}$

Substituting $\alpha=-\dfrac{3}{4}$ and $\beta=\dfrac{5}{8}$ into equation (1) and solving for p gives $p=4$.

Mark allocation: 3 marks

- 1 mark for three equations equating the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components
- 1 mark for correctly evaluating α and β
- 1 mark for the correct answer for p

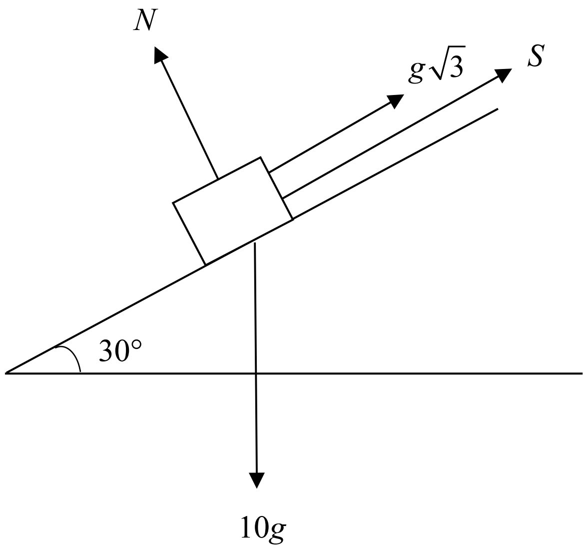
Tip

- *If three vectors are linearly dependent, it is simpler to express any one of the vectors as a linear combination of the other two vectors to solve for p. Alternatively, the vectors* $\underset{\sim}{a}=\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}$, $\underset{\sim}{b}=2\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}$ *and* $\underset{\sim}{c}=p\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$ *are* ***linearly dependent*** *if* $\gamma\underset{\sim}{a}+\alpha\underset{\sim}{b}+\beta\underset{\sim}{c}=\underset{\sim}{0}$, *where* $\gamma,\alpha,\beta\in R$ *and not all of* γ,α,β *are equal to zero. The solution process here is more involved than the one provided.*

Question 10a.

Worked solution

Moving down the plane:



Mark allocation: 1 mark

- 1 mark for all forces correctly labelled

Question 10b.

Worked solution

Resolve forces parallel to the plane (the direction of motion of the box down the plane is taken as the positive direction).

$F_{net} = ma = 10a$.

$F_{net} = 10g\sin(30^{\circ}) - g\sqrt{3} - S$

$= 5g - g\sqrt{3} - 3g$

$= g(2 - \sqrt{3})$.

Therefore, $10a = g(2 - \sqrt{3})$.

Answer: $a = \dfrac{g\left(2 - \sqrt{3}\right)}{10}$ ms^{-1}

Mark allocation: 2 marks

- 1 mark for a correct expression for the resultant force down the plane
- 1 mark for the correct acceleration

Question 10c.

Worked solution

For constant speed the acceleration = 0

$F_{net} = 5g - g\sqrt{3} - S = 0$

$\Rightarrow S = g(5 - \sqrt{3})$

Mark allocation: 1 mark

- 1 mark for the correct answer

END OF WORKED SOLUTIONS