THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
Fax 03 9836 5025
info@theheffernangroup.com.au
www.theheffernangroup.com.au

SPECIALIST MATHS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2016

Section A – Multiple-choice answers

1.	E	**6.**	E	**11.**	C	**16.**	C
2.	D	**7.**	B	**12.**	D	**17.**	A
3.	C	**8.**	C	**13.**	E	**18.**	A
4.	A	**9.**	D	**14.**	B	**19.**	A
5.	D	**10.**	B	**15.**	E	**20.**	D

Section A - Multiple-choice solutions

Question 1

$$y=\frac{x^2-3x-4}{x^2-x-12}$$
$$=\frac{(x-4)(x+1)}{(x-4)(x+3)}$$
$$=\frac{x+1}{x+3} \qquad \text{provided } x-4\neq 0, \quad \text{so } x\neq 4$$
$$=1-\frac{2}{x+3}$$

The graph has an asymptote at $x=-3$. It has a point of discontinuity at $x=4$.
It also has an asymptote at $y=1$ but this is not offered in the possible answers.
The answer is E.

Question 2

For $f(x)=\arccos(3-x)$ to be defined we require

$$-1\leq 3-x\leq 1$$
$$-4\leq -x\leq -2$$
$$2\leq x\leq 4$$

So $d_f=[2,4]$.
The answer is D.

Question 3

$$\frac{2x^2+1}{(x^2-1)(x^2+4)}=\frac{2x^2+1}{(x-1)(x+1)(x^2+4)}$$
$$=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+4}$$

The answer is C.

Question 4

$$\text{period} = \frac{2\pi}{n} = a \quad \text{(from the graph)}$$

$$\text{so} \quad n = \frac{2\pi}{a}$$

So the coefficient of x must be $\frac{2\pi}{a}$.

This eliminates options B, D and E.

The graph of $y = \sec\left(\frac{2\pi x}{a}\right)$ has been translated $\frac{a}{4}$ units to the right to obtain the graph shown.

For example, the point (0,1) lies on the graph of $y = \sec\left(\frac{2\pi x}{a}\right)$.

On the graph shown, the point where the function equals 1 is $\left(\frac{a}{4}, 1\right)$. So the rule for the graph shown must be $y = \sec\left(\frac{2\pi}{a}\left(x - \frac{a}{4}\right)\right)$.

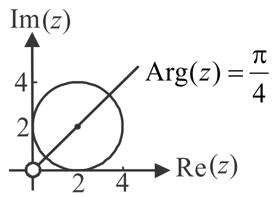
The answer is A.

Question 5

Do a quick sketch.

The graph of $|z - 2 - 2i| = 2$ or $|z - (2 + 2i)| = 2$, is the graph of a circle with centre at $z = 2 + 2i$ and radius of 2 units.

As an example, the graph of $\text{Arg}(z) = \frac{\pi}{4}$ is shown.



If the graph of $\text{Arg}(z) = \theta$ is to intersect with the circle then $0 \le \theta \le \frac{\pi}{2}$.

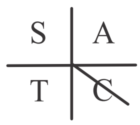
The answer is D.

Question 6

For $z_1 = 1 - \sqrt{3}i$,

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$$



$$= -\frac{\pi}{3}$$

$$z_1 = 2\text{cis}\left(-\frac{\pi}{3}\right)$$

$$\text{So } (z_1)^{12} = 2^{12}\text{cis}\left(12 \times -\frac{\pi}{3}\right) \quad \text{(De Moivre)}$$

$$= 2^{12}\text{cis}(-4\pi)$$

$$\text{So} \quad z_2 = 2^{12}\text{cis}(0)$$

So $\text{Arg}(z_2) = 0$ and option A is incorrect.

For option B, $z_2 = 2^{12}(\cos(0) + i\sin(0))$

$$= 2^{12}(1 + 0)$$

$$= 2^{12}$$

$$\text{Im}(z_2) + \text{Re}(z_2) = 0 + 2^{12} > 0$$

Option B is incorrect.

$|z_2| = 2^{12}$ so option C is incorrect.

$\text{Re}(z_2) = 2^{12}$ so option D is incorrect.

The answer is E.

Question 7

$$\int_1^2 (x-3)(2x+1)^5\,dx$$

$$=\int_3^5 \left(\frac{u-7}{2}\right)u^5 \times \frac{1}{2}\frac{du}{dx}dx$$

$$=\frac{1}{4}\int_3^5 (u-7)u^5\,du$$

$$=\frac{1}{4}\int_3^5 (u^6-7u^5)\,du$$

The answer is B.

Let $u=2x+1$ $\quad x=2,\ u=5$

so $\dfrac{du}{dx}=2$ $\quad x=1,\ u=3$

Also, since $u=2x+1$

$$u-1=2x$$

$$x=\frac{u-1}{2}$$

$$x-3=\frac{u-1}{2}-3$$

$$x-3=\frac{u-7}{2}$$

Question 8

For $x=0, \dfrac{dy}{dx}=0$, so eliminate options A and E.

The slopes $\left(\dfrac{dy}{dx}\right)$ are influenced by x-values only so eliminate option D (option E has already been eliminated).

We are left with $\dfrac{dy}{dx}=xe^x$ and $\dfrac{dy}{dx}=xe^{-x}$.

From the direction field, as $x\to\infty,\ \dfrac{dy}{dx}\to 0$.

For $\dfrac{dy}{dx}=xe^x$, as $x\to\infty, \dfrac{dy}{dx}\to\infty$.

For $\dfrac{dy}{dx}=xe^{-x}$, as $x\to\infty, \dfrac{dy}{dx}\to 0$.

The answer is C.

Question 9

$\dfrac{dy}{dx}=x^{\frac{3}{2}}+2x, \quad h=0.1, \qquad x_{n+1}=x_n+h, \quad y_{n+1}=y_n+hf(x_n)$ (formula sheet)

$x_0=1 \qquad y_0=2$

$x_1=1.1 \qquad y_1=2+0.1(1^{\frac{3}{2}}+2)$

$=2+0.1\times 3$

$=2.3$

$x_2=1.2 \qquad y_2=2.3+0.1(1.1^{\frac{3}{2}}+2.2)$

$=2.63536...$

$=2.635$ (to 3 decimal places)

The answer is D.

Question 10

At $t=2, t=3$ and $t=4$ the particle changes direction. Between $t=0$ and $t=2$, the distance travelled by the particle in the same direction, is greater than the distance travelled between $t=2$ and $t=3$ or between $t=3$ and $t=4$ or between $t=4$ and $t=6$. Also the distance travelled between $t=3$ and $t=4$ is less than the distance travelled between $t=2$ and $t=3$ so the particle is **not** furthest from its initial position in the $t=3$ to $t=4$ period.

The particle is furthest from its initial position during the time interval (1.5, 2.5).

The answer is B.

Question 11

The vector resolute $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by $\left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

$$\text{So } \left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \left((2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}) \bullet \frac{1}{\sqrt{2}}\left(\underset{\sim}{i}-\underset{\sim}{k}\right)\right)\frac{1}{\sqrt{2}}\left(\underset{\sim}{i}-\underset{\sim}{k}\right)$$
$$= \frac{1}{2}(2+0-1)(\underset{\sim}{i}-\underset{\sim}{k})$$
$$= \frac{1}{2}(\underset{\sim}{i}-\underset{\sim}{k})$$

The answer is C.

Question 12

Start by finding the Cartesian equation of the path.

$x = 2\cos(t) \qquad y = \sin(t)$

$\frac{x^2}{4} = \cos^2(t) \qquad y^2 = \sin^2(t)$

$$\frac{x^2}{4} + y^2 = \cos^2(t) + \sin^2(t)$$
$$\frac{x^2}{4} + y^2 = 1$$

The path follows an ellipse with a semi-major axis of 2 and a semi-minor axis of 1.

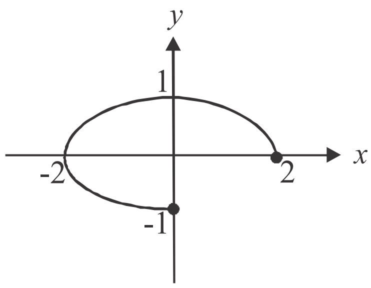
When $t = 0$, $\quad \underset{\sim}{r}(t) = 2\underset{\sim}{i} + 0\underset{\sim}{j}$.

The starting point is $(2,0)$.

When $t = \frac{3\pi}{2}$, $\quad \underset{\sim}{r}(t) = 0\underset{\sim}{i} - \underset{\sim}{j}$.

The finishing point is $(0,-1)$.

The graph is shown to the right.



The answer is D.

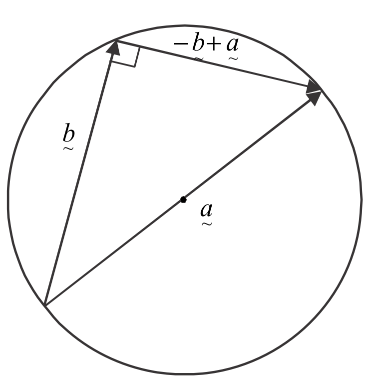
Question 13

The vector $-\underset{\sim}{b}+\underset{\sim}{a}$, together with vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ form a right angled triangle since $\underset{\sim}{a}$ spans the diameter of the circle.

So $(-\underset{\sim}{b}+\underset{\sim}{a}) \bullet \underset{\sim}{b} = 0$

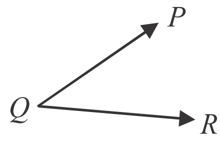
or $\quad (\underset{\sim}{a}-\underset{\sim}{b}) \bullet \underset{\sim}{b} = 0$

The answer is E.



Question 14

The angle PQR is the angle between $\overrightarrow{QP}$ and $\overrightarrow{QR}$.



$$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP} \qquad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$$
$$= -2\underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{i} + \underset{\sim}{k} \qquad = -2\underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$$
$$= \underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k} \qquad = \underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$$

$$\overrightarrow{QP} \bullet \overrightarrow{QR} = 1 + 6 - 2 = 5$$

Also, $\overrightarrow{QP} \bullet \overrightarrow{QR} = |\overrightarrow{QP}||\overrightarrow{QR}|\cos(\angle PQR)$

$$5 = \sqrt{1+4+4}\sqrt{1+9+1}\cos(\angle PQR)$$
$$5 = 3\sqrt{11}\cos(\angle PQR)$$
$$\angle PQR = \cos^{-1}\left(\frac{5}{3\sqrt{11}}\right)$$
$$= 59.83321...^\circ$$

The closest answer is 59.8°.
The answer is B.

Question 15

$$a = \sqrt{v+4} \qquad t = 0,\ v = -3$$

Since the initial conditions are in terms of the variables t and v, we use $a = \frac{dv}{dt}$ (from the formula sheet).

$$\frac{dv}{dt} = \sqrt{v+4}$$
$$\frac{dt}{dv} = \frac{1}{\sqrt{v+4}}$$
$$t = \int \frac{1}{\sqrt{v+4}}\, dv$$
$$t = 2\sqrt{v+4} + c$$

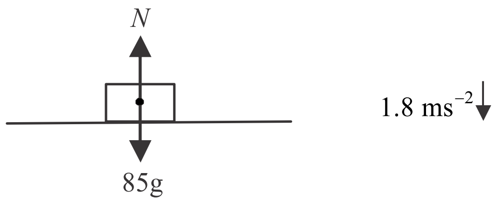
When $t = 0,\ v = -3$

$$0 = 2\sqrt{1} + c$$
$$c = -2$$
$$t = 2\sqrt{v+4} - 2$$
$$\frac{t+2}{2} = \sqrt{v+4}$$
$$v = \frac{(t+2)^2}{4} - 4$$
$$v = \frac{t^2 + 4t - 12}{4}$$

The answer is E.

Question 16

Let N be the reaction force.

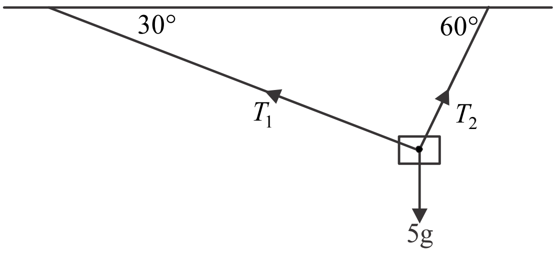


$$85g - N = 1.8 \times 85$$
$$N = 85g - 153$$
$$= 680$$

The answer is C.

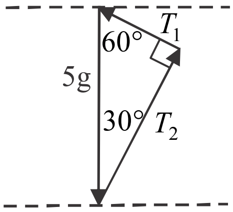
Question 17

Draw in the forces.



Method 1

Add the force vectors head to tail.



A right–angled triangle is formed.

$$\frac{T_1}{T_2} = \tan(30°)$$

$$T_1 = \frac{1}{\sqrt{3}} T_2$$

So $\quad T_2 : T_1$

becomes $\quad \sqrt{3}T_1 : T_1 \quad$ (using the result from above)

so $\quad \sqrt{3} : 1$

and $\quad 1 : \frac{1}{\sqrt{3}}$

The ratio of T_2 to T_1 is $1 : \frac{1}{\sqrt{3}}$.

The answer is A.

Method 2

Resolve the forces horizontally.

$$T_1 \cos(30°) = T_2 \cos(60°)$$

$$T_1 \frac{\sqrt{3}}{2} = \frac{T_2}{2}$$

$$T_2 = \sqrt{3}\, T_1$$

So $\quad T_2 : T_1$

becomes $\quad \sqrt{3}T_1 : T_1$

(using the result from above)

so $\quad \sqrt{3} : 1$

$\quad 1 : \frac{1}{\sqrt{3}}$

The ratio of T_2 to T_1 is $1 : \frac{1}{\sqrt{3}}$.

The answer is A.

Question 18

An approximate 95% confidence interval is $\left(\bar{x}-1.96\frac{s}{\sqrt{n}},\ \bar{x}+1.96\frac{s}{\sqrt{n}}\right)$ (formula sheet).

In this case, $\bar{x}=748.6$, $s=2.7$ and $n=40$.

So the interval is given by $\left(748.6-1.96\frac{2.7}{\sqrt{40}},\ 748.6+1.96\frac{2.7}{\sqrt{40}}\right)$

$$=(747.76...,749.43...)$$

$$=(747.8,\ 749.4) \quad \text{(correct to one decimal place)}$$

The answer is A.

Question 19

Because of the Central Limit Theorem, we can assume that the distribution of the sample mean $\bar{X}$ is approximately normal with

$$\begin{aligned}\mathrm{E}(\bar{X})&=\mu\\&=1554\end{aligned}$$

$$\begin{aligned}\text{and } \mathrm{sd}(\bar{X})&=\frac{\sigma}{\sqrt{n}}\\&=\frac{162}{\sqrt{50}}\\&=22.9102...\end{aligned}$$

$$\begin{aligned}\Pr(\bar{X}<1500)&=\Pr\left(Z<\frac{1500-1554}{22.9102...}\right)\\&=\Pr(Z<-2.35702...)\\&=0.009211...\end{aligned}$$

The closest answer is 0.0092.
The answer is A.

Question 20

If H_0 is rejected, then this **could** mean that

- a correct decision has been made (if H_0 is not true)
- a type 1 error has been made (if H_0 is true)
- an incorrect decision has been made (if it is a type 1 error)
- H_1 is true (if H_0 is not true).

It **could not** mean that a type 2 error has been made because that occurs when H_0 is not rejected.
The answer is D.

SECTION B

Question 1 (11 marks)

a. A point of inflection occurs if $f''(x)=0$ **and** the sign of the function f'' changes on opposite sides of the point (i.e. there is a change in concavity).

At $x=1$, $f''(1)=\dfrac{-8(1-1)}{\pi(1^2-2+2)}=0$ **(1 mark)**

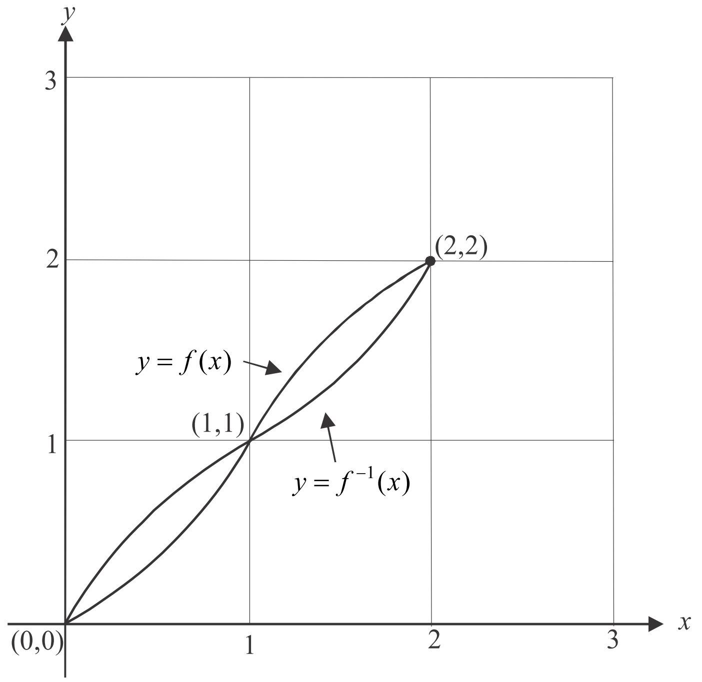
At $x=0$, (for example) $f''(0)=\dfrac{2}{\pi}$

At $x=2$, (for example) $f''(2)=-\dfrac{2}{\pi}$

So $f''(x)=0$ and the sign of f'' changes on opposite sides of the point of inflection.

(1 mark)

b.



(1 mark) correct shapes

(1 mark) – correct endpoints and point of inflection

c. $f(x)=1+\dfrac{4}{\pi}\arctan(x-1)$

Let $y=1+\dfrac{4}{\pi}\arctan(x-1)$

Swap x and y for inverse.

$x=1+\dfrac{4}{\pi}\arctan(y-1)$ **(1 mark)**

$\dfrac{\pi}{4}(x-1)=\arctan(y-1)$

$y-1=\tan\left(\dfrac{\pi}{4}(x-1)\right)$

$y=1+\tan\left(\dfrac{\pi}{4}(x-1)\right)$

$f^{-1}(x)=1+\tan\left(\dfrac{\pi}{4}(x-1)\right)$

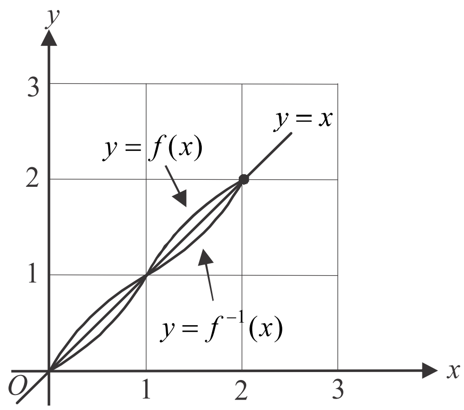
So $a=1,\ b=\pi$ and $c=4$. **(1 mark)**

d. Looking at the graph, we could draw the line $y = x$ through the points of intersection of *f* and f^{-1}.

Because of the symmetry of the graphs of *f* and f^{-1} around the line $y = x$, we can write

$$A = 4\int_0^1 (f^{-1}(x) - x)dx$$

So $g(x) = x$ **(1 mark)**



e. Method 1 – "hence"

$$A = 4\int_0^1 (f^{-1}(x) - x)dx$$ **(1 mark)**

$= 0.234915...$

$= 0.2$ (correct to one decimal place)

(1 mark)

Method 2 – "otherwise"

$$A = 2\int_0^1 (f^{-1}(x) - f(x))dx$$ **(1 mark)**

$= 0.234915...$

$= 0.2$ (correct to one decimal place) **(1 mark)**

f. $\text{volume} = \pi\int_0^1 x^2 dy$

$f(x) = 1 + \frac{4}{\pi}\arctan(x-1)$

so let $y = 1 + \frac{4}{\pi}\arctan(x-1)$.

From part **c**. when we were finding f^{-1}, we know that $f^{-1}(x) = 1 + \tan\left(\frac{\pi}{4}(x-1)\right)$

So rearranging the rule for *f*, without swapping the *x* and *y*, we get

$x = 1 + \tan\left(\frac{\pi}{4}(y-1)\right)$.

$$\text{volume} = \pi\int_0^1 \left(1 + \tan\left(\frac{\pi}{4}(y-1)\right)\right)^2 dy$$ **(1 mark)**

$= 1.2274...$

$= 1.2$ cubic units (correct to one decimal place)

(1 mark)

Question 2 (10 marks)

a. **i.**

$$u = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \qquad \theta = \tan^{-1}\left(\frac{1}{2} \div \frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{\frac{3}{4} + \frac{1}{4}} \qquad = \frac{\pi}{6}$$

$$= 1$$

| S | ~~A~~ |
| T | C |

So $u = \text{cis}\left(\frac{\pi}{6}\right)$

(1 mark) – correct modulus
(1 mark) – correct argument

ii. Since $u = \text{cis}\left(\frac{\pi}{6}\right)$ is one root of the equation $z^3 - i = 0$, then we are effectively finding the cubed roots of i so the 3 roots will be spaced $\frac{2\pi}{3}$ apart.

The second root is $\text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$

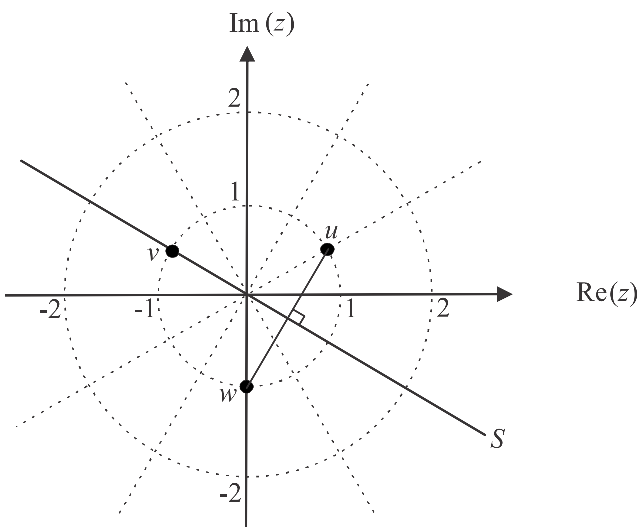
$$= \text{cis}\left(\frac{5\pi}{6}\right)$$

and the third root is $\text{cis}\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)$

$$= \text{cis}\left(\frac{9\pi}{6}\right)$$

$$= \text{cis}\left(\frac{3\pi}{2}\right)$$

So $u = \text{cis}\left(\frac{\pi}{6}\right)$, $v = \text{cis}\left(\frac{5\pi}{6}\right)$ and $w = \text{cis}\left(\frac{3\pi}{2}\right)$ because $\text{Im}(w) = -1$.



(1 mark) – correctly showing u and v
(1 mark) correctly showing w

iii. See graph above. **(1 mark)**

iv. From part **ii.**, $w = \text{cis}\left(\frac{3\pi}{2}\right) = -i$

$$|z-u| = |z-w|$$

$$\left|x+iy-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right| = |x+iy+i|$$

$$\sqrt{\left(x-\frac{\sqrt{3}}{2}\right)^2+\left(y-\frac{1}{2}\right)^2} = \sqrt{x^2+(y+1)^2}$$

$$x^2-\sqrt{3}x+\frac{3}{4}+y^2-y+\frac{1}{4} = x^2+y^2+2y+1$$

$$-\sqrt{3}x-3y=0$$

$$y = -\frac{x}{\sqrt{3}}$$

(1 mark)

From part **ii.**, $v = \text{cis}\left(\frac{5\pi}{6}\right)$

$$= \cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)$$

$$= -\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

Show that $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ satisfies $y = -\frac{x}{\sqrt{3}}$.

$LS = y$ $\qquad$ $RS = -\frac{x}{\sqrt{3}}$

$= \frac{1}{2}$ $\qquad$ $= +\frac{\sqrt{3}}{2} \div \sqrt{3}$

$= \frac{1}{2}$

$= LS$

Have shown.

(1 mark)

b. $z^2+6i\sin(\alpha)z-9=0$

We have a quadratic equation in z.

$$z = \frac{-6i\sin(\alpha)\pm\sqrt{-36\sin^2(\alpha)+36}}{2}$$

$$= \frac{-6i\sin(\alpha)\pm 6\sqrt{1-\sin^2(\alpha)}}{2}$$

$$= -3i\sin(\alpha)\pm 3\sqrt{\cos^2(\alpha)}$$

$$= -3i\sin(\alpha)\pm 3\cos(\alpha)$$

(1 mark)

$z = 3\cos(\alpha)-3i\sin(\alpha)$ $\quad$ or $\quad$ $z = -3\cos(\alpha)-3i\sin(\alpha)$

$= 3(\cos(-\alpha)+i\sin(-\alpha))$ $\quad$ $= -3(\cos(\alpha)+i\sin(\alpha))$

So $z_1 = 3\text{cis}(-\alpha)$ $\quad$ and $\quad$ $z_2 = -3\text{cis}(\alpha)$

or vice versa

(1 mark) $\qquad$ **(1 mark)**

Question 3 (11 marks)

a. At the point of projection, we have:

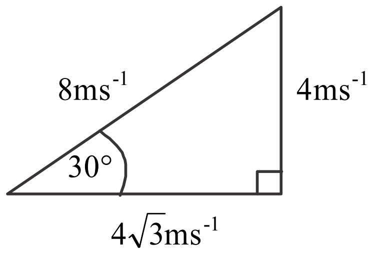
Horizontal component of velocity

$8\cos 30° = 4\sqrt{3}$

Vertical component of velocity

$8\sin 30° = 4$

So initial velocity $= 4\sqrt{3}\,\underset{\sim}{i} + 4\,\underset{\sim}{j}$. **(1 mark)**



b.

$$\ddot{\underset{\sim}{r}}(t) = -\frac{t}{50}\underset{\sim}{i} + \left(\frac{t}{20} - g\right)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}(t) = -\frac{t^2}{100}\underset{\sim}{i} + \left(\frac{t^2}{40} - gt\right)\underset{\sim}{j} + \underset{\sim}{c}_1$$

$\dot{\underset{\sim}{r}}(0) = 4\sqrt{3}\,\underset{\sim}{i} + 4\,\underset{\sim}{j}$ from part a.

So $\underset{\sim}{c}_1 = 4\sqrt{3}\,\underset{\sim}{i} + 4\,\underset{\sim}{j}$

$$\dot{\underset{\sim}{r}}(t) = \left(4\sqrt{3} - \frac{t^2}{100}\right)\underset{\sim}{i} + \left(\frac{t^2}{40} - gt + 4\right)\underset{\sim}{j}$$ **(1 mark)**

$$\underset{\sim}{r}(t) = \left(4\sqrt{3}t - \frac{t^3}{300}\right)\underset{\sim}{i} + \left(\frac{t^3}{120} - \frac{gt^2}{2} + 4t\right)\underset{\sim}{j} + \underset{\sim}{c}_2$$

$\underset{\sim}{r}(0) = 0\,\underset{\sim}{i} + 1.5\,\underset{\sim}{j}$ **(1 mark)**

So $\underset{\sim}{c}_2 = 0\,\underset{\sim}{i} + 1.5\,\underset{\sim}{j}$

$$\underset{\sim}{r}(t) = \left(4\sqrt{3}t - \frac{t^3}{300}\right)\underset{\sim}{i} + \left(\frac{t^3}{120} - \frac{gt^2}{2} + 4t + 1.5\right)\underset{\sim}{j}$$ as required **(1 mark)**

c. The ball hits the ground when $\frac{t^3}{120} - \frac{gt^2}{2} + 4t + 1.5 = 0$

Use CAS to solve this equation for t. **(1 mark)**

$t = 1.097341...$

Substitute into $4\sqrt{3}t - \frac{t^3}{300} = 7.59820...$

The ball travels 7.6 m horizontally before hitting the ground (correct to one decimal place). **(1 mark)**

d.

$$\text{distance} = \int_0^{1.097341...} \sqrt{\left(4\sqrt{3} - \frac{t^2}{100}\right)^2 + \left(\frac{t^2}{40} - gt + 4\right)^2}\, dt$$ (formula sheet – arc length)

(using the expression for $\dot{\underset{\sim}{r}}(t)$ in part b.)**(1 mark)** – correct integrand

$= 8.41767...$ **(1 mark)** – correct terminals

distance = 8.4 metres (correct to one decimal place) **(1 mark)**

e.

$$\text{speed} = \left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{\left(4\sqrt{3} - \frac{t^2}{100}\right)^2 + \left(\frac{t^2}{40} - gt + 4\right)^2}$$

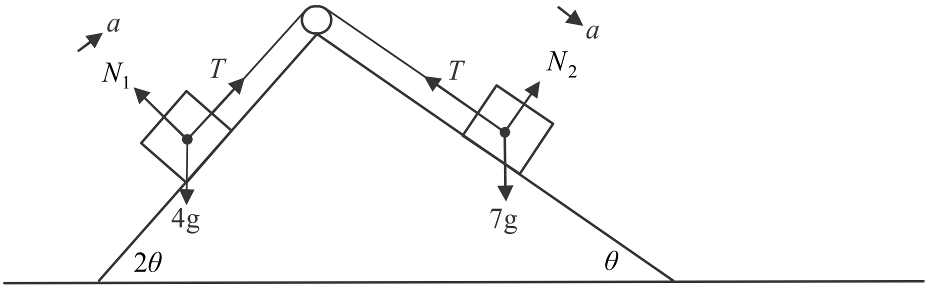
When the ball hits the ground, $t = 1.097341...$ from part **c**. **(1 mark)**

So speed $= 9.64589...$

$= 9.6\,\text{ms}^{-1}$ (correct to one decimal place) **(1 mark)**

Question 4 (9 marks)

a. Draw the forces.



Around the 4kg particle (parallel to its plane)

$$T - 4g\sin(2\theta) = 4a$$

(1 mark)

b. Around the 7kg particle (parallel to its plane)

$$7g\sin(\theta) - T = 7a$$ **(1 mark)**

So $T = 7g\sin(\theta) - 7a$

From part a., $T = 4a + 4g\sin(2\theta)$

So $7g\sin(\theta) - 7a = 4a + 4g\sin(2\theta)$ **(1 mark)**

$$-11a = 8g\sin(\theta)\cos(\theta) - 7g\sin(\theta)$$

$$a = \frac{g\sin(\theta)(8\cos(\theta) - 7)}{-11}$$

$$a = \frac{g\sin(\theta)}{11}(7 - 8\cos(\theta))$$

as required

(1 mark)

c. When the system is in equilibrium, $a = 0$.

So solve $0 = \frac{g\sin(\theta)}{11}(7 - 8\cos(\theta))$ for θ **(1 mark)**

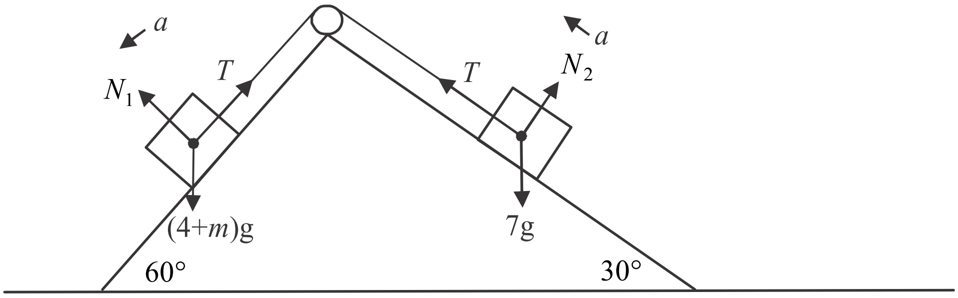
$\sin(\theta) = 0$ or $\cos(\theta) = \frac{7}{8}$

$\theta = 0°, 180°, ...$ $\quad \theta = 28.9550...°$

but $0° < \theta < 45°$ so $\theta = 29.0°$ (correct to one decimal place)

(1 mark)

d.



<u>Around the (4 + *m*) kg particle</u>
(parallel to its plane)

$$(4+m)g\sin(60^\circ)-T=(4+m)a$$

$$T=\frac{\sqrt{3}g}{2}(4+m)-(4+m)a$$

<u>Around the 7 kg particle</u>
(parallel to its plane)

$$T-7g\sin(30^\circ)=7a$$

$$T=\frac{7g}{2}+7a$$

Combining these, we have

$$\frac{\sqrt{3}g}{2}(4+m)-(4+m)a=\frac{7g}{2}+7a$$

$$\frac{\sqrt{3}g}{2}(4+m)-\frac{7g}{2}=7a+(4+m)a$$

$$\frac{4\sqrt{3}g+\sqrt{3}mg-7g}{2}=(11+m)a$$

(1 mark)

$$a=\frac{g\left(4\sqrt{3}+\sqrt{3}m-7\right)}{2(m+11)}$$

Since $a=\dfrac{g\left(11\sqrt{3}-14\right)}{50}$, which is given in the question,

we have $\dfrac{g\left(11\sqrt{3}-14\right)}{50}=\dfrac{g(4\sqrt{3}+\sqrt{3}m-7)}{2(m+11)}$ **(1 mark)**

<u>Method 1</u> – use CAS to solve

$$m=\frac{3}{2}$$

(1 mark)

<u>Method 2</u> – by hand

$$(11\sqrt{3}-14)(m+11)=25(4\sqrt{3}+\sqrt{3}m-7)$$

$$11\sqrt{3}m+121\sqrt{3}-14m-154=100\sqrt{3}+25\sqrt{3}m-175$$

$$11\sqrt{3}m-14m-25\sqrt{3}m=100\sqrt{3}-175-121\sqrt{3}+154$$

$$-14m-14\sqrt{3}m=-21\sqrt{3}-21$$

$$-14m(1+\sqrt{3})=-21(1+\sqrt{3})$$

$$m=\frac{-21(1+\sqrt{3})}{-14(1+\sqrt{3})}$$

$$m=\frac{3}{2}$$

(1 mark)

Question 5 (10 marks)

a. $\text{E}(4R+4C)=4\text{E}(R)+4\text{E}(C)$ (formula sheet)

$=4\times 2.2+4\times 4.9$

$=28.4$

So $\text{E}(W)=28.4\text{kg}$ **(1 mark)**

$\text{var}(4R+4C)=4^2\text{ var}(R)+4^2\text{ var}(C)$ (formula sheet)

$=16\times 0.18+16\times 0.31$

$=7.84$

standard deviation of $W=\sqrt{7.84}=2.8$ kg

(1 mark)

b. $H_0:\ \mu=8.6$ **(1 mark)**

$H_1:\ \mu<8.6$ **(1 mark)**

c. $\text{E}(\overline{X})=\mu=8.6,\quad \text{sd}(\overline{X})=\dfrac{0.54}{\sqrt{36}}=0.09$

p value $=\Pr(\overline{X}\le 8.4|\mu=8.6)$ **(1 mark)**

$=\Pr\left(Z\le\dfrac{8.4-8.6}{0.09}\right)$

$=\Pr(Z\le -2.\dot{2})$

$=0.0131341...$

$=0.013$ (correct to three decimal places)

(1 mark)

d. H_0 **should be** rejected at the 5% level of significance. This is because $p<0.05$, that is, $0.013<0.05$. **(1 mark)**

e. If H_0 is not to be rejected at the 1% level then the p value for the test must be greater than or equal to 0.01. **(1 mark)**

When $0.01=\Pr(Z\le c)$,

then $c=-2.326347...$ (inverse normal) **(1 mark)**

So the minimum value of the sample mean $\bar{x}$ occurs

when $\dfrac{\bar{x}-8.6}{0.09}=-2.326347...$

$\bar{x}=8.390628...$

So $\bar{x}=8.391$ kg (correct to three decimal places)

(1 mark)

Question 6 (9 marks)

a. **i.** concentration of salt in tank

$$= \frac{\text{amount of salt in tank at time } t}{\text{volume of brine in tank at time } t}$$

$$= \frac{x}{500 + 40t}$$

(1 mark)

Note that every minute there is an increase in the volume of the brine of $60 - 20 = 40$ litres.

ii.

$$\frac{dx}{dt} = \frac{dx_{\text{inflow}}}{dl} \cdot \frac{dl_{\text{inflow}}}{dt} - \frac{dx_{\text{outflow}}}{dl} \cdot \frac{dl_{\text{outflow}}}{dt}$$

$$= 0 \times 60 - \frac{x}{500 + 40t} \times 20$$

(1 mark)

$$= \frac{-x}{25 + 2t}$$

So $\frac{dx}{dt} + \frac{x}{25 + 2t} = 0$. **(1 mark)**

b. Since $\frac{dx}{dt} = \frac{-x}{25 + 2t}$,

$$\int -\frac{1}{x} dx = \int \frac{1}{25 + 2t} dt \qquad \text{(separation of variables)}$$

$$-\log_e(x) + c_1 = \frac{1}{2}\log_e(25 + 2t) + c_2 \qquad (\text{since } x > 0,\ t \geq 0 \text{ we don't need mod signs})$$

$$\log_e(x) - c_1 = -\frac{1}{2}\log_e(25 + 2t) - c_2$$

$$\log_e(x) = \log_e(25 + 2t)^{-\frac{1}{2}} + c \quad \text{where} \quad c = c_1 - c_2$$

(1 mark)

When $t = 0$, $x = 100$

$$\log_e(100) = \log_e\left(\frac{1}{\sqrt{25}}\right) + c$$

$$c = \log_e(500)$$

So $\log_e(x) = \log_e\left(\frac{1}{\sqrt{25 + 2t}}\right) + \log_e(500)$

$$= \log_e\left(\frac{500}{\sqrt{25 + 2t}}\right)$$

$$x = \frac{500}{\sqrt{25 + 2t}}$$

So $x(t) = \frac{500}{\sqrt{25 + 2t}}$ as required. **(1 mark)**

c. $x = \dfrac{500}{\sqrt{25+2t}}$

$= 500(25+2t)^{-\frac{1}{2}}$

$\dfrac{dx}{dt} = -250(25+2t)^{-\frac{3}{2}} \times 2$

$= \dfrac{-500}{\sqrt{(25+2t)^3}}$

So for $\dfrac{dx}{dt} + \dfrac{x}{25+2t} = 0$,

$LS = \dfrac{-500}{(25+2t)^{\frac{3}{2}}} + \dfrac{500}{(25+2t)^{\frac{1}{2}}} \times \dfrac{1}{25+2t}$

$= 0$

$= RS$

Have verified. **(1 mark)**

The initial conditions are $t = 0, x = 100$.

$x(t) = \dfrac{500}{\sqrt{25+2t}}$

When $t = 0$,

$x = \dfrac{500}{\sqrt{25}}$

$= 100$

Have verified. **(1 mark)**

d. Method 1

No salt was added to the tank, only water was added.

At $t = 0$, $x = 100\text{g}$

At $t = 20$, $x = 62.0173...\text{g}$ **(1 mark)**

The amount of salt that flowed out in the first 20 minutes was 37.9826… or 38.0 grams (correct to one decimal place). **(1 mark)**

Method 2

outflow $= \dfrac{20x}{500+40t}$ (from part a. ii.)

$= \dfrac{x}{25+2t}$

Amount of salt flowing out in first 20 minutes

$= \int_0^{20} \dfrac{x}{25+2t}\,dt$ **(1 mark)**

$= \int_0^{20} \dfrac{1}{25+2t} \times \dfrac{500}{\sqrt{25+2t}}\,dt$

$= 37.9826...$

$= 38.0$ grams (correct to one decimal place) **(1 mark)**