SPECIALIST MATHEMATICS

Written examination 2

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(*TSSM's *2015 trial exam updated for the current study design)

<u>SOLUTIONS</u>

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: A

Explanation:

$x = 5 - 6\sec(2t)\,, y = 3 + 5\tan(2t) \Rightarrow \dfrac{(x-5)^2}{36} - \dfrac{(y-3)^2}{25} = 1$

$\Rightarrow \text{the equation of the asymptotes: } \dfrac{x-5}{6} \pm \dfrac{y-3}{5} = 0$

$\Rightarrow y = \frac{5}{6}x - \frac{7}{6} \;\; or \;\; y = -\frac{5}{6}x + \frac{43}{6}$

Question 2

Answer: E

Explanation

*F*rom the graph we can see the semi-major axis $a = 7 - 2 = 5$
and the semi-minor axis $b = 3$
Therefore the equation of this ellipse is $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{9} = 1$

Question 3

Answer: B

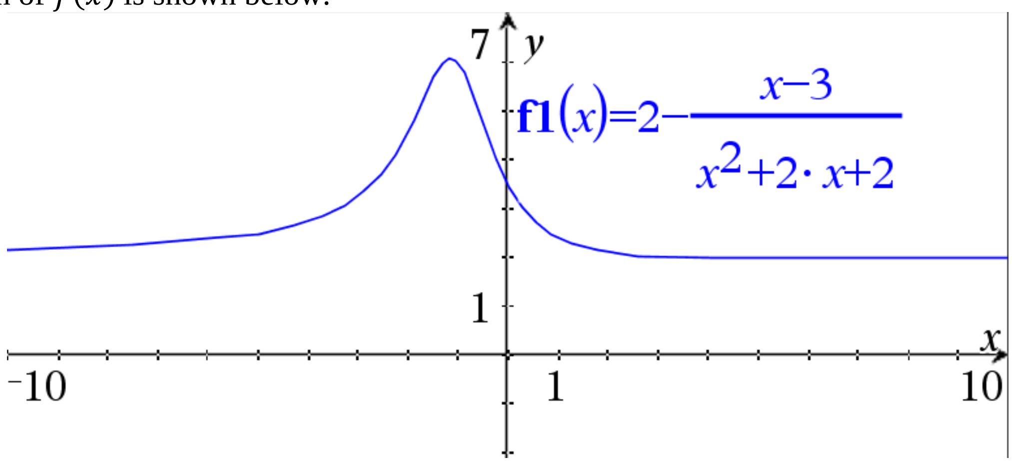
Explanation:

$$f(x) = \frac{2x^2 + 3x + 7}{x^2 + 2x + 2} = \frac{2(x^2 + 2x + 2) - x + 3}{x^2 + 2x + 2} = 2 - \frac{x - 3}{x^2 + 2x + 2}$$

$\Rightarrow y = 2$ is the horizontal asymptote

For $x^2 + 2x + 2 = (x + 1)^2 + 1$, the discriminant $\Delta = 2^2 - 4 \times 1 \times 2 = -4 < 0$. Hence $x^2 + 2x + 2 \neq 0$ and $(-1, 1)$ is the minimal point of $(x + 1)^2 + 1$.
Therefore there are no vertical asymptotes. $f(x)$ has a maximal point at $x = -1$.

The graph of $f(x)$ is shown below.



Question 4

Answer: E

Explanation:

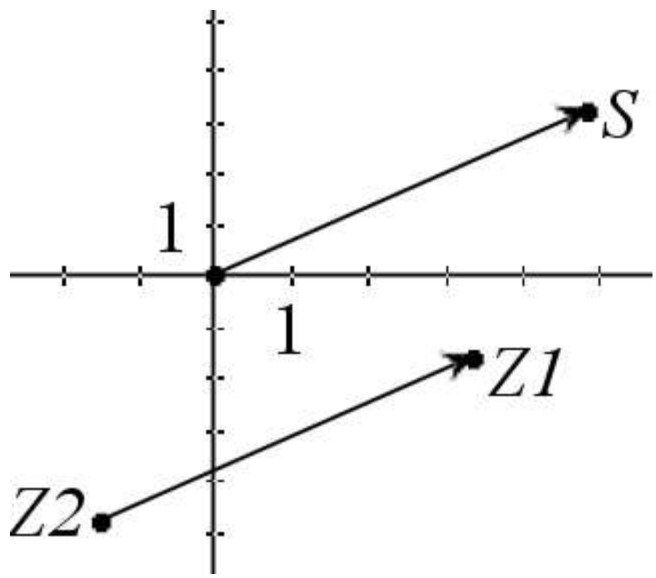
$\arctan(2 - 3x) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow 4\arctan(2 - 3x) + 5 \subset \left(-\frac{\pi}{2} \times 4 + 5, \quad \frac{\pi}{2} \times 4 + 5\right) = (5 - 2\pi, 5 + 2\pi)$

Question 5

Answer: D

Explanation:
Any complex number can be regarded as a position vector.
Hence $z = z_1 - z_2 = \overrightarrow{z_2 z_1} = \overrightarrow{OS}$.



Question 6

Answer: E

Explanation:
$|z - z_1| = |z - z_2|$ represents the locus of points with equal distance to z_1 and z_2, which is the perpendicular bisector between z_1 and z_2.

Question 7

Answer: B

Explanation:

$$z = -5\sqrt{3} - 5i = 10\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = 10cis(-\frac{5\pi}{6})$$

$$\Rightarrow z^{10} = 10^{10}\left(cos\left(-\frac{5\pi}{6} \times 10\right) + sin\left(-\frac{5\pi}{6} \times 10\right)\right) = 10^{10}cis(-\frac{\pi}{3})$$

Question 8

Answer: C

Explanation:
Geometric interpretation of multiplication of complex numbers: $z = 4cis(72°)z_1$ can be obtained from z_1 by a rotation of 72° around the origin in anti-clockwise, followed by a dilation of factor 4 from the origin.

Question 9

Answer: A

Explanation:
For any quadratic equation $az^2 + bz + c, a \neq 0$, the two solutions are $u, v = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Hence $u + v = \frac{-b}{a},\ u \times v = \frac{c}{a}$

$u + v = \frac{3i}{5} = \frac{6i}{10},\ u \times v = \frac{7}{10} \Rightarrow a = 10, b = -6i, c = 7.$

Question 10

Answer: B

Explanation:
$$\int_{\log_e(\frac{\pi}{6})}^{\log_e(\frac{\pi}{2})} \frac{e^x}{1+e^{2x}}\,dx = \int_{\log_e(\frac{\pi}{6})}^{\log_e(\frac{\pi}{2})} \frac{1}{1+(e^x)^2}\,de^x$$
Let $u = e^x$. Then $u = \frac{\pi}{2}$ $when\ x = \log_e(\frac{\pi}{2})$ and $u = \frac{\pi}{6}$ $when\ x = \log_e(\frac{\pi}{6})$
Therefore $\int_{\log_e(\frac{\pi}{6})}^{\log_e(\frac{\pi}{2})} \frac{e^x}{1+(e^x)^2}\,de^x = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{1+u^2}\,du$

Question 11

Answer: E

Explanation:
According the Fundamental Theorem of Calculus, $G(5) - G(1) = \int_1^5 g(x)\,dx = 8$.
Therefore $G(5) = 8 + G(1) = 8 + 2 = 10$

Question 12

Answer: E

Explanation:
A useful mathematical model for setting up differential equations of dynamic systems

$$\frac{dx}{dt} = R_{in} \times C_{in} - R_{out} \times C_{out}$$

where R_{in} and R_{out} are the flowing in and flowing out rate; C_{in} and C_{out} are the concentrations of the solutions which are flowing in and flowing out respectively.
Therefore

$$\frac{dx}{dt} = 2 \times 12 - 1.5 \times \frac{x}{85 + (2 - 1.5)t} = 24 - \frac{3x}{170 + t}$$

Question 13

Answer: C

Explanation:
Use a List and Spread Sheets in CAS to solve

	A x	B y	C dy
=			
1	2	6	1.1275
2	2.2	6.2255	1.14678
3	2.4	6.45486	1.16756

Question 14

Answer: A

Explanation:
Look at the slope field in CAS for each of the differential equations.

Question 15

Answer: D

Explanation:

The component coefficient matrix of the vectors is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & a & 1 \\ 5 & 5 & a \end{bmatrix}$.

Let $\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & a & 1 \\ 5 & 5 & a \end{bmatrix}\right) = 0$. Solve for a by CAS, shown below

$$\text{solve}\left(\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & a & 1 \\ 5 & 5 & a \end{bmatrix}\right)=0,a\right) \qquad a=0 \text{ or } a=3$$

Question 16

Answer: C

Explanation:

$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right|} = \frac{3+2}{\sqrt{13}\sqrt{13}} = \frac{5}{13} \Rightarrow \tan(\theta) = \frac{12}{5} \Rightarrow \tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} = -\frac{120}{119}$$

Question 17

Answer: A

Explanation:

$$\underset{\sim}{r}(t) = \int \left(\frac{e^t+e^{-t}}{2}\underset{\sim}{i} + \frac{e^t-e^{-t}}{2}\underset{\sim}{j} + 2t\underset{\sim}{k}\right)dt = \frac{e^t-e^{-t}}{2}\underset{\sim}{i} + \frac{e^t+e^{-t}}{2}\underset{\sim}{j} + t^2\underset{\sim}{k} + \underset{\sim}{c}$$

$$\underset{\sim}{r}(0) = 2\underset{\sim}{j} + 2\underset{\sim}{k} = \underset{\sim}{j} + \underset{\sim}{c} \Rightarrow \underset{\sim}{c} = \underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{r}(t) = \frac{e^t-e^{-t}}{2}\underset{\sim}{i} + \frac{e^t+e^{-t}+2}{2}\underset{\sim}{j} + (t^2+2)\underset{\sim}{k} = \frac{e^t-e^{-t}}{2}\underset{\sim}{i} + \frac{\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)^2}{2}\underset{\sim}{j} + (t^2+2)\underset{\sim}{k}$$

Question 18

Answer: E

Explanation:
The resultant force of any number of forces acting upon the body is the sum of all the forces.

Question 19

Answer: C

Explanation:
The velocity $\underset{\sim}{v}(t) = -24\cos(3t)\,\underset{\sim}{i} - 45\sin(3t)\underset{\sim}{j},\ t \geq 0$.
The speed=$\left|\underset{\sim}{v}(t)\right| = \sqrt{24^2\cos^2(3t) + 45^2\sin^2(3t)} = \sqrt{24^2 + 1449\sin^2(3t)}$
Therefore the minimum speed is 24 m/s

Question 20

Answer: B

Explanation:

$$v^2 - u^2 = 2as \Rightarrow v = \sqrt{u^2 + 2as} = \sqrt{(-2)^2 + 2 \times 9.8 \times 10} = 10\sqrt{2}$$

Question 21

Answer: B

Explanation:

$$a = v\frac{dv}{dx} = ve^{\frac{x^2}{100}} \Rightarrow \frac{dv}{dx} = e^{\frac{x^2}{100}} \Rightarrow v(5) - v(1) = \int_1^5 e^{\frac{x^2}{100}}\,dx$$

$$v(5) = v(1) + \int_1^5 e^{\frac{x^2}{100}}\,dx \approx 7.45$$

Question 22

Answer: D

Explanation:
Displacement=$\int_0^6 v(t)\,dt = \int_0^6 (t^3 - 4t^2 - 4t + 16)\,dt = 60$

SECTION 2: Extended Response questions

Question 1 (10 marks)

a.

$$x = 2 + 3\tan(t), \qquad y = 4\sec(t) \Rightarrow \tan(t) = \frac{x-2}{3}, \sec(t) = \frac{y}{4} \Rightarrow \boldsymbol{\frac{y^2}{16} - \frac{(x-2)^2}{9} = 1}$$

1 mark

b. The equation of the asymptotes are

$$\frac{y}{4} \pm \frac{x-2}{3} = 0 \Rightarrow y = \pm\frac{4}{3}(x-2)$$

1 mark

c. Differentiate both sides of $\frac{y^2}{16} - \frac{(x-2)^2}{9} = 1$, we get

$$\frac{2yy'}{16} - \frac{2(x-2)}{9} = 0 \Rightarrow y' = \frac{16(x-2)}{9y}$$

$\Rightarrow y' = \frac{16(6-2)}{9\times\frac{20}{3}} = \frac{16}{15}$ when $x = 6, y = \frac{20}{3}$.

The equation of the tangent at $(6, \frac{20}{3})$ is

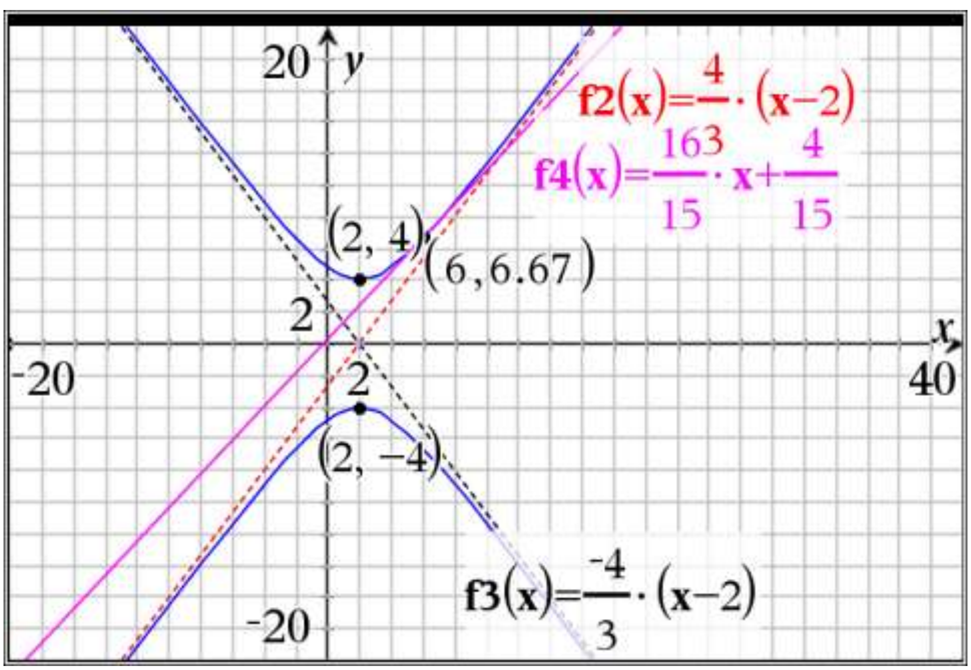
$y - \frac{20}{3} = \frac{16}{15}(x-6)$.

Or

$$y = \frac{16}{15}x + \frac{4}{15}$$

1 mark

d.



4 marks

e. $V = \pi \int_0^4 (16 + \frac{16(x-2)^2}{9}) dx$ 1 mark

f. By CAS
$V = \pi \int_0^4 (16 + \frac{16(x-2)^2}{9}) dx \approx 230.85$ 1 mark

Question 2

a. $z = 2\sqrt{2} + 2\sqrt{2}i = 4cis\left(\frac{\pi}{4}\right) \Rightarrow z^4 = 4^4 cis(\pi) = -256$

1 mark

b. $z^4 = 4^4 cis(\pi) \Rightarrow z = 4cis\left(\frac{\pi}{4}\right), 4cis\left(\frac{\pi}{4} + \frac{\pi}{2}\right), 4cis\left(\frac{\pi}{4} + \frac{2\pi}{2}\right), 4cis\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)$
Therefore the other solutions are

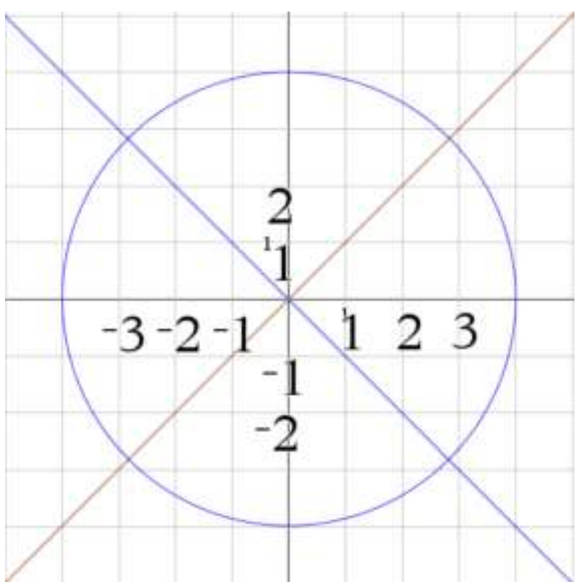
$$z = 4cis\left(\frac{3\pi}{4}\right), \qquad 4cis\left(-\frac{3\pi}{4}\right), \qquad 4cis\left(-\frac{\pi}{4}\right)$$

1 mark

c. The shape of the locus represented by $\left|z - 2\sqrt{2} - 2\sqrt{2}i\right| = \left|z + 2\sqrt{2} + 2\sqrt{2}i\right|$
is the perpendicular bisector of the line segment connecting
$A\left(2\sqrt{2}, 2\sqrt{2}\right)$ and $B(-2\sqrt{2}, -2\sqrt{2})$.
The gradient of AB is 1⇒ the gradient of the bisector is -1.
The midpoint of AB is $(0, 0)$
Therefore the equation of the bisector is $\boldsymbol{y = -x}$

d.

The shape of the locus represented by $|z| = |2\sqrt{2} + 2\sqrt{2}i|$ is a circle with centre (0, 0) and radius $r = \left|2\sqrt{2} + 2\sqrt{2}i\right| = 4$.



1 mark

e. **i.** From the CAS

$$\sqrt{\left(2\cdot\sqrt{2}-\frac{8\cdot\sqrt{42}}{7}\right)^2+\left(2\cdot\sqrt{2}\right)^2}+\sqrt{\left(2\cdot\sqrt{2}+\frac{8\cdot\sqrt{42}}{7}\right)^2+\left(2\cdot\sqrt{2}\right)^2} \quad 16$$

Hence LHS=RHS.

ii. From the equation given in ei, the semi-major axis of the ellipse is 8 and the centre is at (0,0). Let (0, b) be a vertex of the ellipse on the y-axis. Substitute $z = bi$ into the equation then

then we have $2\sqrt{(\frac{8\sqrt{42}}{7})^2 + b^2} = 16$. Solve for b by CAS

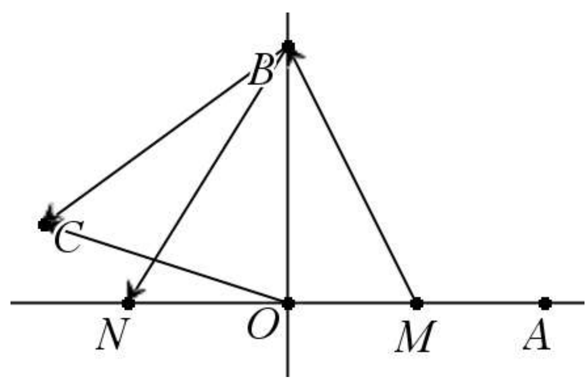
$$\text{solve}\left(\sqrt{b^2+\left(\frac{8\cdot\sqrt{42}}{7}\right)^2}=8,b\right)\Big|b>0 \quad b=\frac{8\cdot\sqrt{7}}{7}$$

Therefore $b = \frac{8\sqrt{7}}{7}$.

Hence the equation of the ellipse is

$$\frac{x^2}{64} + \frac{7y^2}{64} = 1$$

Question 3



a. $\left|\overrightarrow{MB}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$

1 mark

b. $\overrightarrow{ON} = -\left|\overrightarrow{MB}\right| \underset{\sim}{i} + \frac{1}{2}\underset{\sim}{i} = \frac{-\sqrt{5}+1}{2}\underset{\sim}{i}$

2 marks

c. $\left|\overrightarrow{BN}\right| = \sqrt{1^2 + \left(\frac{-\sqrt{5}+1}{2}\right)^2} = \frac{\sqrt{10-2\sqrt{5}}}{2}$

1 mark

d. Let θ be the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Then $\cos(\theta) = \frac{\overrightarrow{OB}\cdot\overrightarrow{OC}}{|\overrightarrow{OB}||\overrightarrow{OC}|} = \frac{\frac{\sqrt{5}-1}{4}}{1\times 1} = \frac{\sqrt{5}-1}{4}.$

While $\cos\left(2\times\frac{\pi}{5}\right) = 2\cos^2(\frac{\pi}{5}) - 1 = 2\times\frac{5+2\sqrt{5}+1}{16} - 1 = \frac{\sqrt{5}-1}{4}$

Therefore $\theta = \frac{2\pi}{5}.$

3 marks

e. $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\underset{\sim}{j} + \frac{-\sqrt{10+2\sqrt{5}}}{4}\underset{\sim}{i} + \frac{\sqrt{5}-1}{4}\underset{\sim}{j} = \frac{\sqrt{10+2\sqrt{5}}}{4}\underset{\sim}{i} + \frac{\sqrt{5}-5}{4}\underset{\sim}{j}$

$$\left|\overrightarrow{BC}\right| = \sqrt{\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 + \left(\frac{\sqrt{5}-5}{4}\right)^2} = \frac{\sqrt{10-2\sqrt{5}}}{2}$$

$\therefore \quad \left|\overrightarrow{BC}\right| = \left|\overrightarrow{BN}\right|$

3 marks

f. The side length can be found using the cosine rule:

$$l = \sqrt{1^2 + 1^2 - 2\times 1\times 1\times\cos(\frac{2\pi}{5})} = \frac{\sqrt{10-2\sqrt{5}}}{2}$$

2 marks

Question 4

a. $\frac{dh}{dr} = \frac{dh/dt}{dr/dt} = \frac{h}{1+h}\times\frac{r+1}{r+2}$ 2 marks

b. $v = \pi r^2 h$

$$\frac{dv}{dt} = 2\pi r\times\frac{dr}{dt}\times h + \pi r^2\times\frac{dh}{dt}$$
$$= 2\pi\times 5\times\frac{7}{6}\times 8 + \pi\times 5^2\times\frac{8}{9}$$
$$= \frac{1040\pi}{9}\ cm^3/s$$

3 marks

c. $\frac{dx}{dt} = 8\times\ 40 - 8\times\frac{x}{400} = \frac{16000-x}{50}$

2 marks

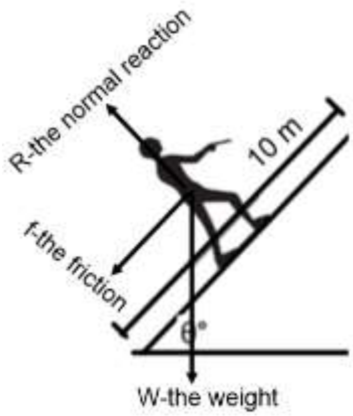
d. $t = \int_0^{200}\frac{50}{16000-x}dx \approx 0.63 minutes$

2 marks

Question 5

a. The forces acting on Nick and his board are shown in the diagram below.

2 marks



b. $mg\sin(\theta) + \mu mg\cos(\theta) = -ma$

$\Rightarrow a = -(g\sin(\theta) + \mu g\cos(\theta))$

$\Rightarrow a = -9.8(\sin(45°) + 0.01\cos(45°)) = -7\ m/s^2$

2 marks

c. $v^2 - u^2 = 2as \Rightarrow v = \sqrt{u^2 + 2as} = \sqrt{15^2 - 2 \times 7 \times 10} = 9.22\ m/s$

2 marks

d. Let t be the time that Nick comes back to the same level after leaving the first skate ramp. Then

$$s = ut - \frac{1}{2}gt^2 \Rightarrow 0 = 9.22\sin(45°)\,t - \frac{1}{2} \times 9.8 \times t^2$$

Solve for t, $t = 0$ (reject) or $t \approx 1.33$ s (accept).

In the 1.33 seconds the horizontal distance travelled by Nick is

$$d = 1.33 \times 9.22 \times \cos(45°) \approx 8.67\ m$$

That is more than 8m. Therefore Nick can land on the other side safely.

3 marks

e. Let t be the time after leaving the first ramp when Nick lands on the second ramp and let h be the vertical distance below the top of the ramps when Nick lands on the second ramp.
Then

$$x = 9.22\cos(45°) \cdot t$$

$$-h = 9.22\sin(45°)\,t - \frac{1}{2} \times 9.8 \times t^2$$

$$\frac{h}{x - 8} = \tan(30°)$$

Solve them simultaneously

$$\begin{cases} x = 4.25\ m \\ h = -2.17\ m \text{ (reject)} \\ t = 0.65\ s \end{cases} \quad \text{OR} \quad \begin{cases} x = 9.44\ m \\ h = 0.83\ m \text{ (accept)} \\ t = 1.45\ s \end{cases}$$

3 marks

f. Let θ be the minimal angle, in degrees, between the first skate ramp and the ground such that Nick can safely land on the second skate ramp t seconds after leaving the first ramp.
Then

$$9.22\cos(\theta)\,t = 8$$
$$9.22\sin(\theta)\,t - \frac{1}{2}\times 9.8\times t^2 = 0$$

Solve them simultaneously,

$$t = 1.04\ s, \qquad \theta = 33.6°$$

Therefore the required minimal angle is 33.6°.

3 marks