The Mathematical Association of Victoria

Trial Examination 2015

SPECIALIST MATHEMATICS

Written Examination 2

Student Name____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

Section	*Number of Questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
		Total	80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 23 pages with a detachable sheet of miscellaneous formulae in the centrefold.
- Answer sheet for multiple choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the test room.

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SECTION 1

Instructions for Section 1
Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

The asymptotes of the hyperbola with equation $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1$ intersect the line with equation $y = 3$ at

A. $(-1,3)$ and $(-3,3)$

B. $(-9,3)$ and $(11,3)$

C. $(3,-9)$ and $(3,11)$

D. $(-1.5,3)$ and $(3.5,3)$

E. $(1-2\sqrt{21},3)$ and $(1+2\sqrt{21},3)$

Question 2

An ellipse has the Cartesian equation $4x^2 + 8x + y^2 - 6y + 12 = 0$. This ellipse could be described by the parametric equations

A. $x = 2\cos(\theta)+1$ and $y = \sin(\theta)-3$

B. $x = 2\cos(\theta)-1$ and $y = \sin(\theta)+3$

C. $x = \frac{1}{2}\cos(\theta)+1$ and $y = \sin(\theta)+3$

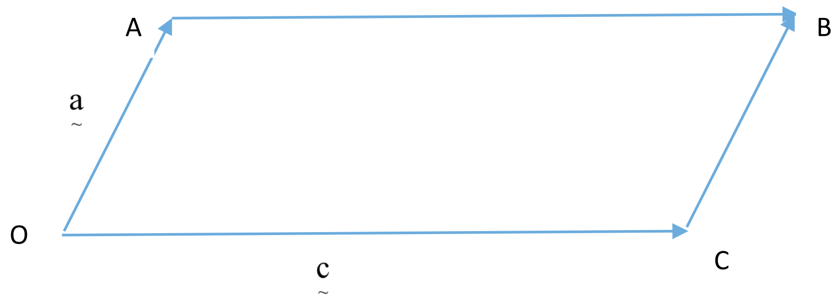
D. $x = \frac{1}{2}\cos(\theta)-1$ and $y = \sin(\theta)+3$

E. $x = \frac{\cos(\theta)-1}{2}$ and $y = \sin(\theta)+3$

Question 3

The diagram below shows a parallelogram OABC spanned by two vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$, where

$\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



It follows that $\overrightarrow{OB} + \overrightarrow{AC}$ is given by

A. $\underset{\sim}{a} + \underset{\sim}{c}$

B. $2\underset{\sim}{a}$

C. $2\underset{\sim}{c}$

D. $2\underset{\sim}{a} + 2\underset{\sim}{c}$

E. $2\underset{\sim}{c} - 2\underset{\sim}{a}$

Question 4

If $\sec(2\theta) = a$ then $\cos(\theta)$ could be given by

A. $\dfrac{1}{2a}$

B. $\dfrac{2}{a^2} - 1$

C. $\dfrac{1+a}{2a}$

D. $\sqrt{\dfrac{a+1}{2a}}$

E. $\sqrt{\dfrac{a-1}{2a}}$

Question 5

The domain and range of the function with the rule $f(x) = \arccos\left(2 - x^2\right)$ are respectively

A. $[-1,1]$ and $[0,\pi]$

B. $[1,3]$ and $[0,\pi]$

C. $[1,3]$ and $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

D. $\left[-\sqrt{3},-1\right]\cup\left[1,\sqrt{3}\right]$ and $[0,\pi]$

E. $\left[-\sqrt{3},-1\right]\cup\left[1,\sqrt{3}\right]$ and $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Question 6

If $z = \sqrt{3}\text{cis}\left(\frac{5\pi}{6}\right)$, then $\frac{1}{z^3}$ is equal to

A. $3\sqrt{3}i$

B. $-3\sqrt{3}i$

C. $-3i$

D. $\frac{\sqrt{3}}{9}i$

E. $-\frac{\sqrt{3}}{9}i$

Question 7

If $A = \{z : |z+1| = |z-i|\}$ and $B = \{z : |z| = 2\}$, then $A \cap B$ is

A. $\{2+2i, -2-2i\}$

B. $\{\sqrt{2}-\sqrt{2}i, -\sqrt{2}+\sqrt{2}i\}$

C. $\{2-2i, -2+2i\}$

D. $\{\sqrt{2}+\sqrt{2}i, -\sqrt{2}-\sqrt{2}i\}$

E. $\{1-i, -1+i\}$

Question 8

For any complex number z, the location on an Argand diagram of the complex number $w = -i^3 \bar{z}$ can be found by

A. Reflecting z about the real axis followed by reflecting about the imaginary axis.

B. Rotating z through $\frac{3\pi}{2}$ in a clockwise direction about the origin and then reflecting about the real axis.

C. Rotating z through $\frac{\pi}{2}$ in an anti-clockwise direction about the origin and then reflecting about the real axis.

D. Reflecting z about the real axis, followed by rotating anti-clockwise $\frac{\pi}{2}$ about the origin.

E. Reflecting z about the real axis, followed by rotating clockwise $\frac{\pi}{2}$ about the origin.

Question 9

The region bounded by the lines $x = 0$, $y = 2$ and the graph of $y = \sqrt[3]{x}$, where $x \geq 0$, is rotated about the y-axis to form a solid of revolution. The volume is

A. 4π

B. $\frac{128\pi}{7}$

C. $\frac{2^{\frac{7}{3}}\pi}{7}$

D. $\frac{3\pi 2^{\frac{7}{3}}}{7}$

E. $\frac{3\pi 2^{\frac{7}{9}}}{7}$

Question 10

A swimming pool with a capacity of 60,000L has 200kg salt dissolved in the water to form a salt solution. Fresh water is added at a rate of 10 litres per minute. The solution is mixed continuously whilst the mixture is drained off at 6 litres per minute. The differential equation for S, the number of kilograms of salt in the pool after t minutes is given by

A. $\frac{dS}{dt} = \frac{3S}{30,000 + 2t}$

B. $\frac{dS}{dt} = 200 - \frac{3S}{30,000 + 2t}$

C. $\frac{dS}{dt} = \frac{-3S}{30,000 - 2t}$

D. $\frac{dS}{dt} = \frac{200S}{30,000 + 2t}$

E. $\frac{dS}{dt} = \frac{-3S}{30,000 + 2t}$

Question 11

Consider the differential equation $\frac{dy}{dx} = \frac{1}{xy}$ with $y_0 = 1$ when $x_0 = 1$. Using Euler's method with a step size of 0.1, an approximation to y when $x = 1.2$ is closest to

A. 1.083
B. 1.182
C. 1.183
D. 1.190
E. 1.191

Question 12

If $\frac{dy}{dx} = \left(3x^4 - 2x\right)^{\frac{2}{3}}$ and $y = 10$ when $x = 5$, then the value of y when $x = 6$ is given by

A. $\int_5^6 \left(3x^4 - 2x\right)^{\frac{2}{3}} dx$

B. $\int_5^6 \left(3x^4 - 2x\right)^{\frac{2}{3}} dx + 10$

C. $\int_5^6 \left(\left(3x^4 - 2x\right)^{\frac{2}{3}} + 10\right) dx$

D. $\int_5^6 \left(3x^4 - 2x\right)^{\frac{2}{3}} dx - 10$

E. $\int_5^6 \left(\left(3x^4 - 2x\right)^{\frac{2}{3}} - 10\right) dx$

Question 13

Using an appropriate substitution, $\int_{-1}^{0} \frac{2x+1}{\sqrt{1-2x}} dx$ can be expressed as

A. $\int_{-1}^{0} \left(\frac{\sqrt{u}}{2} - \frac{1}{\sqrt{u}} \right) du$

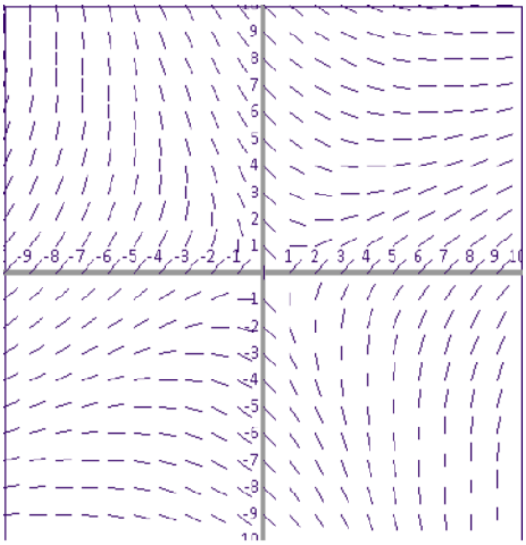
B. $\int_{-1}^{0} \left(\frac{1}{\sqrt{u}} - \frac{\sqrt{u}}{2} \right) du$

C. $\int_{1}^{3} \left(\frac{\sqrt{u}}{2} - \frac{1}{\sqrt{u}} \right) du$

D. $\int_{1}^{3} \left(\frac{1}{\sqrt{u}} - \frac{\sqrt{u}}{2} \right) du$

E. $2\int_{1}^{3} \left(\frac{2}{\sqrt{u}} - \sqrt{u} \right) du$

Question 14



The differential equation that best represents the above direction field is

A. $\dfrac{dy}{dx} = \dfrac{x+y}{y}$

B. $\dfrac{dy}{dx} = \dfrac{x-y}{x}$

C. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

D. $\dfrac{dy}{dx} = \dfrac{x-y}{y}$

E. $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$

Question 15

If θ is the angle between vectors $\underset{\sim}{a} = 2\sqrt{3}\,\underset{\sim}{i} + \sqrt{3}\,\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 2\,\underset{\sim}{j} + 2\,\underset{\sim}{k}$ then $\tan(2\theta)$ is

A. $-\dfrac{1}{6}$

B. $-\dfrac{1}{\sqrt{35}}$

C. $-\dfrac{2}{\sqrt{35}}$

D. $\dfrac{\sqrt{35}}{17}$

E. $-\dfrac{\sqrt{35}}{17}$

Question 16

The vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = m\underset{\sim}{i} + n\underset{\sim}{j}$ and $\underset{\sim}{c} = -\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ are linearly dependent. If $\underset{\sim}{b}$ is a unit vector, then the values of m and n could be

A. $m = \frac{-1}{\sqrt{5}}$ and $n = \frac{2}{\sqrt{5}}$

B. $m = \frac{1}{\sqrt{5}}$ and $n = \frac{2}{\sqrt{5}}$

C. $m = \frac{-1}{\sqrt{5}}$ and $n = \frac{-2}{\sqrt{5}}$

D. $m = \frac{1}{\sqrt{3}}$ and $n = \frac{-2}{\sqrt{3}}$

E. $m = 2$ and $n = 1$

Question 17

The velocity vector of a particle as it passes through a fixed point $(0,0,0)$ when t=0 is given by the equation $\underset{\sim}{v}(t) = 3\cos(2t)\underset{\sim}{i} - 2\sin(t)\underset{\sim}{j} + 4e^{-2t}\underset{\sim}{k}$. The position vector of the particle at any time t is given by

A. $\frac{3}{2}\sin(2t)\underset{\sim}{i} + 2\cos(t)\underset{\sim}{j} - 2e^{-2t}\underset{\sim}{k}$

B. $3\sin(2t)\underset{\sim}{i} + \left(2\cos(t) - 2\right)\underset{\sim}{j} + \left(2 - 2e^{-2t}\right)\underset{\sim}{k}$

C. $\frac{3}{2}\sin(2t)\underset{\sim}{i} + \left(2\cos(t) - 2\right)\underset{\sim}{j} + \left(2 - 2e^{-2t}\right)\underset{\sim}{k}$

D. $-6\sin(2t)\underset{\sim}{i} - 2\cos(t)\underset{\sim}{j} - 8e^{-2t}\underset{\sim}{k}$

E. $-6\sin(2t)\underset{\sim}{i} + \left(2 - 2\cos(t)\right)\underset{\sim}{j} + \left(8 - 8e^{-2t}\right)\underset{\sim}{k}$

Question 18

A body moves in a straight line such that its velocity v ms^{-1} is given by $v = (1-x)^2$, where x metres is its displacement from the origin. If the body starts at the origin, its acceleration after 2 seconds is

A. $-\frac{2}{27}$ ms^{-2}

B. $-\frac{1}{4}$ ms^{-2}

C. $-\frac{2}{3}$ ms^{-2}

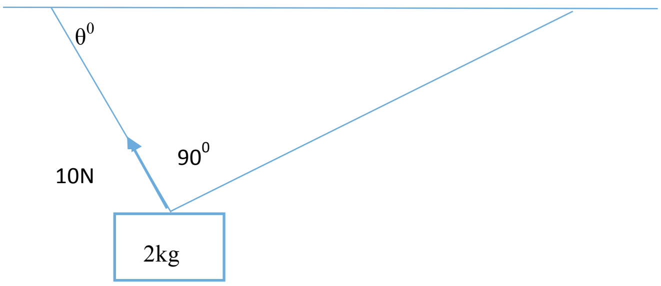
D. -1 ms^{-2}

E. 2 ms^{-2}

Question 19
It takes 4 seconds for a particle of mass 5kg, subject to uniform acceleration, to be brought to rest over a distance of 12 metres. If the motion was in a straight line, then the magnitude of the net force acting on the particle is

A. 1.5N
B. 6.0N
C. 7.5N
D. 9.0N
E. 15.0N

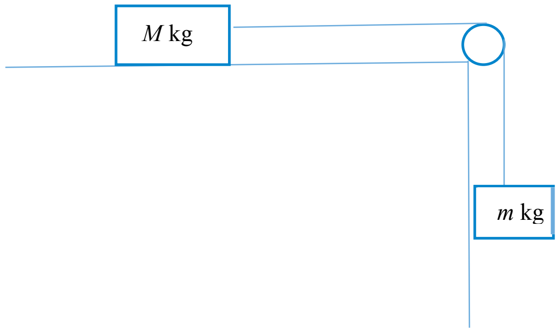
Question 20



The diagram above shows a string is attached two points on a horizontal ceiling. A 2kg mass is tied to the string so that a right-angle is formed. If the tension in one section of the string is 10N, then angle that θ that this string makes with the ceiling is closest to

A. 9^0
B. 11^0
C. 31^0
D. 59^0
E. 79^0

Question 21



The diagram above shows a light inextensible string that passes over a smooth pulley and connects a body of mass M kg sitting on a rough horizontal plane to a body of mass m kg, which is hanging vertically. The coefficient of friction between the body of mass M kg and the plane is $\frac{1}{3}$. When the system is released from rest, it accelerates at 2 ms^{-2}.

The ratio $\dfrac{M}{m}$ is equal to

A. $\dfrac{3(g-2)}{g+6}$

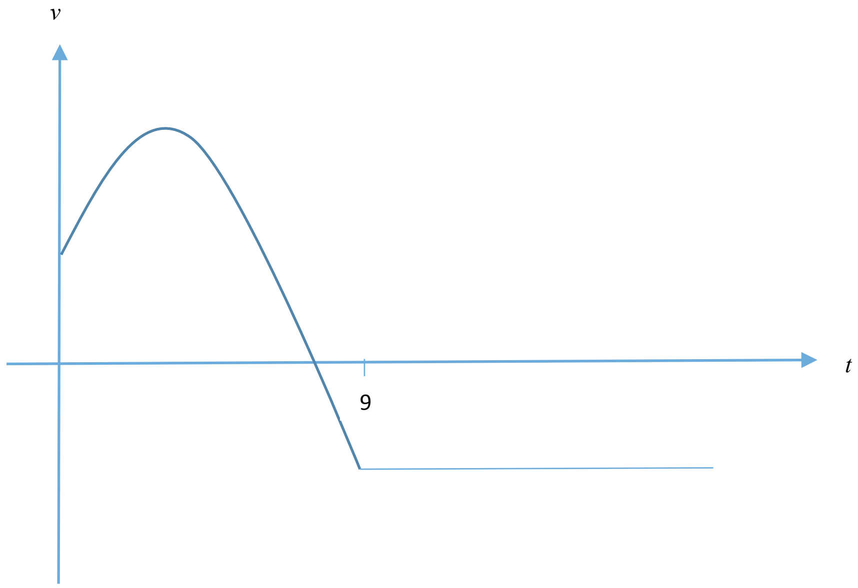
B. $\dfrac{g+6}{3(g-2)}$

C. $\dfrac{3(g-2)}{g+2}$

D. $\dfrac{3(g+2)}{g+6}$

E. $\dfrac{g+6}{3(g+2)}$

Question 22



The velocity-time graph for the motion of a body along a straight line is shown in the diagram above. The motion can be modelled by the equation $v=-\frac{1}{2}(t-3)^2+8$ for the first 9 seconds after which time, the body maintains a constant velocity. The body returns to its original position after

A. 7 seconds

B. $\frac{117}{20}$ seconds

C. $\frac{239}{60}$ seconds

D. $\frac{243}{20}$ seconds

E. $\frac{841}{60}$ seconds

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
Unless otherwise stated, an **exact** answer is requited to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams are **not** drawn to scale.
Take the **acceleration due to gravity** to have a magnitude g m/s^2, where $g = 9.8$.

Question 1 (13 marks)

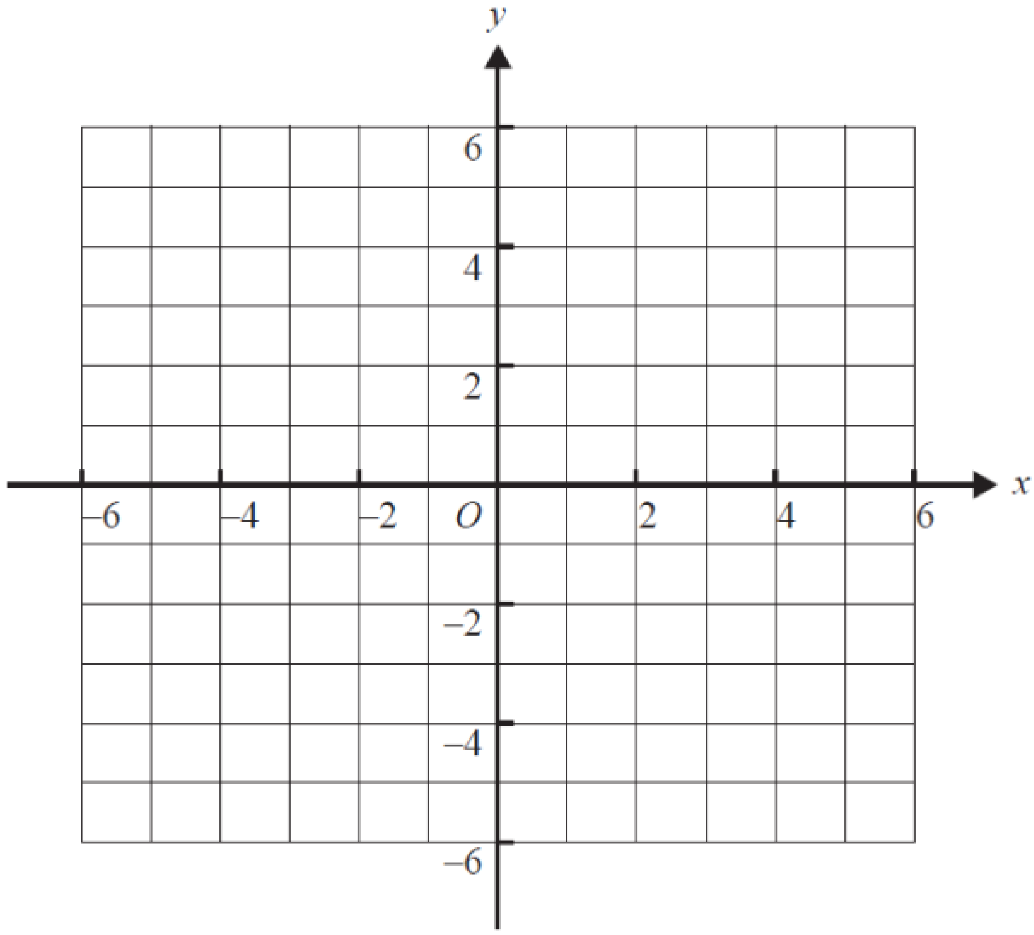
Consider the function f with the rule $f(x) = \dfrac{x^2}{(x+1)(x-3)}$ over its maximal domain.

a. State the equations of all asymptotes. 2 marks

b. Find the coordinates of all stationary points. 2 marks

c. Find the coordinates of the non-stationary point of inflection, correct to two decimal places. 2 marks

d. Sketch the graph of *f* for $x \in [-6, 6]$ on the axis below. Label all asymptotes with their equation and all axes intercepts, stationary points and endpoints with their coordinates. 3 marks



Consider the area, where $x \geq 0$, that is bounded by the graph of *f*, the *y*-axis and the line $y = -\frac{1}{4}$.

e. **i.** Set up a definite integral in terms of *x* that gives the area of this region. 3 marks

ii. Find this area, correct to two decimal places. 1 mark

Question 2 (12 marks)

Consider a triangle with vertices O, A and B, where O is the origin, $\overrightarrow{OA} = \sqrt{3}\,\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{OB} = \left(\sqrt{2}+\sqrt{3}\right)\underset{\sim}{i} + \left(1-\sqrt{2}\right)\underset{\sim}{j}$.

a. Show that triangle OAB is isosceles. 2 marks

b. Find $\cos(\theta)$, where θ is the angle between vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$. 1 mark

Consider the complex numbers $u = \sqrt{3} + i$ and $v = \sqrt{2} - \sqrt{2}i$.

c. **i.** Express u and v in polar form. 2 marks

ii. Find the angle, ϕ , between u and v. 1 mark

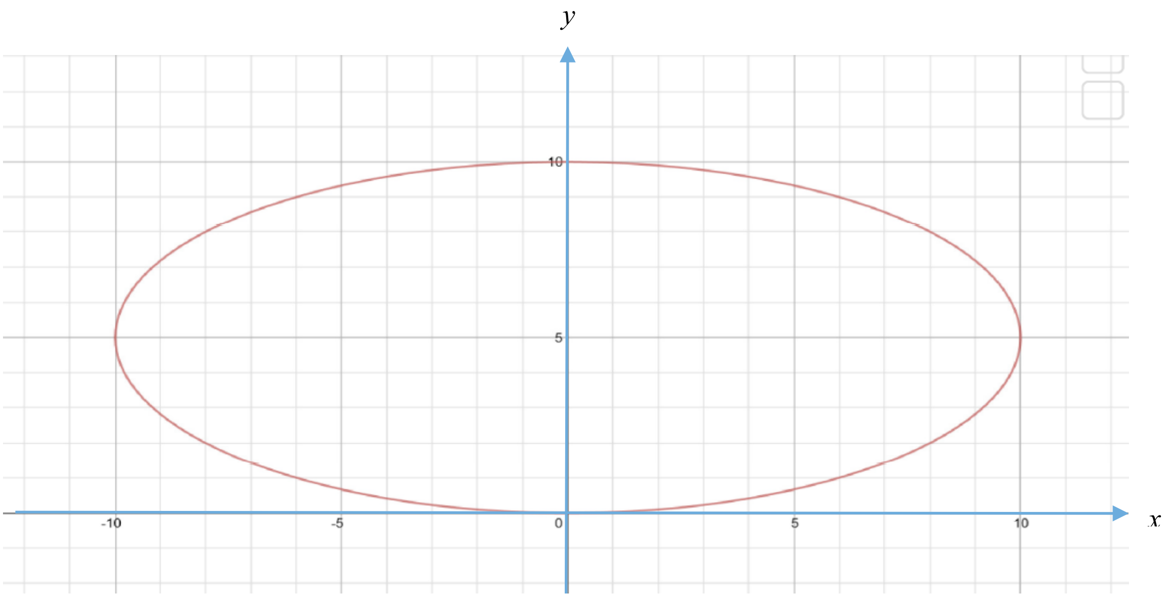
iii. Use a suitable compound angle theorem to find $\cos(\phi)$ and verify that $\phi = \theta$. 2 marks

d. Express $w = i\left(\frac{u}{v}\right)$ in polar form. 2 marks

e. Find the smallest positive value of n, $n \in Z$, such that w^n is a real number. 2 marks

Question 3 (11 marks)

The graph of an ellipse is shown below.



a. Write down the Cartesian equation of the ellipse. 1 mark

b. The ellipse is rotated about the y-axis to form a solid of revolution of volume V.

i. Write down a definite integral in terms of y that when evaluated will give V. 2 marks

ii. Find the value of V. 1 mark

This volume is a tank used to hold a chemical solution under purification treatment. All measurements are in metres. A chemical solution is pumped into the tank at a rate of 100 m^3 per hour

c. If the tank is initially empty, find the rate of change of the depth of the chemical solution when the volume is 1500 m^3. Give your answer correct to two decimal places. 3 marks

When the tank is filled with 1500 m^3 of the chemical solution, the treatment process commences. The initial concentration of the chemical in the solution is 50 kg/m^3. If C is the concentration of chemical, in kilograms per cubic metre, in the tank at any time t hours, the rate of change of C is given by

$$\frac{dC}{dt} = 0.1(10 - C).$$

d. Find an expression for C in terms of t. 2 marks

e. **i.** The concentration of the chemical in the solution reaches an equilibrium value. What is this value? 1 mark

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ii How long, to the nearest hour, does it take for the concentration to be within 5 kg/ m^3 of the equilibrium value? 1 mark

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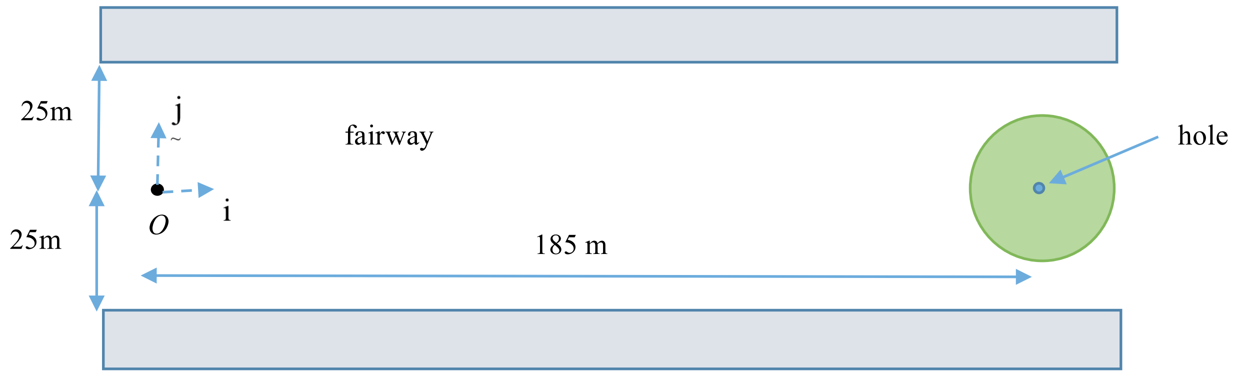
Question 4 (12 marks)



A fairway on a golf course is illustrated above. The fairway is 50 metres wide and 200 metres long. Let $\underset{\sim}{i}$ be the unit vector in the direction of the hole, $\underset{\sim}{j}$ the unit vector perpendicular to $\underset{\sim}{i}$, as shown, and $\underset{\sim}{k}$ the unit vector vertically up.

Cal hits the golf ball from the origin O, which is situated in the middle of the fairway as shown, towards the hole, which is 185 metres away, with a speed of 45ms^{-1} at an angle of 28^0 to the horizontal ($\underset{\sim}{i} - \underset{\sim}{j}$) plane but with an angle of 8^0 to the direction of the hole ($\underset{\sim}{i}$ direction). The ball, once in the air, is only subjected to gravitational acceleration.

a. Show that the velocity vector, at any time t seconds after being hit and correct to two decimal places, is given by

$$\underset{\sim}{v}(t) = 39.35\underset{\sim}{i} + 5.53\underset{\sim}{j} + (21.13 - 9.8t)\underset{\sim}{k}.$$

3 marks

b. Find the position vector $\underset{\sim}{r}(t)$ at any time t seconds. 1 mark

c. How long is the ball in the air? Give your answer correct to two decimal places. 1 mark

d. Did the ball land on the fairway? Justify your response. 2 marks

e. Find the distance from where the ball lands to the hole. Give your answer correct to one decimal place. 1 mark

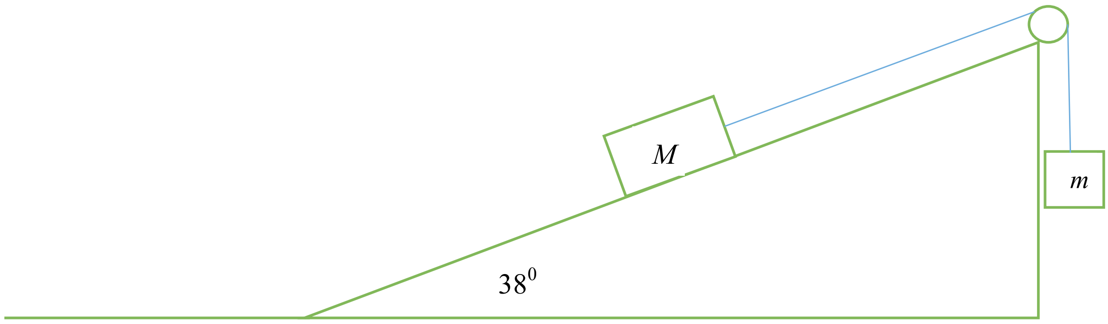
Cal hits the ball again, from where it landed. This time Cal's direction is perfect and the ball lands 5 metres from the hole. The horizontal acceleration of the ball, a ms^{-2}, as it rolls towards the hole is given by $a = 1 - \frac{1}{4}v^2$, where v is the velocity of the ball in ms^{-1}.

f. **i.** If the horizontal velocity of the ball as it lands is 7 ms^{-1}, find an expression for the displacement of the ball in terms of v. 3 marks

ii. Does the ball roll into the hole? Justify your answer. 1 mark

Question 5 (10 marks)

Two masses, M and m, are attached by a light inextensible string over a smooth pulley with mass m hanging freely. Mass M sits on a rough plane which is inclined at an angle of 38^0 to the horizontal. At the end of the inclined plane is a rough horizontal plane. The coefficient of friction between M and both the inclined plane and the horizontal plane is 0.15.



a. The system will remain in equilibrium when $p \le \dfrac{m}{M} \le q$.

Find the values of p and q, correct to two decimal places. 6 marks

The string breaks and mass M slides down the inclined plane.

b. **i.** If mass M was initially 2.5 metres up the inclined plane when the string breaks, find its speed as it reaches the end of the inclined plane. Give your answer correct to two decimal places.

2 marks

ii. Mass M comes to rest 2 seconds after the string breaks. How far did it travel along the rough horizontal plane? Give your answer correct to two decimal places. 2 marks

END OF QUESTION AND ANSWER BOOK