The Mathematical Association of Victoria

SPECIALIST MATHEMATICS

SOLUTIONS - Trial Exam 2015
Written Examination 2

SECTION 1: Multiple Choice Questions

Question			**Question**	
1	B		12	B
2	D		13	D
3	C		14	E
4	D		15	D
5	D		16	A
6	E		17	C
7	B		18	A
8	D		19	C
9	B		20	C
10	E		21	A
11	C		22	D

Question 1 B

$\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1$ has asymptotes at $(x-1) = \pm\frac{4}{2}(y+2)$.

Simplifying gives, asymptotes are located at $x = 2y+5$ and $x = -2y-3$.

$x = 2y+5$ intersects the line $y = 3$ at (11, 3).

$x = -2y-3$ intersects the line $y = 3$ at (-9, 3). B

Question 2 D

By completing the square $4x^2 + 8x + y^2 - 6y + 12 = 0$ can be expressed as $4(x+1)^2 + (y-3)^2 = 1$.

Since $\cos^2(\theta) + \sin^2(\theta) = 1$ we can let $\cos(\theta) = 2(x+1)$ and $\sin(\theta) = y-3$.

Rearranging gives, $x = \frac{1}{2}\cos(\theta) - 1$ and $y = \sin(\theta) + 3$. D

Question 3 C

$\overrightarrow{OB} + \overrightarrow{AC} = (\underset{\sim}{c} + \underset{\sim}{a}) + (-\underset{\sim}{a} + \underset{\sim}{c}) = 2\underset{\sim}{c}$ C

Question 4 D

$\sec(2\theta) = a$ so $\cos(2\theta) = \frac{1}{a}$

$\cos(2\theta) = 2\cos^2(\theta) - 1$

$\therefore \frac{1}{a} = 2\cos^2(\theta) - 1$

$\therefore \cos^2(\theta) = \frac{1+a}{2a}$

$\therefore \cos(\theta) = \pm\sqrt{\frac{a+1}{2a}}$

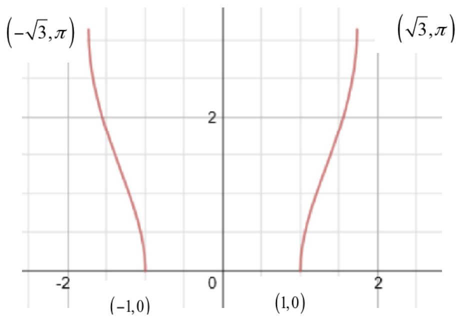
Only positive option given so D.

Question 5 D

$-1 \le 2 - x^2 \le 1$

Using CAS to solve $-\sqrt{3} \le x \le -1$ or $1 \le x \le \sqrt{3}$

So domain of $\arccos(2-x^2)$ is $[-\sqrt{3}, -1] \cup [1, \sqrt{3}]$.



From the graph we can see that the range is $[0, \pi]$. So D.

Question 6 E

$z = \sqrt{3}cis\left(\frac{5\pi}{6}\right)$ so $\frac{1}{z^3} = z^{-3} = \left(\sqrt{3}\right)^{-3} cis\left(\frac{5\pi}{6} \times -3\right)$

Simplifying $z^{-3} = \frac{1}{3\sqrt{3}} cis\left(-\frac{5\pi}{2}\right) = \frac{\sqrt{3}}{9} cis\left(-\frac{\pi}{2}\right)$.

Converting to Cartesian form gives $z^{-3} = -\frac{\sqrt{3}}{9}i$. E

Question 7 B

$\{z : |z+1| = |z-i|\}$ can be expressed by the Cartesian equation $y = -x$.

$\{z : |z| = 2\}$ can be expressed by the Cartesian equation $x^2 + y^2 = 4$.

Solving simultaneously gives $x = \pm\sqrt{2}, y = \mp\sqrt{2}$. So $A \cap B = \left\{\sqrt{2} - \sqrt{2}i, -\sqrt{2} + \sqrt{2}i\right\}$. B

Question 8 D

Let $z = x + yi$. So $w = -i^3 \bar{z} = -i \times i^2 (x - yi) = i(x - yi) = y + xi$.

Reflection in x-axis gives $x - yi$, followed by a rotation of $\frac{\pi}{2}$ anticlockwise about the origin (ie multiplication by i) gives $i(x - iy) = y + ix$. D

Question 9 B

$V = \pi \int x^2 dy$

Since $y = \sqrt[3]{x}$, $x^2 = y^6$, so $V = \pi \int_0^2 y^6 dy = \frac{128\pi}{7}$ B

Question 10 E

$\frac{dS}{dt} = imput - output$

$\frac{dS}{dt} = 0 \times 10 - \frac{S}{60000 + 4t} \times 6$

$\frac{dS}{dt} = -\frac{3S}{30000 + 2t}$ E

Question 11 C

$y_1 \approx y_0 + hf(x_0, y_0)$ where $f(x, y) = \frac{1}{xy}$

$= 1 + \frac{1}{1} \times 0.1$

$= 1.1$ when $x_1=1.1$

$y_2 \approx 1.1 + \frac{1}{1.1 \times 1.1} \times 0.1 = 1.18264463.. = 1.183$ (correct to 3 dec. places) C

Question 12 B

By rule: $y_2 = \int_{x_1}^{x_2} f(x)dx + y_1$ so $y_2 = \int_5^6 \left(3x^4 - 2x\right)^{\frac{2}{3}} dx + 10$ B

Question 13 D

$\int_{-1}^{0} \frac{2x+1}{\sqrt{1-2x}} dx$ let $u = 1 - 2x, \frac{du}{dx} = -2$

Rearranging $u = 1–2x$ to give $2x+1 = 2 – u$

Changing the terminals to u values,

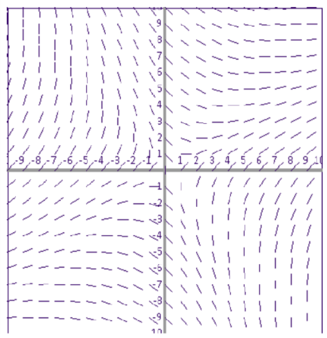
$x = -1 \rightarrow u = 3$

$x = 0 \rightarrow u = 1$

So $\int_{-1}^{0} \frac{2x+1}{\sqrt{1-2x}} dx$ can be expressed as $\int_3^1 \frac{2-u}{\sqrt{u}} \times -\frac{1}{2} du$

Simplifying we get $-\frac{1}{2}\int_3^1\left(\frac{2}{\sqrt{u}}-\sqrt{u}\right)du=\frac{1}{2}\int_1^3\left(\frac{2}{\sqrt{u}}-\sqrt{u}\right)du=\int_1^3\left(\frac{1}{\sqrt{u}}-\frac{\sqrt{u}}{2}\right)du$ D

Question 14 E



Consider $\frac{dy}{dx}=\frac{x-y}{x+y}$.

When $x=0, \frac{dy}{dx}=-1$ as indicated on direction field.

When $y=0, \frac{dy}{dx}=1$ as indicated on direction field.

When $x=5$ and $y=5, \frac{dy}{dx}=0$ as indicated on direction field.

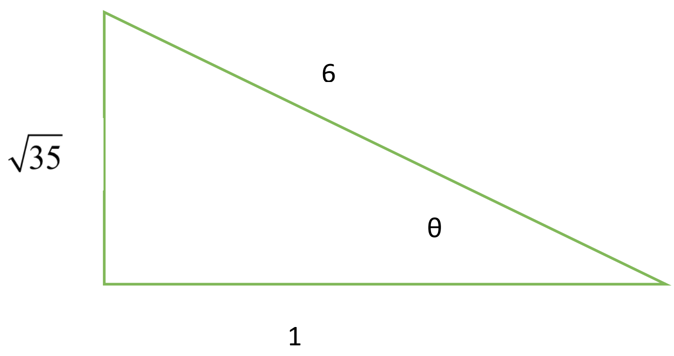
When $x=5$ and $y=-5, \frac{dy}{dx}$ is ∞, as indicated on direction field. E

Alternatively use CAS

Question 15 D

$$\cos(\theta)=\frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}=\frac{2\sqrt{3}-2\sqrt{3}-2}{\sqrt{16}\times\sqrt{9}}=\frac{-2}{12}=\frac{-1}{6}$$

So θ is in the second quadrant.



$$\tan(2\theta)=\frac{2\tan(\theta)}{1-\tan^2(\theta)}=\frac{2\times-\sqrt{35}}{1-35}=\frac{\sqrt{35}}{17}$$ D

Question 16 A

Since $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependant then there exists real numbers *p* and *q* such that

$p\underset{\sim}{a}+q\underset{\sim}{c}=\underset{\sim}{b}$.

So $p\begin{bmatrix}2\\1\\-1\end{bmatrix}+q\begin{bmatrix}-1\\-3\\1\end{bmatrix}=\begin{bmatrix}m\\n\\0\end{bmatrix}$

Therefore we obtain the simultaneous equations:

$2p-q=m$

$p-3q=n$

$-p+q=0$

The third equations gives *p*=*q*.

So $m=p, n=-2p$.

As $\underset{\sim}{b}$ is a unit vector: $m^2+n^2=1$.

Therefore: $p^2+(-2p)^2=1$ gives $p=\pm\frac{1}{\sqrt{5}}$.

So either $m=\frac{1}{\sqrt{5}}, n=-\frac{2}{\sqrt{5}}$ or $m=-\frac{1}{\sqrt{5}}, n=\frac{2}{\sqrt{5}}$.

$m=-\frac{1}{\sqrt{5}}, n=\frac{2}{\sqrt{5}}$ is option A.

Question 17 C

$\underset{\sim}{r}(t)=\int\underset{\sim}{v}(t)\,dt$

$=\frac{3}{2}\sin(2t)\underset{\sim}{i}+2\cos(t)\underset{\sim}{j}-2e^{-2t}\underset{\sim}{k}+\underset{\sim}{c}$

$\underset{\sim}{r}(0)=0\underset{\sim}{i}+0\underset{\sim}{j}+0\underset{\sim}{k}=0\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}+\underset{\sim}{c}$

So $\underset{\sim}{c}=-2\underset{\sim}{j}+2\underset{\sim}{k}$

Therefore $\underset{\sim}{r}(t)=\frac{3}{2}\sin(2t)\underset{\sim}{i}+(2\cos(t)-2)\underset{\sim}{j}+(2-2e^{-2t})\underset{\sim}{k}$ C

Question 18 A

$a=v\frac{dv}{dx}=(1-x)^2\times-2(1-x)=-2(1-x)^3$

Since $v=(1-x)^2$ then $\frac{dx}{dt}=(1-x)^2$. So $t=\int\frac{1}{(1-x)^2}dx$

$\therefore t=\frac{1}{1-x}+c$

Since *x*=0 when *t*=0, gives c=-1, so $t=\frac{1}{1-x}-1$.

Solving for when *t*=2, gives $x=\frac{2}{3}$.

Substituting into the acceleration equation gives $-\frac{2}{27}$ ms^{-2}. A

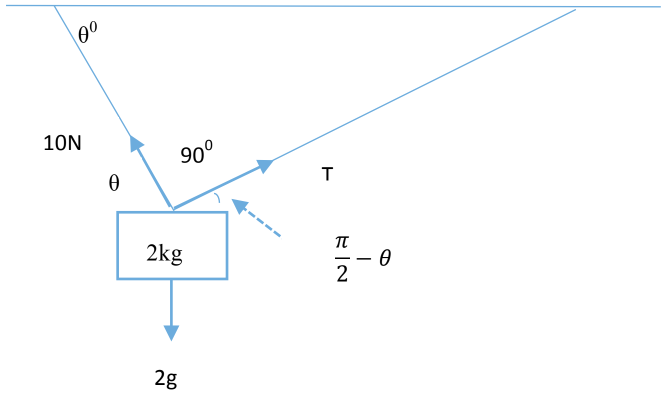
Question 19 C

Substituting $t=4, v=0, s=12$ into $s=\frac{1}{2}(u+v)t$ gives $u=6$.

Substituting into $v=u+at$ gives $a=-1.5$.

So net force that brings the particle to rest is $1.5\times 5=7.5N$ C

Question 20 C



Let the angle made between the string and the ceiling be θ and the tension in the other section of the string be *T*.

Resolving the forces into components:

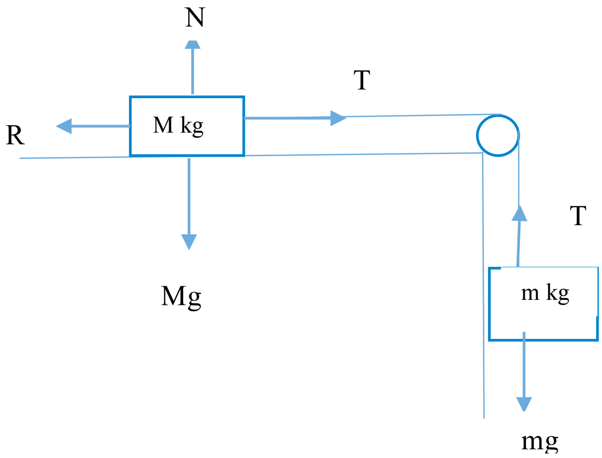
Horizontal components: $10\cos(\theta)=T\sin(\theta)$

Vertical components: $2g=T\cos(\theta)+10\sin(\theta)$

Solving simultaneously gives $\theta=30.67742$

So closest to 31^0. C

Question 21 A



Mass m kg: $mg-T=ma$

So $T=m(g-a)$

Mass M kg: $T-R=Ma$

But $R = \mu N = \frac{1}{3}Mg$

So $T = M\left(a + \frac{1}{3}g\right)$

Equating the tension equations gives $m\left(g - a\right) = M\left(a + \frac{1}{3}g\right)$

Therefore $\frac{M}{m} = \frac{g-a}{a+\frac{1}{3}g}$

Substituting $a = 2$ gives $\frac{M}{m} = \frac{g-2}{2+\frac{1}{3}g} = \frac{3\left(g-2\right)}{6+g} = \frac{3\left(g-2\right)}{g+6}$ A

Question 22 D

When $t = 9, v = -10.$

$$\int_0^9 \left(-\tfrac{1}{2}\left(t-3\right)^2 + 8\right)dt = \left(t-9\right)\times 10$$

Solving gives $t = \frac{243}{20}$ seconds. D

SECTION 2

Question 1

a. $x = -1, x = 3$ ½ mark each [round down] A1

$y = 1$ A1

b. Stationary points occur when $f'\left(x\right) = 0$. A1

$$f'\left(x\right) = \frac{-2x\left(x+3\right)}{\left(x-3\right)^2\left(x+1\right)^2} = 0$$

So $x = 0, x = -3$

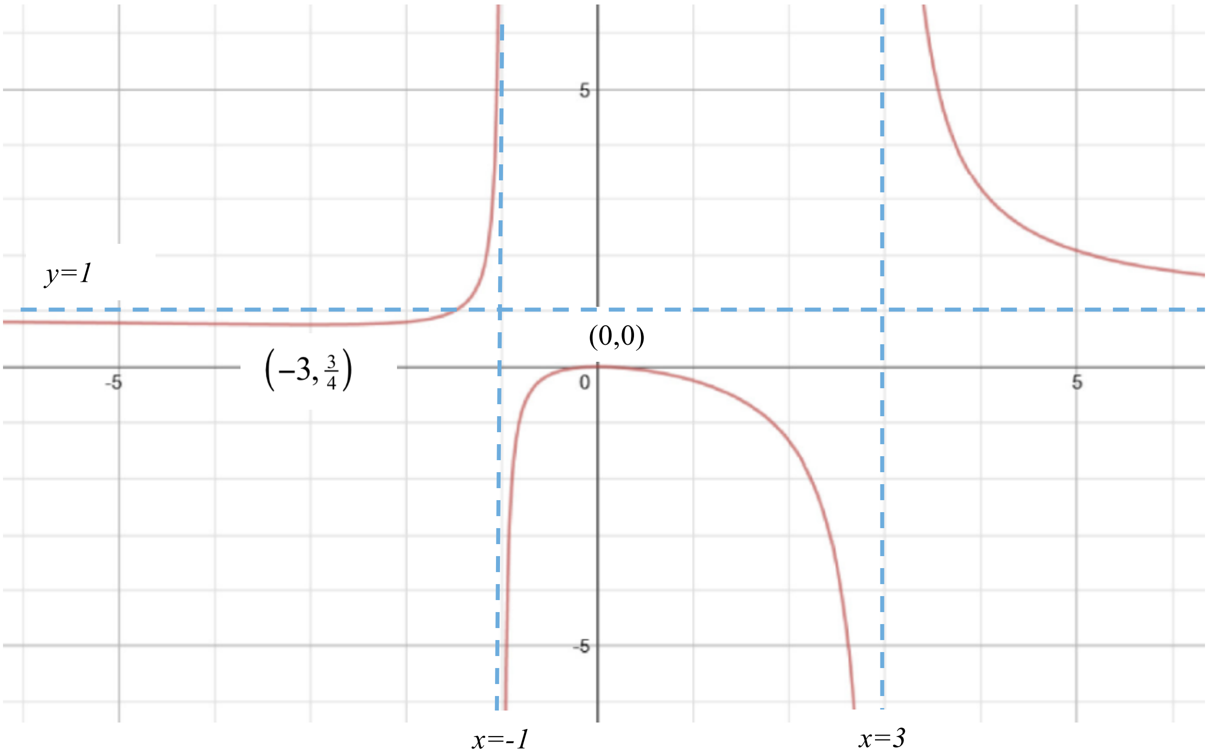
$\left(0,0\right)$ and $\left(-3, \frac{3}{4}\right)$ A1

c. $f''\left(x\right) = 0$ or $\frac{2\left(2x^3 + 9x^2 + 9\right)}{\left(x-3\right)^3\left(x+1\right)^3} = 0$ M1

$x = -4.703416..... \approx -4.70$ (correct to 2 dec. places)

$\left(-4.70, 0.78\right)$ A1

d.



½ mark each asymptote indicated, labelled & graph approaching such
½ mark each, coordinates of turning points indicated & in correct position
½ mark location of sections of curve, especially curve crossing asymptote & approaching $y=1$ from below. Round down.

e. **i.** Solving $\dfrac{x^2}{(x-1)(x+3)} = -\dfrac{1}{4}$ to give $x = 1, -\dfrac{3}{5}$.

But since $x \geq 0$ then $x=1$. A1

Area $= \dfrac{1}{4} \times 1 - \left| \displaystyle\int_0^1 \frac{x^2}{(x+1)(x-3)} dx \right|$ M1 (allocated for the absolute value)

Simplifying we get: Area $= \dfrac{1}{4} + \displaystyle\int_0^1 \frac{x^2}{(x+1)(x-3)} dx$ A1

Alternatively,

$\left| \displaystyle\int_0^1 \left(-\frac{1}{4} - f(x) \right) dx \right|$. A1 terminals, A1 absolute value, A1 integrand expression

ii. Area = 0.1644167…. ie Area = 0.16 square units (correct to two decimal places) A1

Question 2

a. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \sqrt{2}\,\underset{\sim}{i} - \sqrt{2}\,\underset{\sim}{j}$ A1

$\left|\overrightarrow{AB}\right| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = 2$

$\left|\overrightarrow{OA}\right| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$ M1

$\left|\overrightarrow{OB}\right| = \sqrt{\left(\sqrt{2}+\sqrt{3}\right)^2 + \left(1-\sqrt{2}\right)^2} = \sqrt{2\sqrt{6} - 2\sqrt{2} + 8}$

So OAB is an isosceles triangle.

b. $\cos(\theta) = \dfrac{\overrightarrow{OA}.\overrightarrow{AB}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{AB}\right|} = \dfrac{\left(\sqrt{3}\,\underset{\sim}{i} + \underset{\sim}{j}\right).\left(\sqrt{2}\,\underset{\sim}{i} - \sqrt{2}\,\underset{\sim}{j}\right)}{2\times 2} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ A1

c. i. $u = \sqrt{3} + i = 2\text{cis}\left(\dfrac{\pi}{6}\right)$ A1

$v = \sqrt{2} - \sqrt{2}i = 2cis\left(-\dfrac{\pi}{4}\right)$ A1

ii. $\therefore \phi = \dfrac{\pi}{6} + \dfrac{\pi}{4} = \dfrac{5\pi}{12}$ A1

iii. $\cos(\phi) = \cos\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{4}\right) - \sin\left(\dfrac{\pi}{6}\right)\sin\left(\dfrac{\pi}{4}\right)$

M1 (use of compound angle theorem)

$= \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{2}}{2} - \dfrac{1}{2} \times \dfrac{\sqrt{2}}{2}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

From **b.** $\cos(\theta) = \dfrac{\sqrt{6}-\sqrt{2}}{4} \therefore \theta = \pm\dfrac{5\pi}{12}$

but θ is the angle between vectors so $\theta \geq 0, \therefore \theta = \dfrac{5\pi}{12}$ A1

$\phi = \dfrac{\pi}{6} + \dfrac{\pi}{4} = \dfrac{5\pi}{12}$. So $\phi = \theta$.

d. Let $w = i\left(\frac{u}{v}\right)$. Express w in polar form.

In polar form $i = 0 + i = cis\left(\frac{\pi}{2}\right)$ A1

$$w = i\left(\frac{u}{v}\right) = cis\left(\frac{\pi}{2}\right) \times \frac{2cis\left(\frac{\pi}{6}\right)}{2cis\left(-\frac{\pi}{4}\right)} = cis\left(\frac{\pi}{2} + \frac{\pi}{6} - \left(-\frac{\pi}{4}\right)\right) = cis\left(\frac{11\pi}{12}\right)$$ A1

e. $w = cis\left(\frac{11\pi}{12}\right)$ so $w^n = cis\left(\frac{11\pi n}{12}\right) = \cos\left(\frac{11\pi n}{12}\right) + i\sin\left(\frac{11\pi n}{12}\right)$

For w^n to be a real number $\sin\left(\frac{11\pi n}{12}\right) = 0$ H1

$\therefore \frac{11\pi n}{12} = \pi k, k \in Z^+$

$n = \frac{12k}{11}$

Therefore $n = 12$. A1

Question 3

a. $\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$ A1

b. **i** $V = \pi\int x^2 dy$

Rearranging the ellipse equation to make x^2 the subject gives:

$$x^2 = 100\left(1 - \frac{(y-5)^2}{25}\right) = 4\left(25 - (y-5)^2\right)$$ A1

$$\therefore V = 4\pi\int_0^{10}\left(25 - (y-5)^2\right)dy$$ A1

ii $V = \frac{2000\pi}{3}$ cubic units A1

c. Solving $1500 = 4\pi\int_0^{d}\left(25 - (y-5)^2\right)dy$ gives $d = 6.4849763$

$\frac{dy}{dt} = \frac{dy}{dV} \times \frac{dV}{dt}$ where $\frac{dV}{dt} = 100$ cm^3/hr

From **b. ii** $\frac{dV}{dy} = 4\pi\left(25-(y-5)^2\right)$ H1

So $\frac{dy}{dt} = \frac{1}{4\pi\left(25-(y-5)^2\right)} \times 100$ M1

When $y = 6.4849763$ then $\frac{dy}{dt} = 0.34910..$

So rate of change of the depth, correct to two decimal places is 0.35 m/hr A1

d.

$\frac{dC}{dt} = 0.1(10-C) = \frac{10-C}{10}$ so $\frac{dt}{dC} = \frac{-10}{C-10}$

$t = -10\int \frac{1}{C-10} dC$

$t = -10\log_e(C-10)+c$ M1

When t=0, C=50 so $c = 10\log_e(40)$

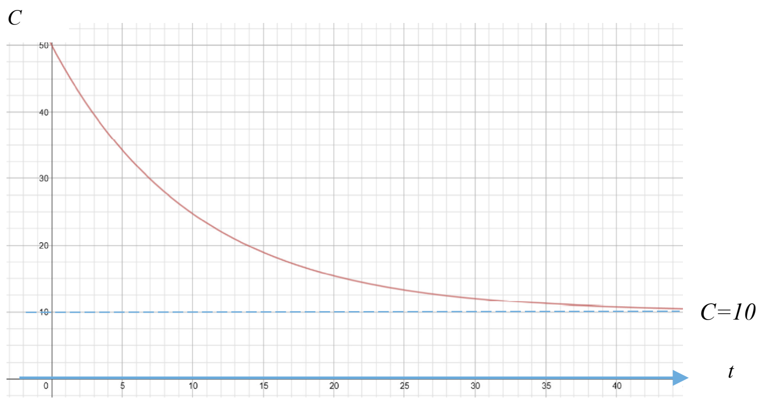
$t = 10\log_e\left(\frac{40}{C-10}\right)$

Rearranging to make C the subject $C = 40e^{-0.1t}+10$ A1

Alternatively, using deSolve to get $C = ke^{-0.1t}+10$ M1

Substituting t=0, C=50 to get k=40, so $C = 40e^{-0.1t}+10$ A1

e. i.



Asymptote at C=10, so chemical concentration settles at 10 kg/m^3. A1

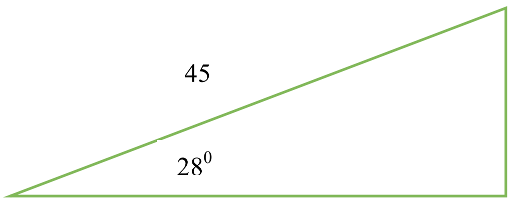
ii. Solve $15 = 40e^{-0.1t}+10$ to give $t = 20.7944..$ So 21 hours A1

Question 4

a. $\underset{\sim}{a} = -9.8\underset{\sim}{k}$

$\therefore \underset{\sim}{v} = \int -9.8\underset{\sim}{k}\,dt = -9.8t\,\underset{\sim}{k} + \underset{\sim}{c}$ A1

$\left|\underset{\sim}{v}(0)\right| = 45$ so resolving in the given directions gives:



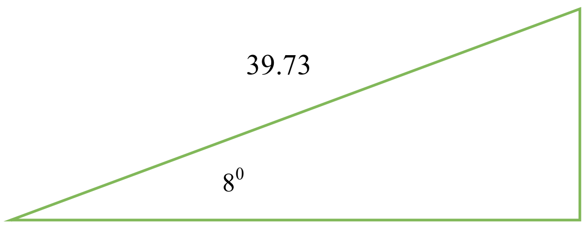
Initial vertical component of velocity

Initial horizontal component of velocity in *i*-*j* plane

Initial vertical component of velocity

$= 45\sin\left(28^0\right) = 21.126... \approx 21.13$ (correct to 2 dec. pl.) A1

Horizontal component of velocity in *i*-*j* plane = $45\cos(28^0) = 39.7326.. \approx 39.73$



Initial component of velocity in *j* direction

$= 39.7326... \times \sin\left(8^0\right) = 5.5297..$

Initial component of velocity in *i* direction

$= 39.7326... \times \cos\left(8^0\right) = 39.3459...$
≈ 39.35 H1

Therefore we get, correct to two decimal places: $\underset{\sim}{v}(t) = 39.35\underset{\sim}{i} + 5.53\underset{\sim}{j} + \left(21.13 - 9.8t\right)\underset{\sim}{k}$

b. $\underset{\sim}{r}(t) = \int \underset{\sim}{v}\,dt = 39.35t\,\underset{\sim}{i} + 5.53t\,\underset{\sim}{j} + \left(21.13t - 4.9t^2\right)\underset{\sim}{k} + \underset{\sim}{c_1}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$ so $\underset{\sim}{r}(t) = 39.35t\,\underset{\sim}{i} + 5.53t\,\underset{\sim}{j} + \left(21.13t - 4.9t^2\right)\underset{\sim}{k}$ A1

c. Ball hits ground when vertical component equals zero.
Solving $21.13t - 4.9t^2 = 0$ where $t > 0$ gives 4.31s (correct to 2 dec. pl.) A1

d. Position at t=4.31s is $\underset{\sim}{r}(4.31) = 169.58\underset{\sim}{i} + 23.83\underset{\sim}{j}$ A1

For the ball to land on the fairway the $\underset{\sim}{j}$ component must be between -25 and 25.

-25< 23.83 < 25 so ball lands on fairway. A1

e.

Distance $=\sqrt{(185-169.58)^2+(23.83-25)^2}=15.4643....\approx 15.5m$ A1

f. **i.** $a=1-\frac{1}{4}v^2$ so $v\frac{dv}{dx}=1-\frac{1}{4}v^2$

$\frac{dv}{dx}=\frac{4-v^2}{4v}$ M1

$\therefore x=\int\frac{4v}{4-v^2}dv$

$x=-2\log_e\left|4-v^2\right|+c$

$x=0,\ v=7$ so $c=2\log_e(45)$ H1

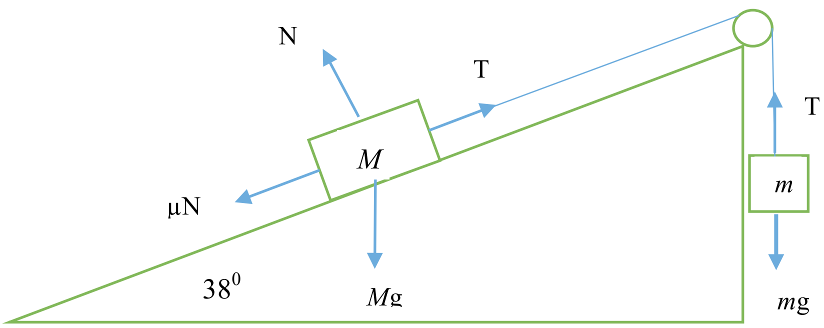
$x=2\log_e\left|\frac{45}{4-v^2}\right|$ A1

ii. Ball stops when v=0 so distance travelled $x=2\log_e\left|\frac{45}{4}\right|\approx 4.84m$ A1

So ball does not roll into the hole which is 5 metres away.

Question 5

a. Mass m is on the point of moving down.



M1 indication of forces on diagram or algebraic equivalent.

In equilibrium so $\sum \underset{\sim}{F}=0$.

Therefore, for mass m: $mg-T=0$

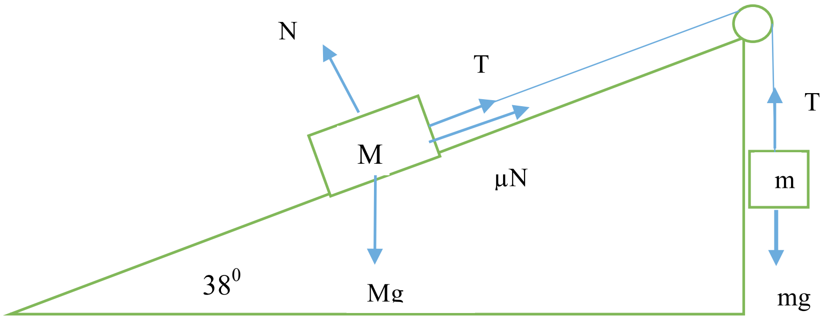
and for mass M: $T-Mg\sin\left(38^0\right)-\mu Mg\cos\left(38^0\right)=0$ M1

Substituting for T and μ, gives: $mg-Mg\sin\left(38^0\right)-0.15Mg\cos\left(38^0\right)=0$

So $\frac{m}{M}=\sin\left(38^0\right)+0.15\cos\left(38^0\right)=0.733863....$

So $p=0.73$ (correct to 2 dec. places) A1

Mass m is on the point of moving up.



M1 indication of forces on diagram or algebraic equivalent.

In equilibrium, so $\sum \underset{\sim}{F} = 0$.

Therefore, for mass m: $mg - T = 0$

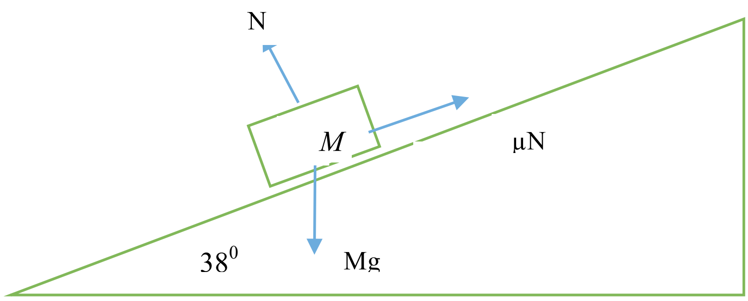
and for mass M: $Mg\sin\left(38^0\right) - \mu Mg\cos\left(38^0\right) - T = 0$ M1

Substituting for T and μ, and solving for $\dfrac{m}{M}$ gives:

So $\dfrac{m}{M} = \sin\left(38^0\right) - 0.15\cos\left(38^0\right) = 0.497....$

So $q = 0.50$ (correct to 2 dec.places) A1

b. i



$\sum \underset{\sim}{F} = Ma$

$\therefore Mg\sin\left(38^0\right) - 0.15Mg\cos\left(38^0\right) = Ma$

$a = 4.8751.......A1$

$v^2 = u^2 + 2as$ where $u = 0$, $s = 2.5$, $a = 4.8751.....$

$\therefore v = \sqrt{2\times 2.5\times 4.8751...} = 4.937....\approx 4.94$ ms^{-1} A1

ii Time to get to end of inclined plane:

$s = ut + \frac{1}{2}at^2$ where $u = 0$, $s = 2.5$, $a = 4.8751.....$

So $t = 1.0127.....$s

Therefore time to come to rest along the horizontal plane is 2-1.1027…s A1

$s = \frac{1}{2}\left(u + v\right)t$ where $u = 4.937....$, $v = 0$, $t = 2\text{-}1.0127...$

So distance travelled = 2.21 m (correct to 2 dec. places) A1