Year 2015

VCE

Specialist Mathematics

Trial Examination 1

Solutions

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Question 1

by partial fractions

$$\frac{2x-5}{x^2-25}=\frac{A}{x-5}+\frac{B}{x+5}$$

$$=\frac{A(x+5)+B(x-5)}{(x-5)(x+5)} \quad \text{M1}$$

$$=\frac{x(A+B)+5(A-B)}{x^2-25}$$

equating coefficients

$(1)\ \ A+B=2 \quad (2)\ \ A-B=-1 \quad \text{adding} \ \ 2A=1 \ \Rightarrow \ A=\frac{1}{2} \quad B=\frac{3}{2}$ A1

$$\int \frac{2x-5}{x^2-25}\,dx=\int \frac{1}{2}\left(\frac{1}{x-5}+\frac{3}{x+5}\right)dx$$

$$=\frac{1}{2}\left(\log_e\left(|x-5|\right)+3\log_e\left(|x+5|\right)\right)+c \quad \text{A1}$$

$$=\frac{1}{2}\log_e\left(|x-5||x+5|^3\right)+c$$

Question 2

a. $z=2\operatorname{cis}\left(-\frac{\pi}{6}\right)=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)=2\cos\left(\frac{\pi}{6}\right)-2i\sin\left(\frac{\pi}{6}\right)=\sqrt{3}-i$

Since c is a real constant, by the conjugate root theorem, $\overline{z}=\sqrt{3}+i$ is also a root,

now $z+\overline{z}=2\sqrt{3}$ and $z\,\overline{z}=3-i^2=4$, M1

the quadratic factor is z^2-(sum of the roots) z + product of the roots

so $\left(z^2-2\sqrt{3}\,z+4\right)$ is the quadratic factor A1

b. $f(z)=z^3+\left(2-2\sqrt{3}\right)z^2+cz+8$

$$=\left(z^2-2\sqrt{3}\,z+4\right)(z+2)=0$$

$$=\left(z-\sqrt{3}+i\right)\left(z-\sqrt{3}-i\right)(z+2)=0$$

expanding gives $c=4-4\sqrt{3}$ A1

and all the roots are $z=\sqrt{3}-i\ ,\ \sqrt{3}+i\ ,\ z=-2$ A1

Question 3

a. $\underset{\sim}{r}(t)=2\left(t+\frac{1}{t}\right)\underset{\sim}{i}+\left(t-\frac{1}{t}\right)\underset{\sim}{j} \quad t>0$

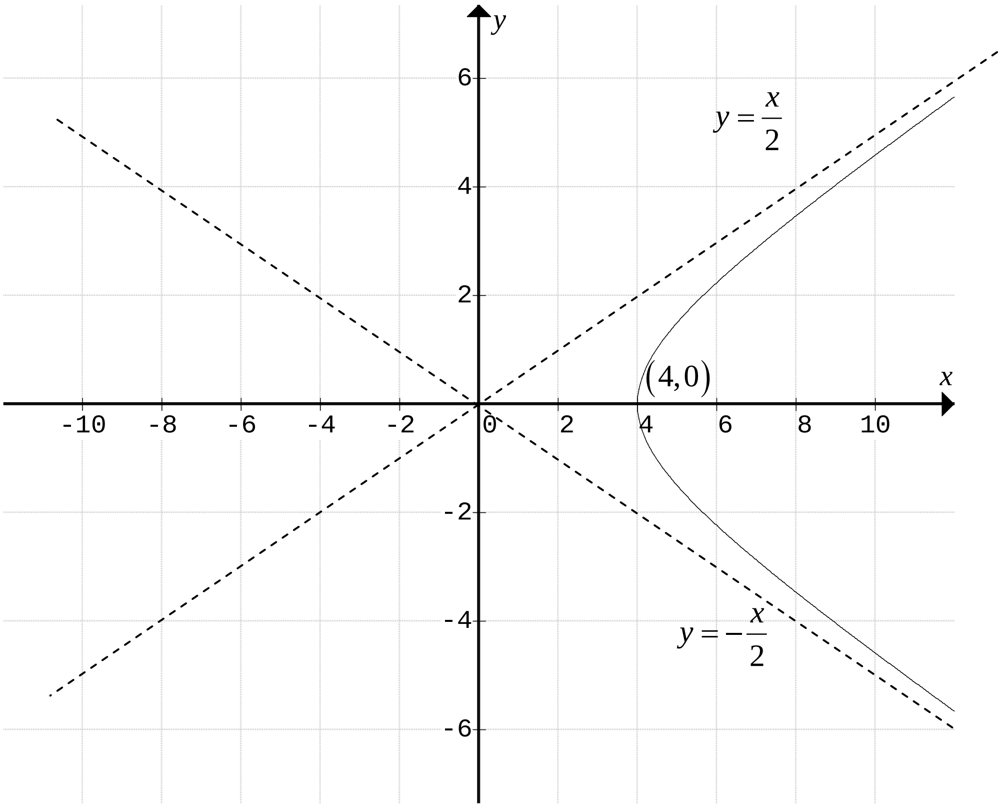
$x=2\left(t+\frac{1}{t}\right)$, $y=t-\frac{1}{t}$, $\frac{x}{2}=t+\frac{1}{t}$, $y=t-\frac{1}{t}$ squaring both equations

$\frac{x^2}{4}=t^2+2+\frac{1}{t^2}$, $y^2=t^2-2+\frac{1}{t^2}$ subtracting to eliminate *t*, $\frac{x^2}{4}-y^2=4$ M1

$\frac{x^2}{16}-\frac{y^2}{4}=1$ so that $b=2$ A1

b. since $t>0 \Rightarrow x\geq 4$ it is only the right hand branch of the hyperbola,

crosses *x*-axis at $(4,0)$ with asymptotes $y=\pm\frac{x}{2}$, correct shape G2



c. $\dot{\underset{\sim}{r}}(t)=2\left(1-\frac{1}{t^2}\right)\underset{\sim}{i}+\left(1+\frac{1}{t^2}\right)\underset{\sim}{j}$ A1

when it moves parallel to the *y*-axis $\dot{\underset{\sim}{r}}(t).\underset{\sim}{i}=0$

$1-\frac{1}{t^2}=0 \Rightarrow t=1$ since $t>0$ A1

Question 4

$y=\log_e\left(x+\sqrt{x^2+4}\right)=\log_e(u)$ $\quad u=x+\sqrt{x^2+4}=x+\left(x^2+4\right)^{\frac{1}{2}}$

$\frac{dy}{du}=\frac{1}{u}$ $\quad \frac{du}{dx}=1+\frac{1}{2}\times 2x\left(x^2+4\right)^{-\frac{1}{2}}$

$$=1+\frac{x}{\sqrt{x^2+4}}=\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}$$ M1

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{x+\sqrt{x^2+4}}\times\frac{x+\sqrt{x^2+4}}{\sqrt{x^2+4}}=\frac{1}{\sqrt{x^2+4}}=\left(x^2+4\right)^{-\frac{1}{2}}$$ A1

$$\frac{d^2y}{dx^2}=-\frac{1}{2}\times 2x\left(x^2+4\right)^{-\frac{3}{2}}=\frac{-x}{\sqrt{\left(x^2+4\right)^3}}$$

$$\left(x^2+4\right)\frac{d^2y}{dx^2}-bx\frac{dy}{dx}=0$$

$$\left(x^2+4\right)\times\frac{-x}{\sqrt{\left(x^2+4\right)^3}}-\frac{bx}{\sqrt{x^2+4}}=\frac{-x(b+1)}{\sqrt{x^2+4}}=0$$

$b=-1$ A1

Question 5

a. $\left(x^2+y^2\right)^2=4\left(x^2-y^2\right)$ expanding $x^4+2x^2y^2+y^4=4x^2-4y^2$

using implicit differentiation and the product rule on the second term.

$$\frac{d}{dx}\left(x^4\right)+\frac{d}{dx}\left(2x^2y^2\right)+\frac{d}{dx}\left(y^4\right)=\frac{d}{dx}\left(4x^2\right)-\frac{d}{dx}\left(4y^2\right)$$

$$4x^3+4xy^2+4x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}=8x-8y\frac{dy}{dx}$$ M1

$$\frac{dy}{dx}\left(4x^2y+4y^3+8y\right)=8x-4x^3-4xy^2$$

$$\frac{dy}{dx}=\frac{x\left(2-x^2-y^2\right)}{y\left(x^2+y^2+2\right)}$$ A1

b. The tangent line is horizontal when $\frac{dy}{dx}=0 \Rightarrow x^2+y^2=2 \quad y^2=2-x^2$

$$\left(x^2+y^2\right)^2=4\left(x^2-y^2\right) \quad\Rightarrow\quad 4=4\left(x^2-\left(2-x^2\right)\right)$$

$$2x^2-2=1 \quad\Rightarrow\quad 2x^2=3 \quad\Rightarrow\quad x^2=\frac{3}{2}$$

$$x=\pm\sqrt{\frac{3}{2}}=\pm\frac{\sqrt{6}}{2} \quad\Rightarrow c=6$$ A1

Question 6

a. $\dfrac{dQ}{dt} = bg - \dfrac{fQ}{V_0 + (g-f)t} = 12 - \dfrac{4Q}{5+2t}$

$V_0 = 5$, $f = 4$, $bg = 12$ $g - f = 2$

$g = 6$, $b = 2$ A1

b. $\dfrac{dQ}{dt} = 12 - \dfrac{4Q}{5+2t}$, $Q(0) = 0$

using Euler's method, with $Q_0 = 0$, $h = \dfrac{1}{2}$, $t_0 = 0$ $f(Q,t) = 12 - \dfrac{4Q}{5+2t}$

$Q_1 = Q_0 + hf(Q_0, t_0)$

$= 0 + \dfrac{1}{2}\left(12 - \dfrac{4\times 0}{5+0}\right) = 6$ M1

$Q_2 = Q_1 + hf(Q_1, t_1)$

$= 6 + \dfrac{1}{2}\left(12 - \dfrac{4\times 6}{5+2\times 0.5}\right) = 10$ A1

Question 7 $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + \underset{\sim}{j} + t\underset{\sim}{k}$

a. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, $\underset{\sim}{a}.\underset{\sim}{b} = 0$

$\underset{\sim}{a}.\underset{\sim}{b} = -6 - 2 - 4t = 0 \quad \Rightarrow \quad 4t = -8$

$t = -2$ A1

b. If the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are equal in length, $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$|\underset{\sim}{a}| = \sqrt{\left(3^2 + (-2)^2 + (-4)^2\right)} = \sqrt{9+4+16} = \sqrt{29}$

$|\underset{\sim}{b}| = \sqrt{(-2)^2 + 1^2 + t^2} = \sqrt{5+t^2}$

$|\underset{\sim}{a}| = |\underset{\sim}{b}| \Rightarrow \sqrt{29} = \sqrt{5+t^2}$ M1

squaring both sides, $\Rightarrow 29 = 5 + t^2 \quad \Rightarrow t^2 = 24$

$t = \pm 2\sqrt{6}$ both answers are acceptable A1

c. If the vector $\underset{\sim}{b}$ makes an angle of 150^0 with the z-axis.

$\cos\left(150^0\right) = \dfrac{\underset{\sim}{b}.\underset{\sim}{k}}{|\underset{\sim}{b}|} \quad \Rightarrow \quad -\dfrac{\sqrt{3}}{2} = \dfrac{t}{\sqrt{5+t^2}}$ so that $t < 0$ M1

$-\sqrt{3}\sqrt{5+t^2} = 2t$ square both sides $3\left(5+t^2\right) = 4t^2 \quad \Rightarrow \quad t^2 = 15$ so $t = \pm\sqrt{15}$

$t = -\sqrt{15}$ only answer A1

Question 8

$y = \frac{x^2+9}{3x} = \frac{x}{3} + \frac{3}{x} = \frac{x}{3} + 3x^{-1}$

crosses x-axis when $y = 0 \Rightarrow x^2 + 9 = 0$ no real solutions,

does not cross the x-axis and does not cross the y-axis. A1

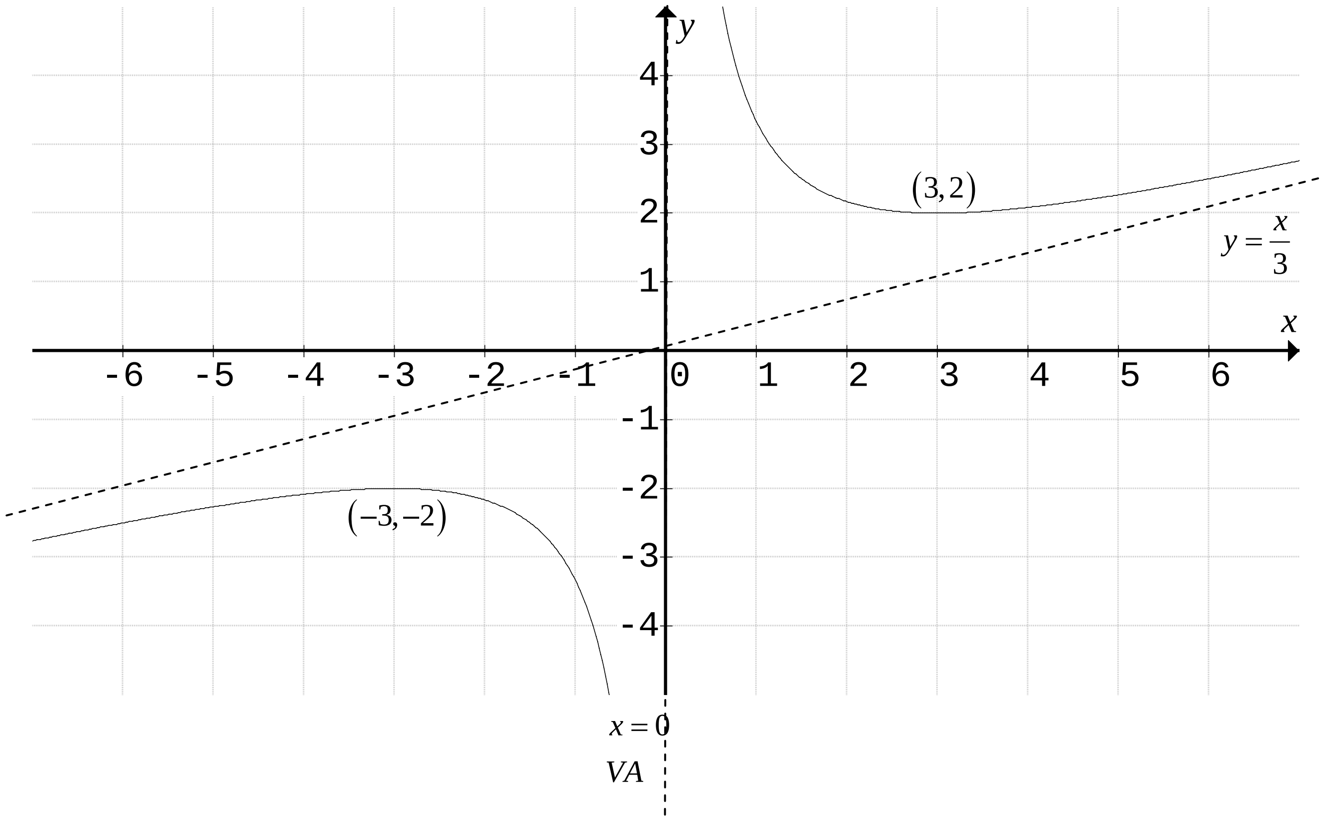
$x = 0$ is a vertical asymptote and $y = \frac{x}{3}$ is an oblique asymptote. A1

For turning points $\frac{dy}{dx} = \frac{1}{3} - 3x^{-2} = \frac{1}{3} - \frac{3}{x^2} = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$,

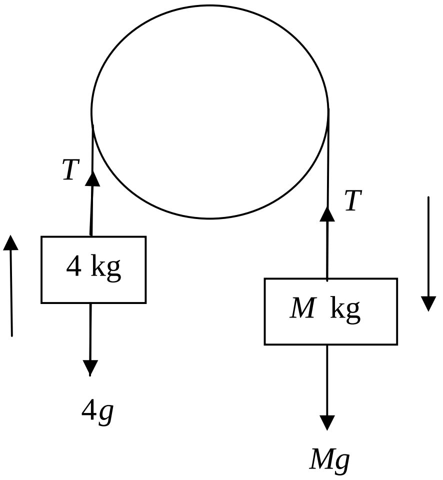
when $x = 3$, $y = 2$ and when $x = -3$, $y = -2$,

$(3, 2)$, $(-3, -2)$ are turning points A1

correct graph, shape asymptotes G1



Question 9



resolving downwards around the M kg mass

$(1)\quad Ma = Mg - T$

resolving upwards around the 4 kg mass M1

$(2)\quad T - 4g = 4a$

adding to eliminate the tension T in the string,

to find the acceleration a, of the system

$(M-4)g = (M+4)a$

$$a = \frac{(M-4)g}{M+4}$$

Now using $v^2 = u^2 + 2as$ with $s = 1$ $u = 0$ $v = \sqrt{g}$ and $a = \frac{(M-4)g}{M+4}$ A1

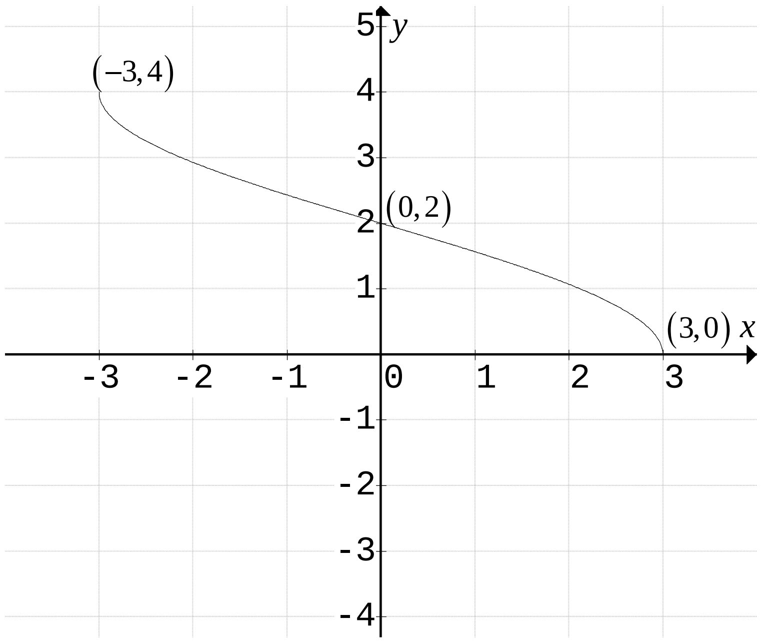
$$g = 0 + 2 \times \frac{(M-4)}{(M+4)} \times g \times 1$$

$$\frac{2(M-4)}{M+4} = 1 \qquad \Rightarrow \qquad 2M - 8 = M + 4$$

$M = 12$ kg A1

Question 10 $\quad f(x) = \frac{4}{\pi}\arccos\left(\frac{x}{3}\right) = \frac{4}{\pi}\cos^{-1}\left(\frac{x}{3}\right)$

a. domain $[-3,3]$ range $[0,4]$, endpoints $(3,0)$, $(-3,4)$ y-intercept $(0,2)$ G2



b. $x\arccos\left(\frac{x}{3}\right)=x\cos^{-1}\left(\frac{x}{3}\right)$

$\frac{d}{dx}\left(x\cos^{-1}\left(\frac{x}{3}\right)\right)=\cos^{-1}\left(\frac{x}{3}\right)\frac{d}{dx}(x)+x\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{3}\right)\right)$ using the product rule

$\frac{d}{dx}\left(x\cos^{-1}\left(\frac{x}{3}\right)\right)=\cos^{-1}\left(\frac{x}{3}\right)-\frac{x}{\sqrt{9-x^2}}$ A1

The required area is $A=\int_0^3\frac{4}{\pi}\arccos\left(\frac{x}{3}\right)dx=\frac{4}{\pi}\int_0^3\cos^{-1}\left(\frac{x}{3}\right)dx$ A1

Hence $\int\left(\cos^{-1}\left(\frac{x}{3}\right)-\frac{x}{\sqrt{9-x^2}}\right)dx=x\cos^{-1}\left(\frac{x}{3}\right)$

$\int\cos^{-1}\left(\frac{x}{3}\right)dx-\int\frac{x}{\sqrt{9-x^2}}dx=x\cos^{-1}\left(\frac{x}{3}\right)$

$\int\cos^{-1}\left(\frac{x}{3}\right)dx=x\cos^{-1}\left(\frac{x}{3}\right)+\int\frac{x}{\sqrt{9-x^2}}dx$

Consider $\int\frac{x}{\sqrt{9-x^2}}dx$ let $u=9-x^2$ $\frac{du}{dx}=-2x$ M1

$\int\frac{x}{\sqrt{9-x^2}}dx=\int xu^{-\frac{1}{2}}\frac{dx}{du}du=\int xu^{-\frac{1}{2}}\frac{1}{-2x}du$

$=-\frac{1}{2}\int u^{-\frac{1}{2}}du=-u^{\frac{1}{2}}=-\sqrt{9-x^2}$

$\int\cos^{-1}\left(\frac{x}{3}\right)dx=x\cos^{-1}\left(\frac{x}{3}\right)-\sqrt{9-x^2}$

$A=\int_0^3\frac{4}{\pi}\arccos\left(\frac{x}{3}\right)dx=\frac{4}{\pi}\int_0^3\cos^{-1}\left(\frac{x}{3}\right)dx$

$=\frac{4}{\pi}\left[x\cos^{-1}\left(\frac{x}{3}\right)-\sqrt{9-x^2}\right]_0^3$

$=\frac{4}{\pi}\left[\left(3\cos^{-1}(1)-\sqrt{9-9}\right)-\left(0\times\cos^{-1}(0)-\sqrt{9}\right)\right]$

$=\frac{4}{\pi}\left[(3\times0-0)+3\right]$

$=\frac{12}{\pi}$ A1

END OF SUGGESTED SOLUTIONS