2015 VCAA Specialist Math Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
E	A	D	D	B	E	A	C	B/C	B	D

12	13	14	15	16	17	18	19	20	21	22
B	C	E	A	A	C	B	C	B	E	D

Q1 $\cos^2 t + \sin^2 t = 1$

Let $\frac{(x-2)^2}{9} = \cos^2 t$ and $\frac{(y-3)^2}{4} = \sin^2 t$

$x = 2 + 3\cos t$ and $y = 3 + 2\sin t$ E

Q2 Sketch the graph of $f(x) = (2-x)\sin^{-1}\left(\frac{x}{2} - 1\right)$ by CAS. A

Q3 $a^2x^2 + (1-a^2)y^2 = c^2$ is a circle when $a^2 = \frac{1}{2}$.

It is a hyperbola when $a > 1$. It is an ellipse when $a < 1$.

It is a pair of straight lines ($x = \pm c$) when $a = 1$. D

Q4 $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = k$ passes through $(5, 5)$

.: $\frac{(5-2)^2}{9} - \frac{(5-1)^2}{4} = k$, .: $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = -3$ D

Q5 $z = \frac{1+i\sqrt{3}}{1+i} = \frac{2\,\text{cis}\,\frac{\pi}{3}}{\sqrt{2}\,\text{cis}\,\frac{\pi}{4}} = \sqrt{2}\,\text{cis}\,\frac{\pi}{12}$.: $z^5 = 4\sqrt{2}\,\text{cis}\,\frac{5\pi}{12}$ B

Q6 $(1+2i) + (1-2i) = 2$ E

Q7 $z = \sqrt{3} + 3i = 2\sqrt{3}\,\text{cis}\,\frac{\pi}{3}$.: $z^{63} = \left(2\sqrt{3}\right)^{63}\text{cis}\,\frac{63\pi}{3}$

$= \left(2\sqrt{3}\right)^{63}\text{cis}21\pi = \left(2\sqrt{3}\right)^{63}\text{cis}\pi = -\left(2\sqrt{3}\right)^{63}$ A

Q8 $|z-i| = |z+2|$ is a perpendicular bisector of the line segment joining $z = 0 + i$ and $z = -2 + 0i$ C

Q9 Based on the locations of z_1 and $z_1 z_2$ shown in the given graph, $r_1 > r_1 r_2$ and $\theta_2 > \theta_1$, .: $\frac{r_1}{r_2} > r_1$ and $\theta_1 < \theta_2$.

.: $\frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right| > r_1$ and $\theta_1 < \theta_2$ B or C

Q10 Let $u = 3x + 1$, $x = \frac{u-1}{3}$, $\frac{du}{dx} = 3$

$u = 1$ when $x = 0$, $u = 4$ when $x = 1$.: $\int_0^1 x^2\sqrt{3x+1}\,dx$

$= \int_1^4 \frac{(u-1)^2\sqrt{u}}{9}\frac{du}{3} = \frac{1}{27}\int_1^4\left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$ B

Q11 By estimation, displacement $= 0$ when $t \approx 3.3$ D

Q12 $\frac{dy}{dx} = 1 - \frac{y}{3} = \frac{3-y}{3}$, $\frac{dx}{dy} = 3 \times \frac{1}{3-y}$

$\frac{x}{3} = \int \frac{1}{3-y}dy = -\log_e|3-y| + c$ and $y = 4$ when $x = 2$

.: $c = \frac{2}{3}$ and $|3-y| = e^{\frac{2-x}{3}}$.: $3 - y = \pm e^{\frac{2-x}{3}}$, $y = \pm e^{\frac{-(x-2)}{3}} + 3$ B

Q13 A smooth curve can be drawn joining $(-2.5, 1.5)$ and $(3, 1)$ with the slope field line segments as tangents to the curve. C

Q14 $y = x\sin x$, $\frac{dy}{dx} = x\cos x + \sin x$

$\frac{d^2y}{dx^2} = -x\sin x + 2\cos x$.: $\frac{d^2y}{dx^2} = -y + 2\cos x$ E

Q15 $\hat{w} = \frac{\tilde{w}}{\sqrt{2}} = \frac{\tilde{i} + \tilde{j}}{\sqrt{2}}$, $\tilde{F}.\tilde{w} = \frac{a+b}{\sqrt{2}}$

$(\tilde{F}.\tilde{w})\hat{w} = (\tilde{F}.\tilde{w})\frac{\tilde{w}}{\sqrt{2}} = \frac{a+b}{2}\tilde{w}$ A

Q16 Resultant force $= \tilde{0}$, i.e. $\tilde{T}_1 + \tilde{T}_2 + \tilde{W} = \tilde{0}$ A

Q17 $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = -\tilde{i} + 2\tilde{j} - \tilde{k}$, $|\overrightarrow{BA}| = \sqrt{6}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -3\tilde{i} - 2\tilde{j}$, $|\overrightarrow{BC}| = \sqrt{13}$

$\overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\angle ABC$

$\cos\angle ABC = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{3-4}{\sqrt{6}\sqrt{13}} = \frac{-1}{\sqrt{6}\sqrt{13}}$ C

Q18 Let $\tilde{r}_1(t) = \tilde{r}_2(t)$.

.: $2 + 4t^2 = 6t$ and $3t + 2 = 4 + t$, .: $t = 1$, $\tilde{r}(1) = 6\tilde{i} + 5\tilde{j}$ B

Q19 1 kg mass: $T - 9.8 = 1 \times 4.9$, .: $T = 14.7$

m kg mass: $m \times 9.8 - 14.7 = m \times 4.9$, .: $m = 3$ C

Q20 $|s| = \left|\frac{1}{2}(u+v)t\right| = \left|\frac{1}{2}\left({}^{+}5 + {}^{-}11\right) \times 16\right| = 48$ B

Q21 Vertical: $F\sin\theta + N - Mg = 0$, .: $N = Mg - F\sin\theta$

Horizontal: $F\cos\theta - \mu N = Ma$

.: $F\cos\theta - \mu(Mg - F\sin\theta) = Ma$ E

Q22 $\frac{dv}{dt} = -(9.8 + 0.1v^2)$, $\frac{dt}{dv} = -\frac{1}{9.8 + 0.1v^2}$

$t = \frac{-10}{\sqrt{98}}\tan^{-1}\frac{v}{\sqrt{98}} + c$ and given $v = 7\sqrt{6}$ when $t = 0$

.: $c = \frac{10}{\sqrt{98}}\tan^{-1}\frac{7\sqrt{6}}{\sqrt{98}} = \frac{10}{\sqrt{98}}\tan^{-1}\sqrt{3} = \frac{10}{\sqrt{98}} \times \frac{\pi}{3} = \frac{10\pi}{21\sqrt{2}}$

When $v = 0$, $t = c = \frac{10\pi}{21\sqrt{2}}$. D

SECTION 2

Q1a $y=\sqrt{2-\sin^2 x}$, $y^2=2-\sin^2 x$

$y\frac{dy}{dx}=-2\sin x\cos x=-\sin(2x)$, $\frac{dy}{dx}=-\frac{\sin(2x)}{y}$

Q1bi $x=0$, $y=\sqrt{2-\sin^2 0}=\sqrt{2}$; $x=\frac{\pi}{2}$, $y=\sqrt{2-\sin^2\frac{\pi}{2}}=1$

Q1bii $x=0$, $\frac{dy}{dx}=-\frac{\sin(0)}{y}=0$; $x=\frac{\pi}{2}$, $\frac{dy}{dx}=-\frac{\sin\pi}{y}=0$

Q1c $y=\sqrt{2-\sin^2 x}$, $y^2=2-\sin^2 x$, dom $\left[0,\frac{\pi}{2}\right]$, range $\left[1,\sqrt{2}\right]$

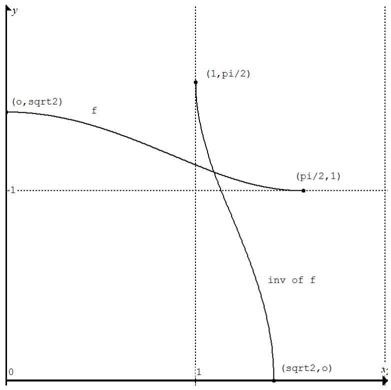
The inverse is $x^2=2-\sin^2 y$, $\sin^2 y=2-x^2$

.: $1-2\sin^2 y=1-2\left(2-x^2\right)=2x^2-3$

.: $\cos(2y)=2x^2-3$, $y=\frac{1}{2}\cos^{-1}\left(2x^2-3\right)$,

Its domain is $\left[1,\sqrt{2}\right]$ and range is $\left[0,\frac{\pi}{2}\right]$.

Q1d

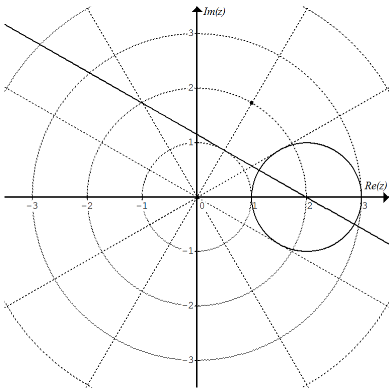


Q1e Let $\sqrt{2-\sin^2 x}=x$, by CAS $a=x\approx 1.099$

Q1fi $V=\int_0^1 \pi y^2\,dx=\int_0^1 \pi\left(2-\sin^2 x\right)dx$

Q1fii By CAS, $V=5.4$ cubic units

Q2ai and Q2aii



Q2aiii The line is a perpendicular bisector of the line segment joining $0+0i$ and $1+i\sqrt{3}$, .: its gradient $=-\frac{1}{\sqrt{3}}$

The equation: $\frac{y-0}{x-2}=-\frac{1}{\sqrt{3}}$, i.e. $x+\sqrt{3}y=2$

Q2aiv Circle: $(x-2)^2+y^2=1$, line: $x-2=-\sqrt{3}y$

.: $\left(-\sqrt{3}y\right)^2+y^2=1$, $3y^2+y^2=1$, $4y^2=1$, $y=\pm\frac{1}{2}$

.: $x=2-\sqrt{3}\left(\pm\frac{1}{2}\right)=2\mp\frac{\sqrt{3}}{2}$

The points of intersection are $\left(2-\frac{\sqrt{3}}{2}\right)+\frac{1}{2}i$ and $\left(2+\frac{\sqrt{3}}{2}\right)-\frac{1}{2}i$.

Q2bi $z^2-(4\cos\alpha)z+4=0$,

$z=\frac{4\cos\alpha\pm\sqrt{16\cos^2\alpha-16}}{2}$

$=\frac{4\cos\alpha\pm i4\sin\alpha}{2}=2(\cos\alpha\pm i\sin\alpha)$

.: $z_1=2\text{cis}\alpha$ and $z_2=2\text{cis}(-\alpha)$

Q2bii $\frac{z_1}{z_2}=\text{cis}(2\alpha)$, $\left|\text{Arg}\left(\frac{z_1}{z_2}\right)\right|=2\alpha=\frac{5\pi}{6}$, .: $\alpha=\frac{5\pi}{12}$

Q3a $x=\sin t$, $\frac{dx}{dt}=\cos t$

$y=\frac{1}{2}\sin t\tan t$, $\frac{dy}{dt}=\frac{1}{2}\left(\sin t\sec^2 t+\cos t\tan t\right)$

.: $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1}{2}\left(\frac{\sin t\sec^2 t+\cos t\tan t}{\cos t}\right)=\frac{1}{2}\tan t\left(\sec^2 t+1\right)$

Q3b At $t=\frac{\pi}{6}$, the slope $=\frac{dy}{dx}=\frac{1}{2}\tan\frac{\pi}{6}\left(\sec^2\frac{\pi}{6}+1\right)=\frac{7\sqrt{3}}{18}$

Q3ci $\cos t=\sqrt{1-\sin^2 t}=\sqrt{1-x^2}$

$y=\frac{1}{2}\sin t\tan t=\frac{1}{2}x\frac{x}{\sqrt{1-x^2}}=\frac{x^2}{2\sqrt{1-x^2}}$

Q3cii $t\in\left[-\frac{\pi}{3},\frac{\pi}{3}\right]$, $x=\sin t\in\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$, the domain.

Q3d RHS $=\frac{2x^2}{\sqrt{1-x^2}}+\frac{d}{dx}\left(x\sqrt{1-x^2}\right)$

$=\frac{2x^2}{\sqrt{1-x^2}}+\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}=\frac{d}{dx}(\arcsin x)=\text{LHS}$

Q3e Area $=4\int_0^{\frac{\sqrt{3}}{2}}\frac{x^2}{2\sqrt{1-x^2}}dx=\int_0^{\frac{\sqrt{3}}{2}}\frac{2x^2}{\sqrt{1-x^2}}dx$

$=\left[\arcsin x-x\sqrt{1-x^2}\right]_0^{\frac{\sqrt{3}}{2}}=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

Q4ai $\frac{2t}{5}=60$, $t=150$, the time required is 150 seconds

Q4aii $\underset{\sim}{r}(150)=(50+25\cos 5\pi)\underset{\sim}{i}+(50+25\sin 5\pi)\underset{\sim}{j}+60\underset{\sim}{k}$

$=25\underset{\sim}{i}+50\underset{\sim}{j}+60\underset{\sim}{k}$

.: $\tan\theta=\frac{60}{\sqrt{25^2+50^2}}$, $\theta\approx 47°$

Q4b After one period $=\frac{2\pi}{\frac{\pi}{30}}=60$ seconds

Q4c $\underset{\sim}{r}(t)=\left(50+25\cos\frac{\pi t}{30}\right)\underset{\sim}{i}+\left(50+25\sin\frac{\pi t}{30}\right)\underset{\sim}{j}+\frac{2t}{5}\underset{\sim}{k}$

$\underset{\sim}{v}(t)=\dot{\underset{\sim}{r}}(t)=\left(-\frac{5\pi}{6}\sin\frac{\pi t}{30}\right)\underset{\sim}{i}+\left(\frac{5\pi}{6}\cos\frac{\pi t}{30}\right)\underset{\sim}{j}+\frac{2}{5}\underset{\sim}{k}\neq\underset{\sim}{0}$

$\underset{\sim}{a}(t)=\ddot{\underset{\sim}{r}}(t)=\left(-\frac{\pi^2}{36}\cos\frac{\pi t}{30}\right)\underset{\sim}{i}+\left(-\frac{\pi^2}{36}\sin\frac{\pi t}{30}\right)\underset{\sim}{j}\neq\underset{\sim}{0}$

$\underset{\sim}{v}(t)\,.\,\underset{\sim}{a}(t)=\frac{5\pi^3}{246}\sin\frac{\pi t}{30}\cos\frac{\pi t}{30}-\frac{5\pi^3}{246}\sin\frac{\pi t}{30}\cos\frac{\pi t}{30}=0$

.: $\underset{\sim}{v}(t)\perp\underset{\sim}{a}(t)$, i.e. the velocity of the helicopter is perpendicular to its acceleration

Q4d Speed $=|\underset{\sim}{v}(t)|=\sqrt{\left(-\frac{5\pi}{6}\sin\frac{\pi t}{30}\right)^2+\left(\frac{5\pi}{6}\cos\frac{\pi t}{30}\right)^2+\left(\frac{2}{5}\right)^2}$

$=\sqrt{\left(\frac{5\pi}{6}\right)^2+\left(\frac{2}{5}\right)^2}\approx 2.65\ \text{ms}^{-1}$

Q4e $\underset{\sim}{r}_h(45)=\left(50+25\cos\frac{3\pi}{2}\right)\underset{\sim}{i}+\left(50+25\sin\frac{3\pi}{2}\right)\underset{\sim}{j}+18\underset{\sim}{k}$

$=50\underset{\sim}{i}+25\underset{\sim}{j}+18\underset{\sim}{k}$

$\underset{\sim}{r}_t=60\underset{\sim}{i}+40\underset{\sim}{j}+8\underset{\sim}{k}$

Distance $=|\underset{\sim}{r}_h-\underset{\sim}{r}_t|=\sqrt{(-10)^2+(-15)^2+10^2}\approx 20.6$ m

Q5a $F=250g\sin 10°\approx 425.438\approx 425$ N

Q5b $250a=425.438-200$, $a\approx 0.902\ \text{ms}^{-2}$

Q5c $v^2=u^2+2as\approx 0+2(0.902)(30)$, $|v|\approx 7.36\ \text{ms}^{-1}$

Q5di $a=1.4(7-v)$, $v\frac{dv}{dx}=1.4(7-v)$, $\frac{dv}{dx}=1.4\left(\frac{7-v}{v}\right)$

.: $1.4\frac{dx}{dv}=\frac{v}{7-v}=\frac{7-(7-v)}{7-v}$, .: $1.4\frac{dx}{dv}=-1+\frac{7}{7-v}$

Q5dii $1.4\int\frac{dx}{dv}dv=\int\left(-1+\frac{7}{7-v}\right)dv$

$1.4x=-v-7\log_e(7-v)+c$ and $v=0$ when $x=0$, .: $c=7\log_e 7$

.: $1.4x=-v-7\log_e(7-v)+7\log_e 7$

Q5diii When $x=D$, $v=5$

.: $1.4D=-5-7\log_e(7-5)+7\log_e 7$, $D\approx 2.7$ m

Q5div $a=1.4(7-v)$, $\frac{dv}{dt}=1.4(7-v)$, $\frac{dt}{dv}=\frac{1}{1.4}\times\frac{1}{7-v}$

$v=0$ when $t=0$, .: $v=5$ when

$t=\frac{1}{1.4}\times\int_0^5\frac{1}{7-v}dv=\frac{1}{1.4}\times\left[-\log_e(7-v)\right]_0^5\approx 0.9$ s

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