2015 VCAA Specialist Mathematics Exam 1 Solutions

Q1a $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OC}\right|$, $a = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Q1b $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(1-\sqrt{3}\right)\tilde{i} + \tilde{j} + \tilde{k}$

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{OA} = \left(1+\sqrt{3}\right)\tilde{i} + \tilde{j} + \tilde{k}$

$\overrightarrow{AC}.\overrightarrow{OB} = \left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right) + 1 + 1 = 0$, .: $\overrightarrow{AC} \perp \overrightarrow{OB}$

Hence the diagonals are perpendicular.

Q2a Let R newtons be the reaction force.

$R - 20\times 9.8 = 20\times 1.2$, $R = 220$

Q2b Let a m s^{-2} be the acceleration.

$166 - 20\times 9.8 = 20a$, $a = -1.5$

The downward acceleration is 1.5 m s^{-2}.

Q3a $\tilde{r}(t) = (4t-3)\tilde{i} + 2t\tilde{j} - 5\tilde{k}$

$\tilde{r}(t) = \left(2t^2 - 3t\right)\tilde{i} + t^2\,\tilde{j} - 5t\,\tilde{k} + \tilde{c}$

$\tilde{r}(0) = \tilde{c} = \tilde{i} - 2\,\tilde{k}$, .: $\tilde{r}(t) = \left(2t^2 - 3t + 1\right)\tilde{i} + t^2\,\tilde{j} - (5t+2)\tilde{k}$

When $t = 2$, $\tilde{r}(2) = 3\,\tilde{i} + 4\,\tilde{j} - 12\,\tilde{k}$

$\left|\tilde{r}(2)\right| = \sqrt{3^2 + 4^2 + (-12)^2} = 13$

The distance from the origin $= 13$ metres

Q4a $z^3 = 8i$, $z^3 - 8i = 0$, $-2i$ is a solution by inspection

.: $z^3 - 8i = (z+2i)\left(z^2 - 2iz - 4\right) = 0$

.: $z^2 - 2iz - 4 = 0$, $z = \dfrac{2i \pm \sqrt{-4+16}}{2} = \pm\sqrt{3} + i$

The solutions are: $-2i$, $\pm\sqrt{3} + i$

Q4b Let $z - 2i = -2i$, $z - 2i = \pm\sqrt{3} + i$

.: $z = 0$ or $z = \pm\sqrt{3} + 3i$

Q5 $y = 2x^2 - 3$ cuts the y-axis at -3, $x^2 = \dfrac{y+3}{2}$

$$V = \int_{-3}^{5} \pi x^2\, dy = \int_{-3}^{5} \frac{\pi}{2}(y+3)\,dy = \frac{\pi}{2}\left[\frac{(y+3)^2}{2}\right]_{-3}^{5} = 16\pi$$

Q6a $a = 4v^2$, $\dfrac{1}{2}\dfrac{dv^2}{dx} = 4v^2$, $\dfrac{dx}{dv^2} = \dfrac{1}{8}\times\dfrac{1}{v^2}$, $x = \dfrac{1}{8}\displaystyle\int \frac{1}{v^2}\,dv^2$

.: $x = \dfrac{1}{8}\log_e v^2 + c$, when $x = 1$, $v = e$

.: $c = \dfrac{3}{4}$ and $x = \dfrac{1}{8}\log_e v^2 + \dfrac{3}{4}$

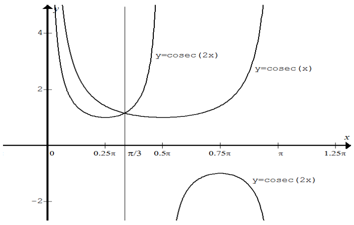
When $x = 2$, $2 = \dfrac{1}{8}\log_e v^2 + \dfrac{3}{4}$, $\log_e v^2 = 10$, $v^2 = e^{10}$, $v = e^5$

Q7a $\sin(2x) = \sin x$, $2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$, .: $\sin x = 0$ or $\cos x = \dfrac{1}{2}$

.: $x = 0, \pi, 2\pi$ or $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

Q7b Sketch the graphs of $y = \text{cosec}(2x)$ and $y = \text{cosec}(x)$.



$\text{cosec}(2x) < \text{cosec}(x)$ for $x \in \left(0, \dfrac{\pi}{3}\right) \cup \left(\dfrac{\pi}{2}, \pi\right)$

Q8a $\displaystyle\int \tan(2x)\,dx = \int \frac{\sin(2x)}{\cos(2x)}\,dx$

$= -\dfrac{1}{2}\displaystyle\int \frac{1}{u}\,du = -\frac{1}{2}\log_e|u| + c$

$= -\dfrac{1}{2}\log_e|\cos(2x)| + c = \dfrac{1}{2}\log_e|\sec(2x)| + c$

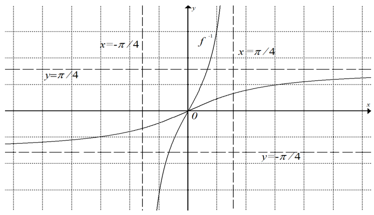
$u = \cos(2x)$

$\dfrac{du}{dx} = -2\sin(2x)$

$-\dfrac{1}{2}\times\dfrac{du}{dx} = \sin(2x)$

Q8bi $y = -\dfrac{\pi}{4}$, $y = \dfrac{\pi}{4}$

Q8bii



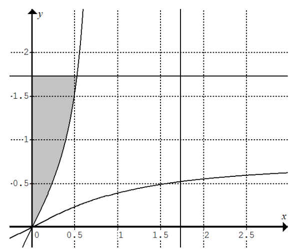
Q8c $f\left(\sqrt{3}\right) = \dfrac{1}{2}\arctan\left(\sqrt{3}\right) = \dfrac{1}{2}\times\dfrac{\pi}{3} = \dfrac{\pi}{6}$

Q8d Area of the required region $=$ area of the shaded region

$= \sqrt{3}\times\dfrac{\pi}{6} - \displaystyle\int_0^{\frac{\pi}{6}} \tan(2x)\,dx$

$= \dfrac{\sqrt{3}\,\pi}{6} - \left[\dfrac{1}{2}\log_e|\sec(2x)|\right]_0^{\frac{\pi}{6}}$

$= \dfrac{\sqrt{3}\,\pi}{6} - \log_e\sqrt{2}$



Q9a $x^2 - xy + \dfrac{3}{2}y^2 = 9$.: $2x - y - x\dfrac{dy}{dx} + 3y\dfrac{dy}{dx} = 0$

.: $2x - y = (x - 3y)\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{2x-y}{x-3y}$

Q9b At $(3,0)$, $m = \dfrac{dy}{dx} = 2$. Tangent: $y = 2(x-3)$, $y = 2x - 6$

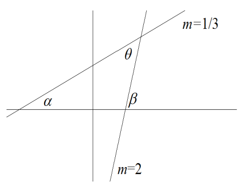
At $\left(0,\sqrt{6}\right)$, $m = \dfrac{1}{3}$. Tangent: $y = \dfrac{1}{3}x + \sqrt{6}$

Q9c $\beta = \tan^{-1}2$, $\alpha = \tan^{-1}\left(\frac{1}{3}\right)$, $\theta = \beta - \alpha = \tan^{-1}2 - \tan^{-1}\left(\frac{1}{3}\right)$

.: $\tan\theta = \tan\left(\tan^{-1}2 - \tan^{-1}\left(\frac{1}{3}\right)\right)$

$= \dfrac{\tan\left(\tan^{-1}2\right) - \tan\left(\tan^{-1}\left(\frac{1}{3}\right)\right)}{1 + \tan\left(\tan^{-1}2\right)\tan\left(\tan^{-1}\left(\frac{1}{3}\right)\right)} = \dfrac{2 - \frac{1}{3}}{1 + 2\times\frac{1}{3}} = 1$

.: $\theta = \frac{\pi}{4}$



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