Year 12 *Trial Exam Paper*

2015

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

SECTION 1

Question 1

Answer is C.

Worked solution

$$y = \frac{9}{x^2 + bx + c}$$
$$y = 9(x^2 + bx + c)^{-1}$$
$$\frac{dy}{dx} = -9(x^2 + bx + c)^{-2}(2x + b)$$
$$\frac{dy}{dx} = \frac{-9(2x + b)}{(x^2 + bx + c)^2}$$

When $x = \frac{5}{2}$, $y = -4$ and $\frac{dy}{dx} = 0$ gives

$$-4 = \frac{9}{(\frac{5}{2})^2 + b(\frac{5}{2}) + c} \quad \text{and} \quad 0 = \frac{-9(2(\frac{5}{2}) + b)}{((\frac{5}{2})^2 + b(\frac{5}{2}) + c)^2}$$

Solving simultaneously (using CAS or otherwise) gives $b = -5$ and $c = 4$.

Tip

- *It is easier if the derivative equation is first simplified to* $0 = 2\left(\frac{5}{2}\right) + b$ *before entering it into a CAS calculator.*

Question 2

Answer is E.

Worked solution

A domain of $[-6, 2]$ and a range of $[-4, 12]$ gives a centre of $(-2, 4)$ for this ellipse.
The semi-major axis is parallel to the *y*-axis and has a length of 8 units.
The semi-minor axis is parallel to the *x*-axis and has a length of 4 units.
Therefore, $a = 4$ and $b = 8$.
The equation of the ellipse is

$$\frac{(x+2)^2}{16} + \frac{(y-4)^2}{64} = 1$$
$$\frac{4(x+2)^2}{64} + \frac{(y-4)^2}{64} = 1$$
$$4(x^2 + 4x + 4) + (y^2 - 8y + 16) = 64$$
$$4x^2 + 16x + 16 + y^2 - 8y + 16 = 64$$
$$4x^2 + y^2 + 16x - 8y - 32 = 0$$

Question 3

Answer is E.

Worked solution

$\tan^2(2\theta)+1=\sec^2(2\theta)$

$\tan^2(2\theta)=(\frac{-5}{4})^2-1$

$\tan^2(2\theta)=\frac{25}{16}-1$

$\tan^2(2\theta)=\frac{9}{16}$

$\tan(2\theta)=\pm\frac{3}{4}$

Because $2\theta\in(\frac{\pi}{2},\pi)$, $\tan(2\theta)=\frac{-3}{4}$.

Now

$\tan(2\theta)=\frac{2\tan(\theta)}{1-\tan^2(\theta)}$

$\frac{-3}{4}=\frac{2\tan(\theta)}{1-\tan^2(\theta)}$

Because $\tan(\theta)=y$:

$\frac{-3}{4}=\frac{2y}{1-y^2}$

$-3(1-y^2)=8y$

$-3+3y^2=8y$

$3y^2-8y-3=0$

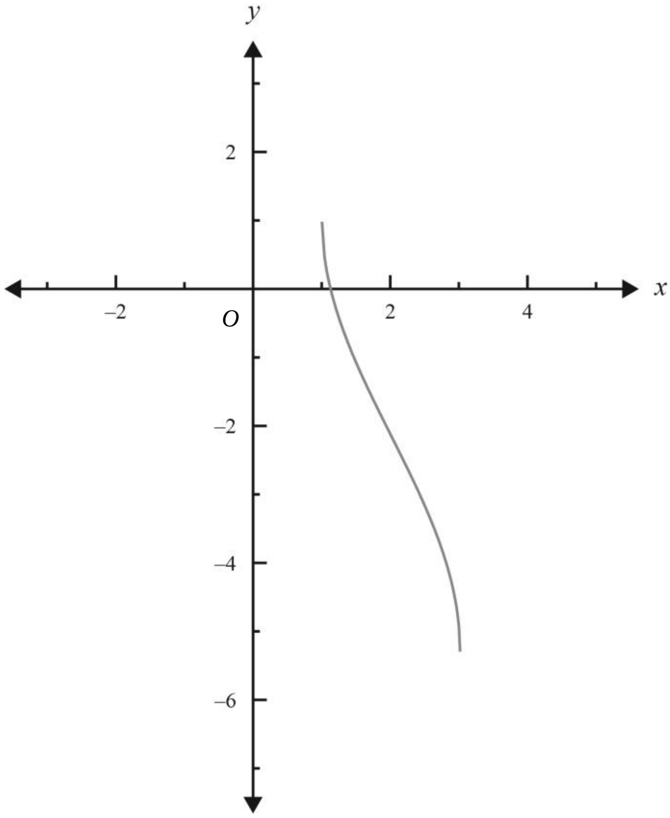
Question 4

Answer is B.

Worked solution

The domain of $y = \cos^{-1}(x)$ is [–1, 1].
The domain of $y = 1 - 2\cos^{-1}(2 - x)$ can be determined by solving

$$-1 \le (2 - x) \le 1$$
$$-3 \le -x \le -1$$
$$-x \ge -3 \quad \text{and} \quad -x \le -1$$
$$x \le 3 \qquad\qquad x \ge 1$$



Therefore, the maximal implied domain is [1, 3].

When $x = 1$:
$y = 1 - 2\cos^{-1}(1)$
$y = 1 - 0$
$y = 1$

When $x = 3$:
$y = 1 - 2\cos^{-1}(-1)$
$y = 1 - 2\pi$

∴ range is $[1 - 2\pi, 1]$.

Tip

- *It would be quicker to graph the function on a CAS calculator and use the* graph trace *function to determine the maximal domain and range.*

Question 5

Answer is A.

Worked solution

$$(a-bi)(2+i)=i^3$$
$$2a+ai-2bi+b=-i$$
$$(2a+b)+(a-2b)i=0-i$$

Hence:

$$2a+b=0$$
$$a-2b=-1$$

Solving simultaneously gives:

$$a=\frac{-1}{5},\ b=\frac{2}{5}$$

Question 6

Answer is C.

Worked solution

$$\frac{i\,[\text{Re}(z)-z]}{\text{Im}(z)}$$
$$=\frac{i\ [a-(a-bi)}{-b}$$
$$=\frac{i\ (a-a+bi)}{-b}$$
$$=\frac{bi^2}{-b}$$
$$=\frac{-b}{-b}$$
$$=1$$

Question 7

Answer is C.

Worked solution

$z^3 = -a$

Let $z = r\,\text{cis}(\theta)$, giving

$[r\,\text{cis}(\theta)]^3 = -a$

$$r^3\text{cis}(3\theta) = a\,\text{cis}(\pi)$$

$$r^3 = a$$

$$r = a^{\frac{1}{3}}$$

and

$3\theta = \pi + 2k\pi,\ k \in Z$

$$\theta = \frac{\pi + 2k\pi}{3}$$

When $k = 0,\ \theta = \dfrac{\pi}{3}$.

When $k = 1,\ \theta = \pi$.

When $k = -1,\ \theta = \dfrac{-\pi}{3}$.

The solutions are

$$z = a^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{3}\right)$$

$$z = a^{\frac{1}{3}}\text{cis}(\pi) = -a^{\frac{1}{3}}$$

$$z = a^{\frac{1}{3}}\text{cis}\left(\frac{-\pi}{3}\right)$$

Tip

- *The answer can be obtained also by realising that for all equations of the form* $z^3 = -a$, *all three solutions will be placed equally around a circle of radius* $|a|^{\frac{1}{3}}$.

Question 8

Answer is E.

Worked solution

$u = \frac{2}{v}$

$\therefore v = \frac{2}{u}$

$v = \frac{2\,\text{cis}(0)}{\text{cis}\left(\frac{\pi}{4}\right)}$

$v = \left(\frac{2}{1}\right)\text{cis}\left(0 - \frac{\pi}{4}\right)$

$v = 2\,\text{cis}\left(\frac{-\pi}{4}\right)$

$p = 2,\ \theta = \frac{-\pi}{4}$

Question 9

Answer is D.

Worked solution

Alternative **A** is a hyperbola and alternative **C** is a circle.
An expression of the form $|z+c|+|z-c| = 2a$, where $|a| > |c|$, represents an ellipse on a complex number plane. Only alternative **D** is of this form.

$|z+4|+|z-4| = 10$ can be converted to Cartesian form $\frac{x^2}{25}+\frac{y^2}{9}=1$, but this is a lengthy process and not recommended for a multiple-choice question. In fact, $|z+4|+|z-4| = 10$ can be described as a locus of points whereby the distance to the point (4, 0) plus the distance to the point (–4, 0) is a constant. These points represent the foci of the ellipse.

Tip

- *Students should be familiar with general equations to represent lines, rays, circles, ellipses and hyperbolas on a complex number plane.*

Question 10

Answer is E.

Worked solution

Let $u = \cos(2x)$

$\therefore \dfrac{du}{dx} = -2\sin(2x)$

$$\int_0^{\frac{\pi}{6}} \sin(4x)\cos(2x)dx$$

$$= \int_0^{\frac{\pi}{6}} 2\sin(2x)\cos(2x)\cos(2x)dx$$

$$= \int_0^{\frac{\pi}{6}} 2\sin(2x)\cos^2(2x)dx$$

$$= -\int_1^{\frac{1}{2}} \cos^2(2x)\frac{du}{dx}dx$$

$$= -\int_1^{\frac{1}{2}} u^2 du$$

$$= \int_{\frac{1}{2}}^{1} u^2 du$$

Question 11

Answer is C.

Worked solution

$$\frac{1-x^2}{(x-1)^2}$$

$$= \frac{-x^2 + 0x + 1}{x^2 - 2x + 1}$$

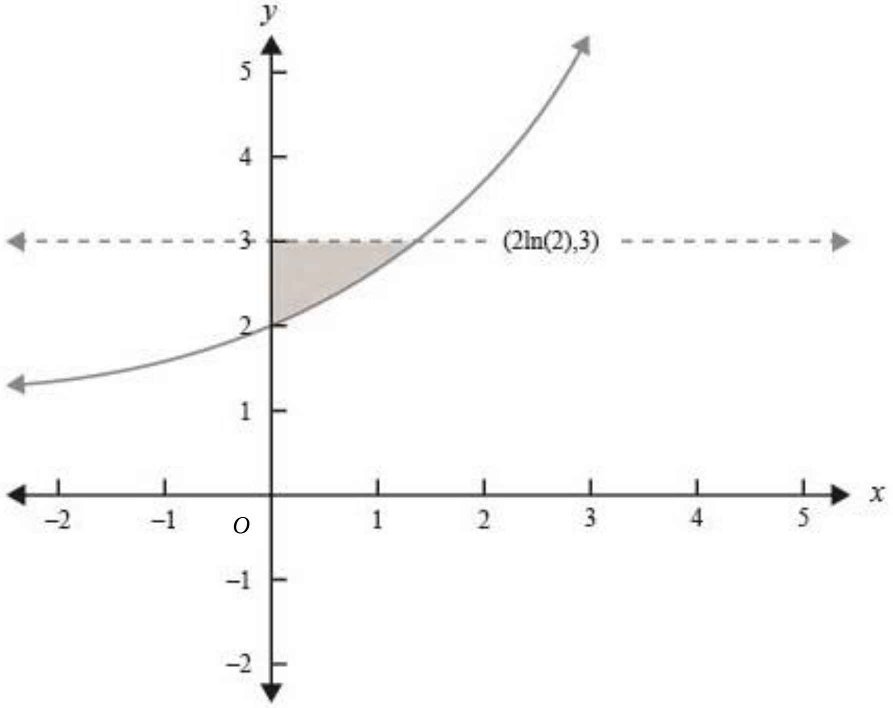
When dividing, this becomes $-1 + \dfrac{2-2x}{(x-1)^2}$ whereby the expression $\dfrac{2-2x}{(x-1)^2}$, with the repeated linear factor in the denominator, can be further divided as $\dfrac{A}{(x-1)^2} + \dfrac{B}{(x-1)}$.

Therefore, the expression $\dfrac{1-x^2}{(x-1)^2}$ can be written as $\dfrac{A}{(x-1)^2} + \dfrac{B}{x-1} - 1$.

Question 12

Answer is A.

Worked solution



$$y = 1 + e^{\frac{1}{2}x}$$

$$e^{\frac{1}{2}x} = y - 1$$

$$\frac{1}{2}x = \log_e(y-1)$$

$$x = 2\log_e(y-1)$$

$$A = \int_2^3 2\log_e(y-1)\,dy$$

Question 13

Answer is D.

Worked solution

$$y = 4e^{\frac{1}{2}x} - 4e^{-\frac{1}{2}x}$$

$$y' = 2e^{\frac{1}{2}x} + 2e^{-\frac{1}{2}x}$$

$$y'' = e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}$$

$$\Rightarrow y'' = \frac{1}{4}y$$

Question 14

Answer is E.

Worked solution

If $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$,
then, $\underset{\sim}{a} = \lambda \underset{\sim}{b}$.

$$5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} = \lambda(3m\underset{\sim}{i} - 6\underset{\sim}{j} + 6n\underset{\sim}{k})$$

$$5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} = 3\lambda m\underset{\sim}{i} - 6\lambda\underset{\sim}{j} + 6\lambda n\underset{\sim}{k}$$

$$\Rightarrow -6\lambda = -1$$

$$\therefore \lambda = \frac{1}{6}$$

$$3\lambda m = 5$$

$$\frac{1}{2}m = 5$$

$$m = 10$$

and

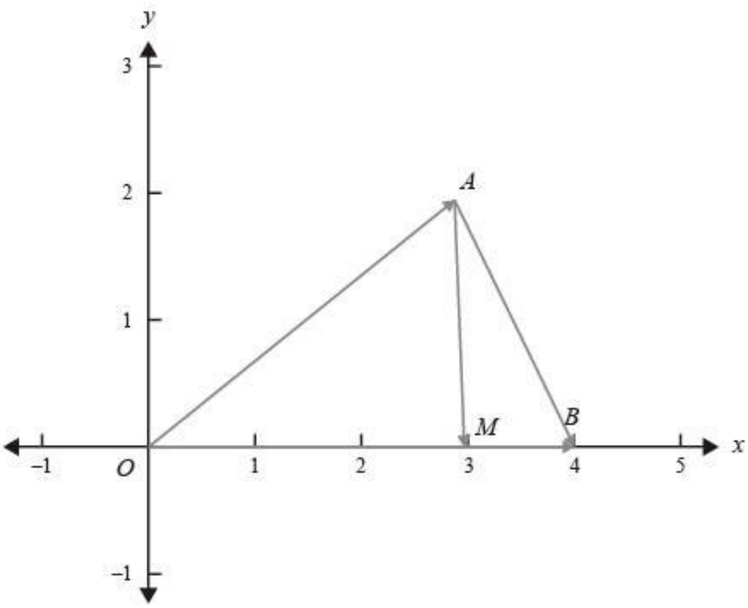
$$6\lambda n = 2$$

$$n = 2$$

Question 15

Answer is A.

Worked solution



$$\overrightarrow{AM} = -2\underset{\sim}{j}$$

$$\overrightarrow{AB} = \underset{\sim}{i} - 2\underset{\sim}{j}$$

$$\overrightarrow{AM}.\overrightarrow{AB} = \left|\overrightarrow{AM}\right|.\left|\overrightarrow{AN}\right|\cos(\theta)$$

$$4 = 2\sqrt{5}\cos(\theta)$$

$$\cos(\theta) = \frac{2}{\sqrt{5}}$$

$$\theta = 27^{\circ}$$

Question 16

Answer is C.

Worked solution

$$x = 35 + 20t - 5t^2$$
$$\therefore \frac{dx}{dt} = 20 - 10t$$
$$\therefore v = 20 - 10t$$

At the maximum height, v = 0.

$$20 - 10t = 0$$
$$t = 2$$

When t = 2:

$$x = 35 + 20(2) - 5(2)^2$$
$$x = 55$$

Therefore, the object is 20 m above the building when it reaches its maximum height of 55 m.

Question 17

Answer is D.

Worked solution

$$\underset{\sim}{r}(t) = 2\cos(3t)\underset{\sim}{i} - 3\sin(3t)\underset{\sim}{j}$$
$$\dot{\underset{\sim}{r}}(t) = -6\sin(3t)\underset{\sim}{i} - 9\cos(3t)\underset{\sim}{j}$$

When t = 0, $\dot{\underset{\sim}{r}}(t) = -9\underset{\sim}{j}$.

Speed = $\left|\dot{\underset{\sim}{r}}(t)\right|$

Speed = 9

Question 18

Answer is E.

Worked solution

$$a = v\frac{dv}{dx}$$
$$v = 5 - x$$
$$\frac{dv}{dx} = -1$$
$$a = -(5 - x)$$
$$a = x - 5$$

Question 19

Answer is E.

Worked solution

$$\frac{dV}{dt} = -\sqrt{h}$$
$$\therefore V = \pi a^2 h$$
$$\frac{dV}{dh} = \pi a^2$$
$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$
$$\frac{dh}{dt} = -\sqrt{h} \times \frac{1}{\pi a^2}$$
$$\frac{dh}{dt} = \frac{-\sqrt{h}}{\pi a^2}$$

Question 20

Answer is A.

Worked solution

$y(0) = 2$

$\Rightarrow x_0 = 0, y_0 = 2$

$y_{n+1} = y_n + hf(x_n, y_n)$ and $x_{n+1} = x_n + h$

When $x = 0.5$:

$y_1 = y_0 + 0.5 f(x_0, y_0)$

$y_1 = 2 + 0.5\left(\dfrac{0}{2}\right)$

$y_1 = 2$

When $x = 1$:

$y_2 = y_1 + 0.5 f(x_1, y_1)$ and $x_1 = x_0 + h = 0.5$

$y_2 = 2 + 0.5\left(\dfrac{0.5}{2}\right)$

$y_2 = 2.125$

Question 21

Answer is D.

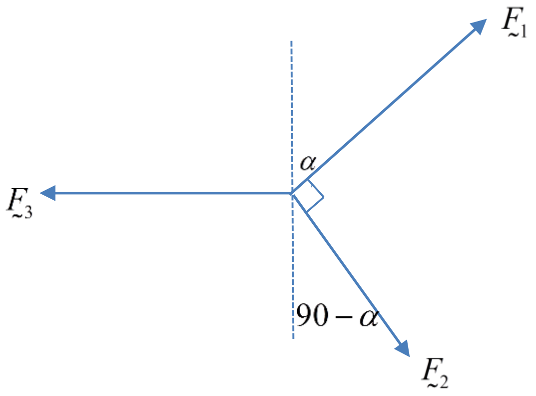
Worked solution

The weight force is Mg newton and acts in a direction vertically down.
The normal reaction is N newton and acts in a direction that is perpendicular to the surface of the plane.
The frictional force is μN newton and acts in a direction that is parallel to the plane opposing the direction of motion.

Question 22

Answer is C.

Worked solution



Alternative **A** is correct.

$$F_1\cos(\alpha) = F_2\cos(90-\alpha)$$
$$F_1\cos(\alpha) = F_2\sin(\alpha)$$

Alternative **B** is correct.

If the particle is in equilibrium, then

$$\sum \underset{\sim}{F} = \underset{\sim}{0}$$
$$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$$

Alternative **C** is incorrect.

From **A**,

$$F_1\cos(\alpha) = F_2\sin(\alpha)$$
$$\cot(\alpha) = \frac{F_2}{F_1}$$

Alternative **D** is correct.

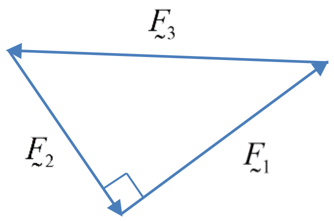
The net horizontal force is zero, so

$$F_3 = F_1\sin(\alpha) + F_2\sin(90-\alpha)$$
$$F_3 = F_1\sin(\alpha) + F_2\cos(\alpha)$$

Alternative **E** is correct.

If the particle is in equilibrium, then the vector sum forms a right-angled triangle, so

$$F_3^{\,2} = F_1^{\,2} + F_2^{\,2}$$



SECTION 2

Question 1a.

Worked solution

$x = 2\sec(\theta)$

$x = 2[\cos(\theta)]^{-1}$

$\frac{dx}{d\theta} = -2[\cos(\theta)]^{-2}[-\sin(\theta)]$

$\frac{dx}{d\theta} = \frac{2\sin(\theta)}{\cos^2(\theta)}$

$y = \tan(\theta)$

$\frac{dy}{d\theta} = \sec^2(\theta)$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos^2(\theta)}.\frac{\cos^2(\theta)}{2\sin(\theta)}$

$\frac{dy}{dx} = \frac{1}{2\sin(\theta)}$

When $\theta = \frac{\pi}{6}$, $x = \frac{4}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}}$.

When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = 1$.

The equation of the tangent is

$y - \frac{1}{\sqrt{3}} = 1\left(x - \frac{4}{\sqrt{3}}\right)$

$y = x - \frac{3}{\sqrt{3}}$

$y = x - \sqrt{3}$

Mark allocation: 4 marks

- 1 method mark for $\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx}$
- 1 mark for correctly obtaining $\frac{dy}{dx} = \frac{1}{2\sin(\theta)}$
- 1 mark for $m_T = 1$ and point $\left(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
- 1 mark for $y = x - \sqrt{3}$

Question 1b.

Worked solution

$$x = 2\sec(\theta)$$

$$\Rightarrow \sec(\theta) = \frac{x}{2}$$

$$y = \tan(\theta)$$

Now, $\tan^2(\theta) + 1 = \sec^2(\theta)$

$$y^2 + 1 = \left(\frac{x}{2}\right)^2$$

$$\frac{x^2}{4} - y^2 = 1$$

When $\theta = 0,\ x = 2 \Rightarrow a = 2.$

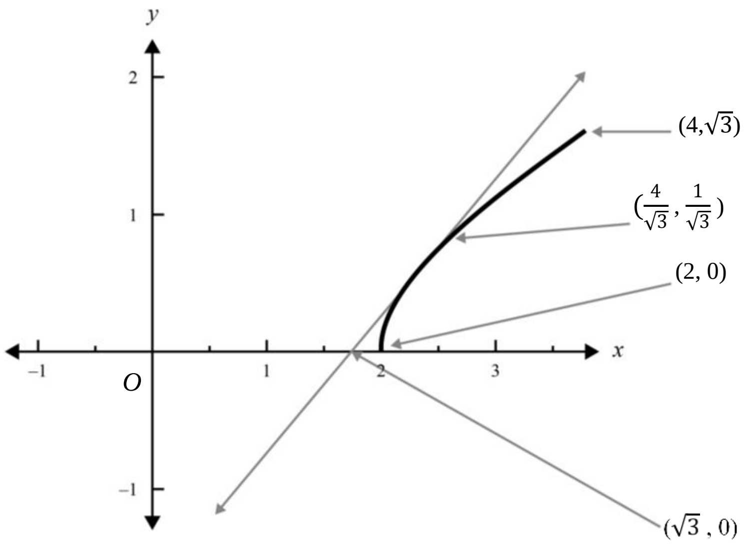
When $\theta = \frac{\pi}{3},\ x = 4 \Rightarrow b = 4.$

Mark allocation: 3 marks

- 1 method mark for using the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$
- 1 mark for showing the Cartesian equation is $\frac{x^2}{4} - y^2 = 1$
- 1 mark for finding $a = 2$ and $b = 4$

Question 1c.

Worked solution



Mark allocation: 2 marks

- 1 mark for the hyperbola with the endpoints (2, 0) and $(4, \sqrt{3})$ labelled clearly
- 1 mark for the tangent $y = x - \sqrt{3}$ with the points $\left(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $(\sqrt{3}, 0)$ labelled clearly

Question 1d.

Worked solution

$$V = \pi \int_{\sqrt{3}}^{\frac{4}{\sqrt{3}}} (x - \sqrt{3})^2 \, dx - \pi \int_{2}^{\frac{4}{\sqrt{3}}} \left(\frac{x^2}{4} - 1 \right) dx$$

$$= \frac{\pi\left(7\sqrt{3} - 12\right)}{9} \text{ cubic units}$$

Mark allocation: 2 marks

- 1 mark for $V = \pi \int_{\sqrt{3}}^{\frac{4}{\sqrt{3}}} (x - \sqrt{3})^2 \, dx - \pi \int_{2}^{\frac{4}{\sqrt{3}}} \left(\frac{x^2}{4} - 1 \right) dx$
- 1 mark for $\frac{\pi\left(7\sqrt{3} - 12\right)}{9}$ cubic units

Question 2a.

Worked solution

$$\frac{dN}{dt} = kN(1500 - N)$$

$$\frac{dt}{dN} = \frac{1}{kN(1500 - N)}$$

$$t = \frac{1}{k}\int \frac{1}{N(1500 - N)}\,dN$$

Resolving $\frac{1}{N(1500 - N)}$ into partial fractions:

$$\frac{1}{N(1500 - N)} = \frac{A}{N} + \frac{B}{1500 - N}$$

$$\frac{1}{N(1500 - N)} = \frac{A(1500 - N) + BN}{N(1500 - N)}$$

$$\frac{1}{N(1500 - N)} = \frac{1500A - AN + BN}{N(1500 - N)}$$

So $1500A = 1 \Rightarrow A = \frac{1}{1500}$

$-A + B = 0,\; B = \frac{1}{1500}$

$$t = \frac{1}{k}\int \frac{\frac{1}{1500}}{N} + \frac{\frac{1}{1500}}{1500 - N}\,dN$$

$$t = \frac{1}{1500k}\int \frac{1}{N} + \frac{1}{1500 - N}\,dN$$

$$t = \frac{1}{1500k}(\log_e|N| - \log_e|1500 - N|) + c$$

$$t = \frac{1}{1500k}\log_e\left|\frac{N}{1500 - N}\right| + c$$

When $t = 0$, $N = 1$.

$$0 = \frac{1}{1500k}\log_e\left(\frac{1}{1499}\right) + c$$

$$\Rightarrow c = \frac{-1}{1500k}\log_e\left(\frac{1}{1499}\right)$$

$$t = \frac{1}{1500k}\log_e\left|\frac{N}{1500-N}\right| - \frac{1}{1500k}\log_e\left(\frac{1}{1499}\right)$$

$$t = \frac{1}{1500k}\log_e\left|\frac{1499N}{1500-N}\right|$$

$$1500kt = \log_e\left|\frac{1499N}{1500-N}\right|$$

$$\frac{1499N}{1500-N} = e^{1500kt},$$ as t=0, N=1 is satisfied.

$$1499N = e^{1500kt}(1500-N)$$

$$1499N + Ne^{1500kt} = 1500e^{1500kt}$$

$$N(1499 + e^{1500kt}) = 1500e^{1500kt}$$

$$N = \frac{1500e^{1500kt}}{1499 + e^{1500kt}}$$

Therefore, A = 1500 and B = 1499.

Mark allocation: 5 marks

- 1 mark for inverting $\frac{dN}{dt} = kN(1500-N)$
- 1 mark for finding $A = \frac{1}{1500}$ and $B = \frac{1}{1500}$ by resolving $\frac{1}{N(1500-N)}$ into partial fractions
- 1 mark for correctly integrating to find $t = \frac{1}{1500k}\log_e\left|\frac{N}{1500-N}\right|$
- 1 mark for finding $c = \frac{-1}{1500k}\log_e\left(\frac{1}{1499}\right)$
- 1 mark for transposing to $N = \frac{1500e^{1500kt}}{1499 + e^{1500kt}}$ and stating that A = 1500 and B = 1499

Question 2b.

Worked solution

When $k = \dfrac{1}{3000}$, $t = 10$.

$$N = \frac{1500e^5}{1499 + e^5}$$

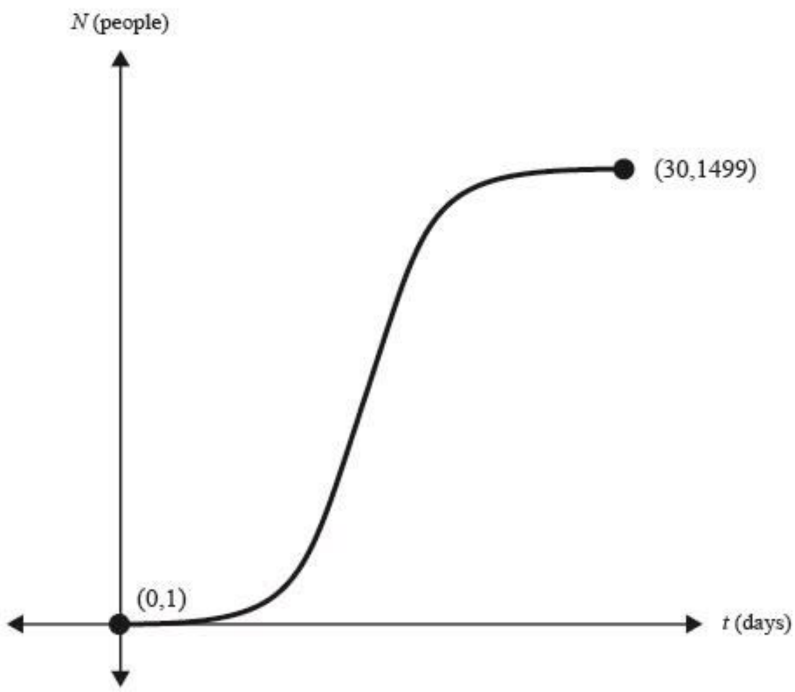
$N = 135$ people

Mark allocation: 1 mark

- 1 mark for $N = 135$ people

Question 2c.

Worked solution



Mark allocation: 2 marks

- 1 mark for graph
- 1 mark for correctly labelled endpoints

Question 3a.

Worked solution

$$(z-i)(\overline{z}+i)=4$$

$$z\overline{z}-i\overline{z}+iz-i^2=4$$

Let $z=x+yi,$ giving

$$(x+yi)(x-yi)-i(x-yi)+i(x+yi)+1=4$$

$$x^2+y^2-xi-y+xi-y+1=4$$

$$x^2+y^2-2y+1=4$$

$$x^2+(y-1)^2=4$$

Mark allocation: 2 marks

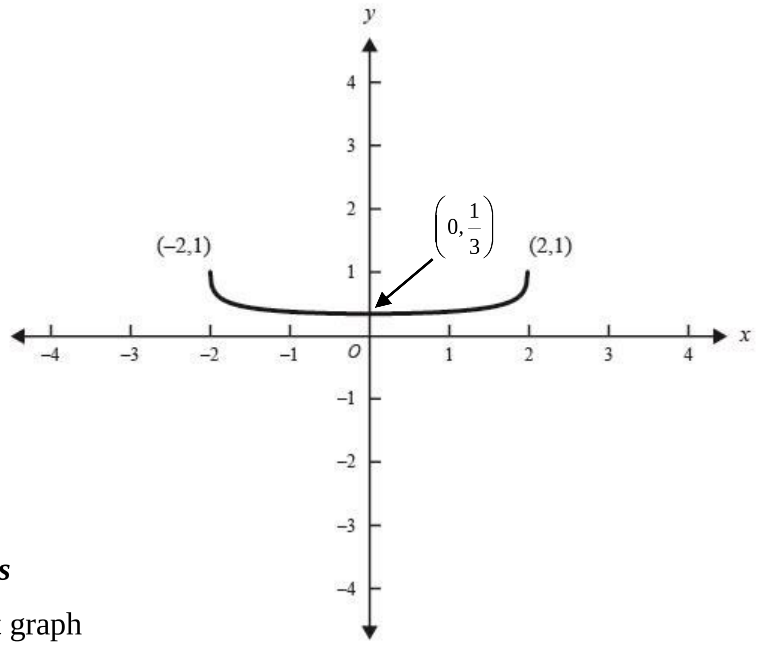
- 1 mark for correctly expanding
  $$(z-i)(\overline{z}+i)=4$$
  $$z\overline{z}-i\overline{z}+iz-i^2=4$$
- 1 mark for substituting $z=x+yi$ and simplifying to $x^2+(y-1)^2=4$

Question 3b.

Worked solution

The domain is [–2, 2].

The range is $\left[\frac{1}{3},1\right]$.



Mark allocation: 3 marks

- 1 mark for correct graph
- 1 mark for labelled endpoints and intercept
- 1 mark for domain and range

Question 3c.

Worked solution

$z+3=k(iz+i)$

Let $z=x+yi$, giving

$$x+yi+3=k[i(x+yi)+i]$$
$$x+yi+3=k(xi-y+i)$$
$$(x+3)+yi=kxi-ky+ki$$
$$(x+3)+yi=-ky+ik(x+1)$$

Equating real and imaginary coefficients:

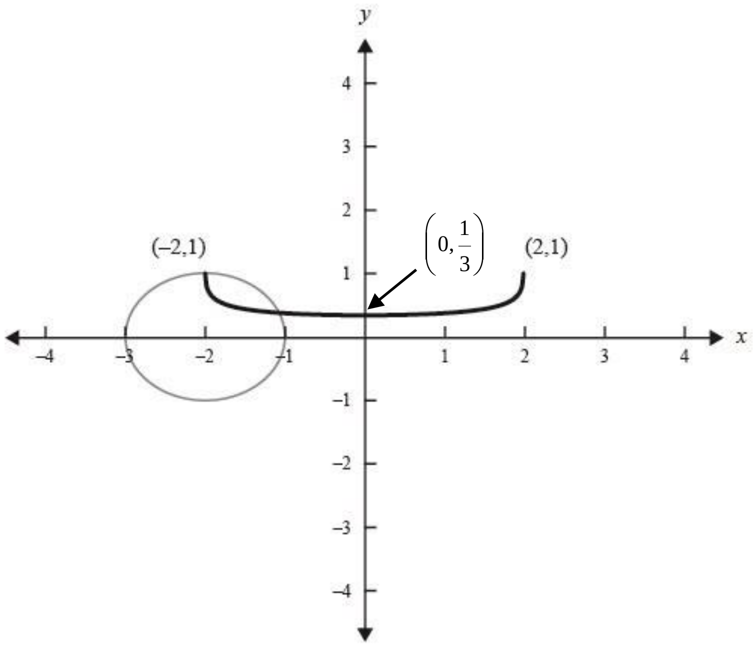
$x+3=-ky$ (1) and

$y=k(x+1)$ (2)

From (1): $k=\dfrac{x+3}{-y}$

Substitute into (2):

$$y=\frac{x+3}{-y}(x+1)$$
$$-y^2=(x+3)(x+1)$$
$$-y^2=x^2+4x+3$$
$$1-y^2=x^2+4x+4$$
$$1-y^2=(x+2)^2$$
$$(x+2)^2+y^2=1$$



Mark allocation: 4 marks

- 1 mark for obtaining $(x+3)+yi=-ky+ik(x+1)$
- 1 mark for equating real and imaginary coefficients $x+3=-ky$ and $y=k(x+1)$
- 1 mark for eliminating *k* to obtain $(x+2)^2+y^2=1$
- 1 mark for sketching graph of circle with centre (–2, 0) and radius of 1

Question 3d.

Worked solution

Solving $y=\dfrac{1}{1+\sqrt{4-x^2}}$ and $(x+2)^2+y^2=1$ using CAS gives $x=-2,\ x=-1.07174$

Area = $\displaystyle\int_{-2}^{-1.07174}\left(\sqrt{1-(x+2)^2}-\frac{1}{1+\sqrt{4-x^2}}\right)dx$

Area = 0.32

Mark allocation: 3 marks

- 1 mark for solving equations simultaneously to obtain terminals $x=-2,\ x=-1.07174$
- 1 mark for Area = $\displaystyle\int_{-2}^{-1.07174}(\sqrt{1-(x+2)^2}-\frac{1}{1+\sqrt{4-x^2}})dx$
- 1 mark for Area = 0.32

Question 4a.

Worked solution

$\underset{\sim}{v}_A = 4\underset{\sim}{i} + \underset{\sim}{j}$

The speed of A is:

$\left|\underset{\sim}{v}_A\right| = \sqrt{4^2 + 1^2}$

$\left|\underset{\sim}{v}_A\right| = \sqrt{17}$

$\underset{\sim}{v}_B = -8\alpha\cos(\alpha t)\underset{\sim}{i} - 8\alpha\sin(\alpha t)\underset{\sim}{j}$

The speed of B is:

$\left|\underset{\sim}{v}_B\right| = \sqrt{64\alpha^2\cos^2(\alpha t) + 64\alpha^2\sin^2(\alpha t)}$

$\left|\underset{\sim}{v}_B\right| = \sqrt{64\alpha^2(\cos^2(\alpha t) + \sin^2(\alpha t))}$

$\left|\underset{\sim}{v}_B\right| = \sqrt{64\alpha^2}$

$\left|\underset{\sim}{v}_B\right| = 8\alpha$ since $\alpha > 0$.

Mark allocation: 3 marks

- 1 mark for differentiating $\underset{\sim}{r}_A$ and $\underset{\sim}{r}_B$ correctly
- 1 mark for $\left|\underset{\sim}{v}_A\right| = \sqrt{17}$
- 1 mark for $\left|\underset{\sim}{v}_B\right| = 8\alpha$

Question 4b.

Worked solution

Ant (A)

$x = 4t,\ y = t,\ t \geq 0$

$y = \dfrac{x}{4},\ x \geq 0$

Beetle (B)

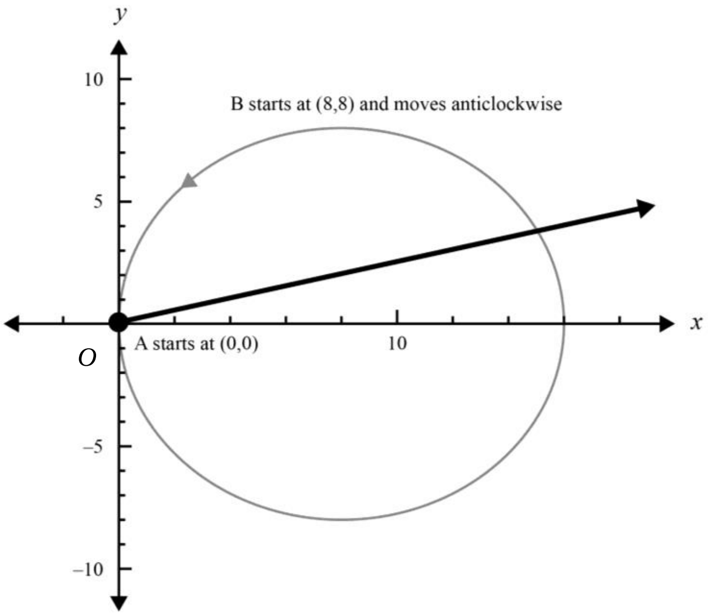
$x = 8 - 8\sin(\alpha t),\ y = 8\cos(\alpha t),\ t \geq 0$

$\sin(\alpha t) = \dfrac{8 - x}{8},\ \cos(\alpha t) = \dfrac{y}{8}$

$$\sin^2(\alpha t) + \cos^2(\alpha t) = 1$$

$$\frac{(8 - x)^2}{64} + \frac{y^2}{64} = 1$$

$(x - 8)^2 + y^2 = 64,\ 0 \leq x \leq 16$



Mark allocation: 5 marks

- 1 mark for the Cartesian equation of the ant's path: $y = \dfrac{x}{4},\ x \geq 0$
- 1 mark for the Cartesian equation of the beetle's path: $(x - 8)^2 + y^2 = 64,\ 0 \leq x \leq 16$
- 1 mark for the graph of the ant's path: $y = \dfrac{x}{4},\ x \geq 0$
- 1 mark for the graph of the beetle's path: $(x - 8)^2 + y^2 = 64,\ 0 \leq x \leq 16$
- 1 mark for indicating the starting position and direction of motion for both insects

Question 4c.

Worked solution

$y=\frac{x}{4}, x\geq 0$ intersecting $(x-8)^2+y^2=64,\ 0\leq x\leq 16$

$$(x-8)^2+\left(\frac{x}{4}\right)^2=64$$

$$x^2-16x+64+\frac{x^2}{16}=64$$

$$16x^2-256x+x^2=0$$

$$17x^2-256x=0$$

$$x(17x-256)=0$$

$$x=0,\ x=\frac{256}{17}$$

When $x=0,\ y=0.$

When $x=\frac{256}{17},\ y=\frac{64}{17}.$

The points of intersection are $(0, 0)$ and $(\frac{256}{17}, \frac{64}{17})$.

Mark allocation: 2 marks

- 1 mark for solving equations simultaneously to obtain $x=0,\ x=\frac{256}{17}$
- 1 mark for finding coordinates of intersecting points $(0, 0)$ and $\left(\frac{256}{17}, \frac{64}{17}\right)$

Question 4d.

Worked solution

The insects cannot collide at (0, 0) because the ant starts at (0, 0) and the beetle starts at (8, 8).

If the insects collide at $\left(\frac{256}{17}, \frac{64}{17}\right)$, they must be there at the same time.

The ant (A) is at $\left(\frac{256}{17}, \frac{64}{17}\right)$ when $t = \frac{64}{17}$.

For the beetle (B):

$$\frac{256}{17} = 8 - 8\sin(\alpha t),\ \frac{64}{17} = 8\cos(\alpha t),\ t \geq 0$$

$$\sin(\alpha t) = \frac{\left(8 - \frac{256}{17}\right)}{8},\ \cos(\alpha t) = \frac{\left(\frac{64}{17}\right)}{8}$$

When $t = \frac{64}{17}$:

$$\sin\left(\frac{64\alpha}{17}\right) = \frac{\left(8 - \frac{256}{17}\right)}{8}$$

$$\sin\left(\frac{64\alpha}{17}\right) = \frac{-15}{17} \qquad (1)$$

$$\cos\left(\frac{64\alpha}{17}\right) = \frac{\left(\frac{64}{17}\right)}{8}$$

$$\cos\left(\frac{64\alpha}{17}\right) = \frac{8}{17} \qquad (2)$$

Using CAS to solve equation (1) for positive values of α:

$\alpha = 1.12, 1.38, 2.79, 3.05\ ...$

Using CAS to solve equation (2) for positive values of α:

$\alpha = 0.29, 1.38, 1.96, 3.05\ ...$

The smallest positive value of α that satisfies both equations (1) and (2) is 1.38.

Mark allocation: 4 marks

- 1 mark for finding that when A is at $\left(\frac{256}{17},\frac{64}{17}\right)$, $t = \frac{64}{17}$
- 1 mark for substituting into $\sin(\alpha t)=\frac{8-x}{8}$, $\cos(\alpha t)=\frac{y}{8}$ to obtain

  $\sin\left(\frac{64\alpha}{17}\right)=\frac{-15}{17}$ (1)

  $\cos\left(\frac{64\alpha}{17}\right)=\frac{8}{17}$ (2)
- 1 mark for solving equations (1) and (2) to obtain positive solutions for α:
- 1 mark for finding that the smallest positive value of α is 1.38

Question 4e.

Worked solution

The insects collide when $t = \frac{64}{17}$.

Distance travelled = speed of insect × time taken

For the ant (A)

Distance = $\sqrt{17} \times \frac{64}{17} = 15.5$

For the beetle (B):

Distance = $8\alpha \times \frac{64}{17}$

Distance = 41.6

Mark allocation: 2 marks

- 1 mark for 15.5
- 1 mark for 41.6

Question 5a.

Worked solution

For the *M* kg object:
$T - Mg = Ma$ (1)

For the 2*M* kg object:
$2Mg - T = 2Ma$ (2)

Adding equations (1) and (2) gives:
$Mg = 3Ma$

$$a = \frac{g}{3}$$

Mark allocation: 2 marks

- 1 mark for $T - Mg = Ma$ and $2Mg - T = 2Ma$
- 1 mark for showing that $a = \frac{g}{3}$ ms^{-2}

Question 5b.

Worked solution

Consider the motion of the smaller object before the string breaks (upwards is positive)

$u = 0,\ t = 3,\ a = \frac{g}{3},\ s = ?$

$s = ut + \frac{1}{2}at^2$

$s = 0 + \left(\frac{1}{2} \times \frac{g}{3} \times 3^2\right)$

$s = 14.7$

When the string breaks, the velocity of the smaller object is

$v = u + at$

$v = 0 + \left(\frac{g}{3} \times 3\right)$

$v = 9.8$

Consider the motion of the smaller object when the string breaks to now determine how long it takes to return to its original position.

$u = 9.8,\ t = ?,\ s = -14.7,\ a = -g(-9.8)$

$$s = ut + \frac{1}{2}at^2$$

$$-14.7 = 9.8t - \left(\frac{1}{2} \times 9.8 \times t^2\right)$$

$$-14.7 = 9.8t - 4.9t^2$$

$$-147 = 98t - 49t^2$$

$$-3 = 2t - t^2$$

$$t^2 - 2t - 3 = 0$$

$$(t-3)(t+1) = 0$$

$$t = 3 \text{ or } t = -1$$

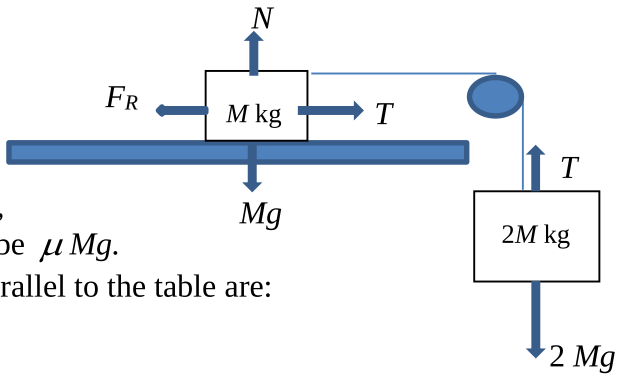
Because $t \geq 0$, it takes 3 seconds for the smaller object to return to its original position.

Mark allocation: 3 marks

- 1 mark for finding the distance travelled by the smaller object before the string breaks
- 1 mark for finding the velocity of the smaller object when the string breaks
- 1 mark for determining the time taken for the smaller object to return to its original position

Question 5c.

Worked solution



Consider the motion of the smaller object.

Resolving the forces perpendicular to the table $N = Mg$,
the frictional force F_R on the smaller mass would then be μMg.
If the tension in the string is T, then resolving forces parallel to the table are:

$$T - \mu Mg = Ma \qquad (1)$$

Consider the motion of the larger object.

$$2Mg - T = 2Ma \qquad (2)$$

Adding equations (1) and (2) gives:

$$2Mg - \mu Mg = 3Ma$$
$$Mg(2-\mu) = 3Ma$$
$$a = \frac{g(2-\mu)}{3}$$

Substituting into equation (1) gives:

$$T - \mu Mg = \frac{Mg(2-\mu)}{3}$$
$$3T - 3\mu Mg = Mg(2-\mu)$$
$$3T = 2Mg - \mu Mg + 3\mu Mg$$
$$3T = 2Mg(1+\mu)$$
$$T = \frac{2Mg(1+\mu)}{3}$$

Mark allocation: 4 marks

- 1 mark for resolving all forces on the smaller object to obtain $T - \mu Mg = Ma$
- 1 mark for resolving forces on the larger object to obtain $2Mg - T = 2Ma$
- 1 mark for solving equations simultaneously to obtain $a = \frac{g(2-\mu)}{3}$
- 1 mark for finding $T = \frac{2Mg(1-\mu)}{3}$

Question 5d.

Worked solution

The system will not move if $a \le 0$.

$$\frac{g(2-\mu)}{3} \le 0$$

$$2-\mu \le 0$$

$$\mu \ge 2$$

$$\therefore k = 2$$

Mark allocation: 2 marks

- 1 method mark for determining $a \le 0$
- 1 mark for showing $k = 2$

END OF SOLUTIONS BOOK