YEAR 12 *Trial Exam Paper*

2015

SPECIALIST MATHEMATICS

Written examination 1

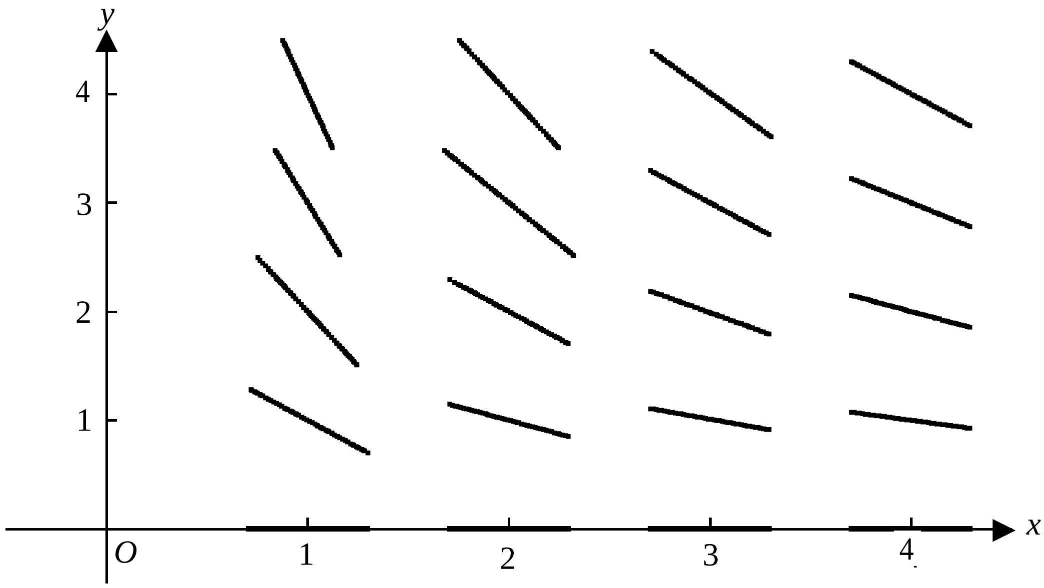
Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

Question 1a.

Worked solution

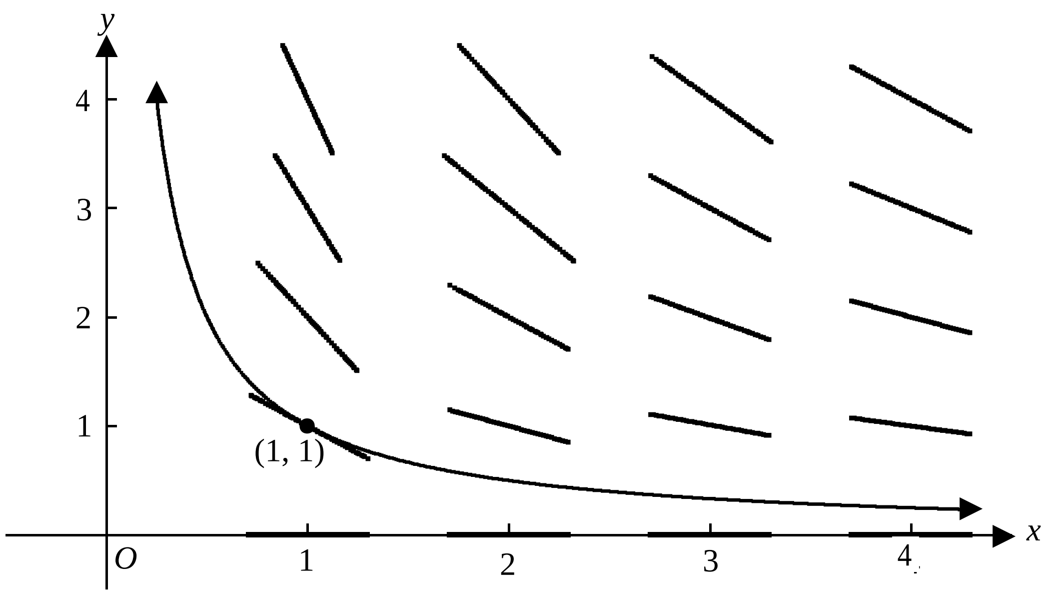


Mark allocation: 1 mark

- 1 mark for the four correct tangents

Question 1b.

Worked solution



Mark allocation: 1 mark

- 1 mark for the correct curve passing through the point (1, 1)

Question 2a.

Worked solution

$z^4 + 3z^2 - 4 = 0$

$\Rightarrow (z^2 + 4)(z^2 - 1) = 0$

$\Rightarrow (z^2 - 4i^2)(z^2 - 1) = 0$

$\Rightarrow (z - 2i)(z + 2i)(z - 1)(z + 1) = 0$

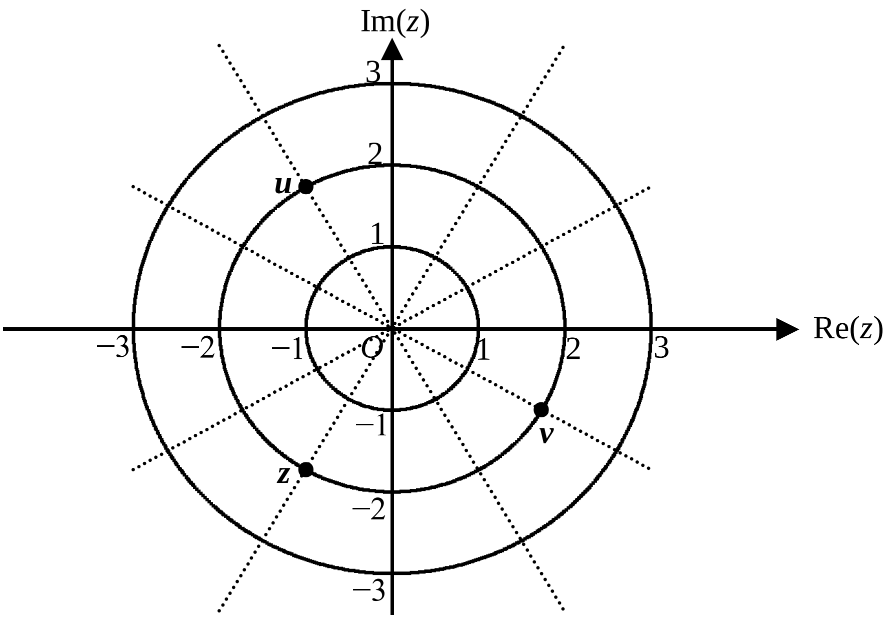
$\therefore z = \pm 2i, \pm 1$

Mark allocation: 3 marks

- 1 mark for factorising as two quadratic expressions
- 1 mark for factorising as four linear expressions
- 1 mark for the correct solutions

Question 2b.

Worked solution



Mark allocation: 1 mark

- 1 mark for the correct answer

Tip

- *The point z is obtained by reflecting $u = \bar{z}$ through the real axis. Then, the point $v = iz$ is obtained by rotating z 90° anti-clockwise.*

Question 2c.

Worked solution

$$u = 2cis\frac{2\pi}{3}$$

$$\sqrt{u} = \left(2cis\frac{2\pi}{3}\right)^{\frac{1}{2}}$$

$$= \sqrt{2}cis\left(\frac{\pi}{3}\right) = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}$$

Mark allocation: 2 marks

- 1 mark for obtaining $\sqrt{u}$ correctly in polar form
- 1 mark for the correct answer

Question 3

Worked solution

$x^2y^2 - 4y - \log_e(x-1)^3 = 8$

$\frac{d}{dx}(x^2y^2) - \frac{d}{dx}(4y) - \frac{d}{dx}\left(\log_e(x-1)^3\right) = \frac{d}{dx}(8)$

or $\frac{d}{dx}(x^2y^2) - \frac{d}{dx}(4y) - 3.\frac{d}{dx}\left(\log_e(x-1)\right) = \frac{d}{dx}(8)$

$\Rightarrow 2xy^2 + x^2 2y\frac{dy}{dx} - 4\frac{dy}{dx} - \frac{3}{x-1} = 0$

$\frac{dy}{dx}\left(2x^2y - 4\right) = \frac{3}{x-1} - 2xy^2$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{3}{x-1} - 2xy^2}{2x^2y - 4}$

Substitute $(2,\ 2)$

$\frac{dy}{dx} = \frac{3-16}{12} = -\frac{13}{12}$, which is the gradient of the tangent at $x = 2$.

$\therefore$ the gradient of the normal is $\frac{12}{13}$.

Mark allocation: 3 marks

- 1 mark for implicitly differentiating the relation correctly
- 1 mark for the correct gradient function
- 1 mark for the correct answer

Tip

- *The relation cannot be explicitly expressed as a function of x, so implicit differentiation is required to obtain* $\frac{dy}{dx}$.

Question 4

Worked solution

$$\tan(x) = \sin(2x)$$

$$\Rightarrow \frac{\sin(x)}{\cos(x)} = \sin(2x)$$

$$\Rightarrow \frac{\sin(x)}{\cos(x)} = 2\sin(x)\cos(x)$$

$$\Rightarrow \sin(x) = 2\sin(x)\cos^2(x)$$

$$\Rightarrow 2\sin(x)\cos^2(x) - \sin(x) = 0$$

$$\sin(x)(2\cos^2(x) - 1) = 0$$

$$\Rightarrow \sin(x) = 0 \text{ or } \cos^2(x) = \frac{1}{2}$$

$$\Rightarrow \sin(x) = 0 \text{ or } \cos(x) = \pm\frac{1}{\sqrt{2}}$$

$$\therefore\ x = n\pi \text{ or } x = \frac{(2n+1)\pi}{4},\ n \in Z$$

Mark allocation: 3 marks

- 1 mark for obtaining a correct equation in terms of sin(x) and cos(x)
- 1 mark for obtaining correct equations, one in terms of sin(x) only and the other in terms of cos(x) only
- 1 mark for the correct solutions

Question 5

Worked solution

$$f(x) = \arcsin\left(\frac{x}{2}\right)$$

$$f'(x) = \frac{1}{\sqrt{4-x^2}}$$

$$= \left(4-x^2\right)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}(-2x)\left(4-x^2\right)^{-\frac{3}{2}}$$

$$= \frac{x}{\left(4-x^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow f''(\sqrt{3}) = \frac{\sqrt{3}}{1^{\frac{3}{2}}} = \sqrt{3}$$

Mark allocation: 3 marks

- 1 mark for the correct first derivative, $f'(x)$
- 1 mark for the correct second derivative, $f''(x)$
- 1 mark for the correct answer

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Question 6a.

Worked solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} - \left(\underset{\sim}{i} - \underset{\sim}{k}\right)$$

$$= \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$$

Mark allocation

- 1 mark for the correct answer

Question 6b.

Worked solution

$$\overrightarrow{AB} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$$

$$\overrightarrow{AB} \cdot \overrightarrow{OB} = 2 + 1 + 2 = 5$$

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$$

$$\left|\overrightarrow{OB}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{OB}}{\left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{OB}\right|} = \frac{5}{\sqrt{6}.\sqrt{6}} = \frac{5}{6}$$

Mark allocation: 2 marks

- 1 mark for the correct values of $\overrightarrow{AB}.\overrightarrow{OB}, \left|\overrightarrow{AB}\right|$ and $\left|\overrightarrow{OB}\right|$
- 1 mark for correct evaluation of cos θ

Question 6c.

Worked solution

$$\text{Area} = \frac{1}{2} \times \left|\overrightarrow{AB}\right| \times \left|\overrightarrow{OB}\right| \times \sin\theta$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\left(\frac{5}{6}\right)^2} = \sqrt{1-\frac{25}{36}} = \sqrt{\frac{11}{36}} = \frac{\sqrt{11}}{6}$$

$$\therefore \text{ Area} = \frac{1}{2} \times \sqrt{6} \times \sqrt{6} \times \frac{\sqrt{11}}{6}$$

$$= \frac{\sqrt{11}}{2} \text{ square units}$$

Mark allocation: 2 marks

- 1 mark for the correct value of $\sin\theta$
- 1 mark for the correct answer

Question 7a.

Worked solution

$$\int_{e^{\frac{-\pi}{2}}}^{e^{\frac{\pi}{2}}} \frac{1}{x\sec(\log_e x)}\,dx$$

$$= \int_{e^{\frac{-\pi}{2}}}^{e^{\frac{\pi}{2}}} \frac{\cos(\log_e x)}{x}\,dx$$

Let $u = \log_e x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$x = e^{\frac{\pi}{2}} \Rightarrow u = \log_e e^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$x = e^{\frac{-\pi}{2}} \Rightarrow u = \log_e e^{\frac{-\pi}{2}} = \frac{-\pi}{2}$$

$$\therefore \int_{e^{\frac{-\pi}{2}}}^{e^{\frac{\pi}{2}}} \frac{\cos(\log_e x)}{x}\,dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(u)\frac{du}{dx}\,dx$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(u)\,du$$

$$= \left[\sin u\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)$$

$$= 1 - (-1) = 2$$

Mark allocation: 3 marks

- 1 mark for the correct substitution
- 1 mark for the correct integrand in terms of *u*
- 1 mark for the correct answer

Tip

- *To anti-differentiate, use the substitution $u = \log_e x$ because its derivative $\frac{1}{x}$ is a factor of the integrand.*

Question 7b.

Worked solution

$$y_1 = y_0 + h \times f(x_0, y_0)$$

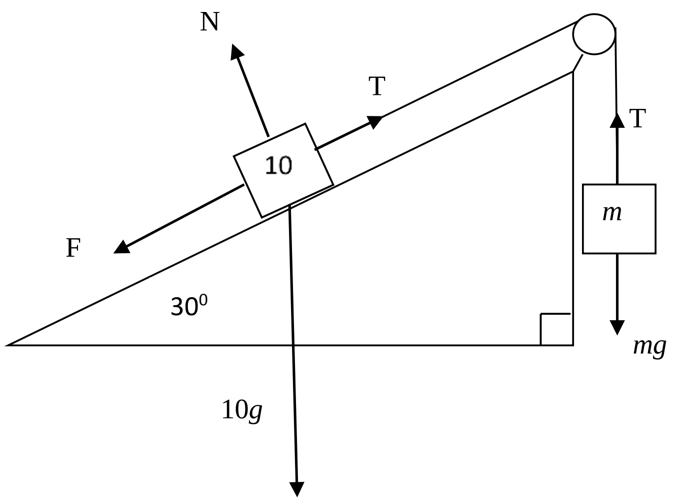
$$y_1 = 1 + 0.2 \times \frac{-2}{2^2} = 1 - 0.1 = 0.9$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 8a.

Worked solution

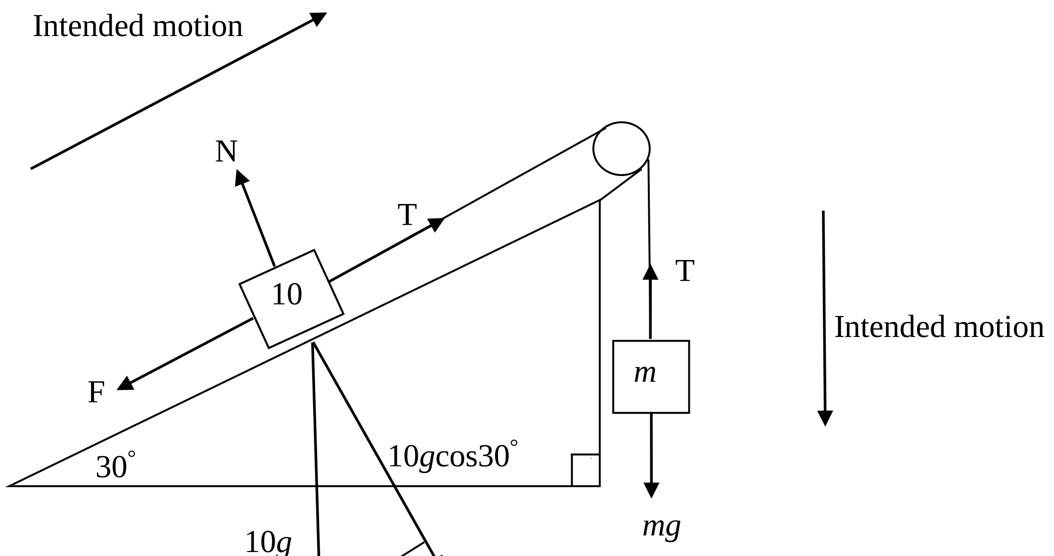


Mark allocation: 1 mark

- 1 mark for showing the four forces acting on the 10 kg mass and the two forces acting on mass *m*

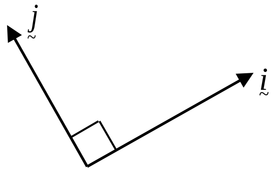
Question 8b.

Worked solution



Forces acting on mass *m*:

$$R = mg - T = 0$$
$$\Rightarrow T = mg$$



Forces acting on mass of 10 kg:

$$\underset{\sim}{R} = (T - F - 10g\sin 30°)\underset{\sim}{i} + (N - 10g\cos 30°)\underset{\sim}{j} = \underset{\sim}{0}$$

$$\Rightarrow (T - F - 5g)\underset{\sim}{i} + (N - 5\sqrt{3}g)\underset{\sim}{j} = \underset{\sim}{0}$$

$$N - 5\sqrt{3}g = 0$$
$$\Rightarrow N = 5\sqrt{3}g$$

$\therefore$ Friction, $F = N\mu = 5\sqrt{3}g \times \dfrac{1}{\sqrt{3}} = 5g$

$$T - F - 5g = 0$$
$$\Rightarrow mg - 5g - 5g = mg - 10g = 0$$
$$\Rightarrow (m - 10)g = 0$$
$$\Rightarrow m = 10$$

$\therefore$ 10 kg is the maximum value of *m* for which the 10 kg mass will not move up the incline.

Mark allocation: 3 marks

- 1 mark for correctly resolving the forces acting on the mass *m* and the 10 kg mass
- 1 mark for correctly calculating the friction force
- 1 mark for the correct answer

Tip

- *The maximum value of m occurs when the 10 kg mass is on the verge of moving up the plane and the friction is acting down the plane.*

Question 9a.

Worked solution

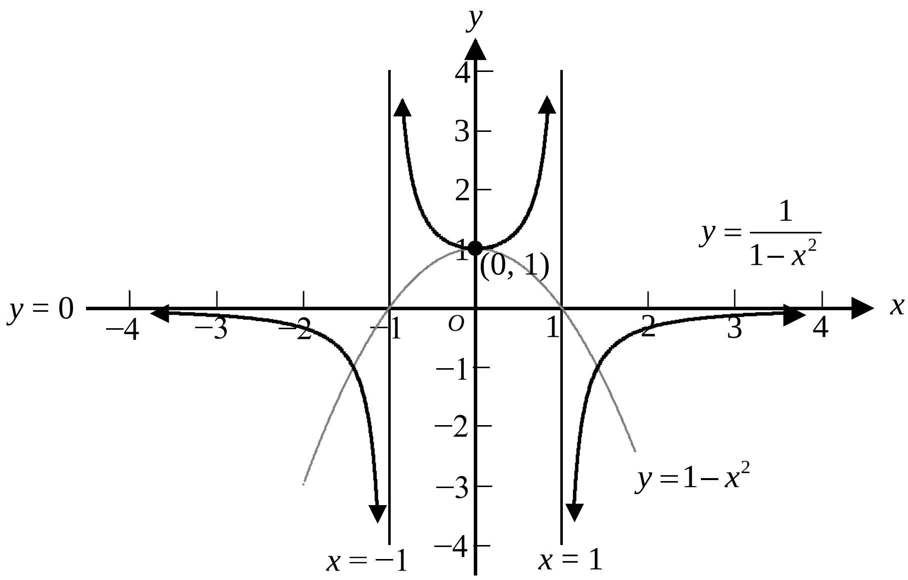
$$\frac{1}{2(1-x)}+\frac{1}{2(1+x)}=\frac{2(1+x)+2(1-x)}{4(1-x^2)}$$
$$=\frac{4}{4(1-x^2)}=\frac{1}{1-x^2}$$

Mark allocation: 1 mark

- 1 mark for correctly expressing the two rational expressions with a common denominator

Question 9b.

Worked solution



Mark allocation: 2 marks

- 1 mark for the correct asymptotes y = 0, x = -1 and x = 1 and the y-intercept (0, 1)
- 1 mark for the correct curve $y=\frac{1}{1-x^2}$

Tip

- *Sketch the graph of* $y=1-x^2$*, and then obtain the graph of its reciprocal* $y=\frac{1}{1-x^2}$.

Question 9c.

Worked solution

$$\text{Volume} = \pi\int_{-3}^{-1} x^2 dy$$

$$y = \frac{1}{1-x^2}$$

$$\Rightarrow 1 - x^2 = \frac{1}{y}$$

$$\therefore x^2 = 1 - \frac{1}{y}$$

$$\text{Volume} = \pi\int_{-3}^{-1}\left(1 - \frac{1}{y}\right)dy$$

$$= \pi\left[y - \log_e|y|\right]_{-3}^{-1}$$

$$= \pi\left[-1 - \log_e 1 - (-3 - \log_e 3)\right]$$

$$= \pi(2 + \log_e 3) \text{ cubic units}$$

Mark allocation: 3 marks

- 1 mark for the correct integrand representing the volume required
- 1 mark for correctly anti-differentiating
- 1 mark for the correct answer

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Question 10a.

Worked solution

$\underset{\sim}{r}(t) = e^t \underset{\sim}{i} + \log_e(t+1)\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{v}(t) = e^t \underset{\sim}{i} + \dfrac{1}{t+1}\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{v}(0) = \underset{\sim}{i} + \underset{\sim}{j}$

$\therefore v(0) = \sqrt{2}$

Mark allocation: 2 marks

- 1 mark for the correct velocity vector
- 1 mark for the correct answer

Question 10b.

Worked solution

$\underset{\sim}{v}(t) = e^t \underset{\sim}{i} + \dfrac{1}{t+1}\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{a}(t) = e^t \underset{\sim}{i} - \dfrac{1}{(t+1)^2}\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{a}(0) = \underset{\sim}{i} - \underset{\sim}{j}$

$\Rightarrow \underset{\sim}{v}(0) \cdot \underset{\sim}{a}(0) = (\underset{\sim}{i} + \underset{\sim}{j}) \cdot (\underset{\sim}{i} - \underset{\sim}{j}) = 1 - 1 = 0$

$\therefore$ the initial acceleration of the particle is perpendicular to its initial velocity.

Mark allocation: 2 marks

- 1 mark for the correct acceleration vector
- 1 mark for showing that the dot product of the initial acceleration vector and the initial velocity vector equals zero

END OF SOLUTIONS BOOK