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SPECIALIST MATHS TRIAL EXAMINATION 2 SOLUTIONS 2015

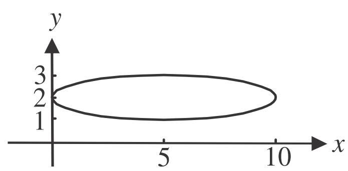
SECTION 1 – Multiple-choice answers

1.	A	7.	E	13.	A	19.	E
2.	C	8.	B	14.	D	20.	A
3.	E	9.	E	15.	D	21.	D
4.	C	10.	B	16.	C	22.	E
5.	B	11.	B	17.	D		
6.	C	12.	C	18.	D		

SECTION 1 - Multiple-choice solutions

Question 1

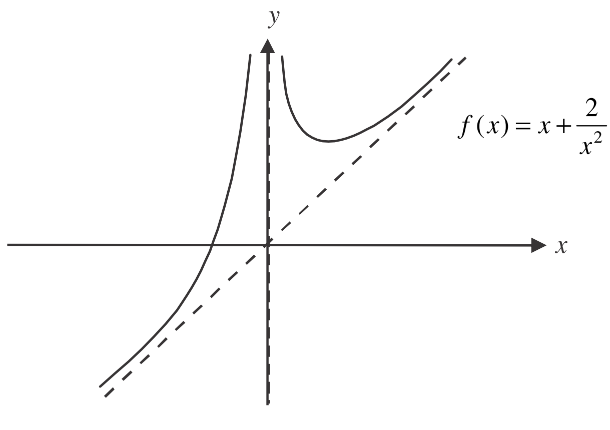
$$x^2 - 10x + 25y^2 - 100y + 100 = 0$$
$$x^2 - 10x + 25 - 25 + 25(y^2 - 4y + 4) = 0$$
$$(x-5)^2 + 25(y-2)^2 = 25$$
$$\frac{(x-5)^2}{25} + \frac{(y-2)^2}{1} = 1$$



The only axis intercept occurs at $(0,2)$.
The answer is A.

Question 2

The graph of $f(x) = x + \frac{2}{x^2}$ has two asymptotes, one with equation $y = x$ and the other with equation $x = 0$. Immediately we see that option C cannot be true as both the asymptotes are straight lines.



The answer is C.

Question 3

$$x = 3\sec(\theta) - 1 \qquad y = \frac{4}{\cot(\theta)}$$
$$\frac{x+1}{3} = \sec(\theta) \qquad \frac{y}{4} = \tan(\theta)$$
$$\frac{(x+1)^2}{9} = \sec^2(\theta) \qquad \frac{y^2}{16} = \tan^2(\theta)$$
$$1 + \tan^2(\theta) = \sec^2(\theta) \qquad \text{(formula sheet)}$$
$$1 + \frac{y^2}{16} = \frac{(x+1)^2}{9}$$
$$\frac{(x+1)^2}{9} - \frac{y^2}{16} = 1$$

The answer is E.

Question 4

For $f(x) = a\sin^{-1}(bx+1)$

$$-1 \le bx + 1 \le 1$$
$$-2 \le bx \le 0$$
$$\frac{-2}{b} \le x \le 0$$

Since $d_f = \left[-\frac{2}{b}, 0\right]$ and $d_f = [-6, 0]$,

then $-\frac{2}{b} = -6$

$$b = \frac{1}{3}$$

Also $-\frac{\pi}{2} \le \frac{y}{a} \le \frac{\pi}{2}$

$$-\frac{\pi a}{2} \le y \le \frac{\pi a}{2}$$

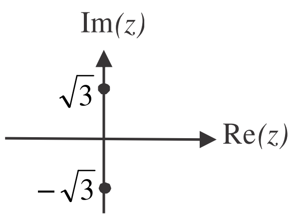
Since $r_f = [-\pi, \pi]$

$$\frac{\pi a}{2} = \pi$$
$$a = 2$$

The answer is C.

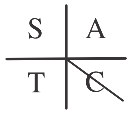
Question 5

The modulus of z is the distance from the origin to the point on the Argand diagram represented by z. Since z lies on the imaginary axis, there are two possible values of z, as shown on the diagram.
One is $\sqrt{3}i$ and the other is $-\sqrt{3}i$.
Only the latter is offered in the answers.
The answer is B.



Question 6

$$
\begin{aligned}
z &= \text{cis}\left(-\frac{\pi}{3}\right)\\
&= \cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\\
&= \frac{1}{2}-\frac{\sqrt{3}}{2}i
\end{aligned}
$$



$$
\begin{aligned}
z+\frac{i}{\sqrt{3}} &= \frac{1}{2}-\frac{\sqrt{3}}{2}i+\frac{i}{\sqrt{3}}\\
&= \frac{1}{2}+\frac{2i-3i}{2\sqrt{3}}\\
&= \frac{1}{2}-\frac{1}{2\sqrt{3}}i
\end{aligned}
$$

The imaginary part is $-\frac{1}{2\sqrt{3}}$.

The answer is C.

Question 7

$z_1 = a\text{cis}(\theta)$ and $z_2 = b\text{cis}(2\theta)$

${z_1}^3 = a^3\text{cis}(3\theta)$ so $\bar{z}_2 = b\text{cis}(-2\theta)$

$$
\begin{aligned}
\arg\left(\frac{{z_1}^3}{\bar{z}_2}\right) &= 3\theta - {}^{-}2\theta\\
&= 5\theta
\end{aligned}
$$

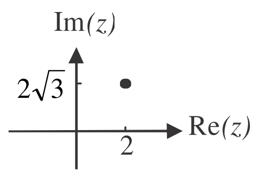
The answer is E.

Question 8

$$
\begin{aligned}
z^2 &= 2+2\sqrt{3}i\\
&= 4\text{cis}\left(\frac{\pi}{3}+2k\pi\right) \qquad k\in Z
\end{aligned}
$$

$r=\sqrt{4+4\times 3}=4$

$$
\begin{aligned}
\theta &= \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)\\
&= \frac{\pi}{3}
\end{aligned}
$$



$$
\begin{aligned}
\text{So } z &= \left(4\text{cis}\left(\frac{\pi}{3}+2k\pi\right)\right)^{\frac{1}{2}}\\
&= 2\text{cis}\left(\frac{1}{2}\left(\frac{\pi}{3}+2k\pi\right)\right)\\
&= 2\text{cis}\left(\frac{\pi}{6}+k\pi\right)
\end{aligned}
$$

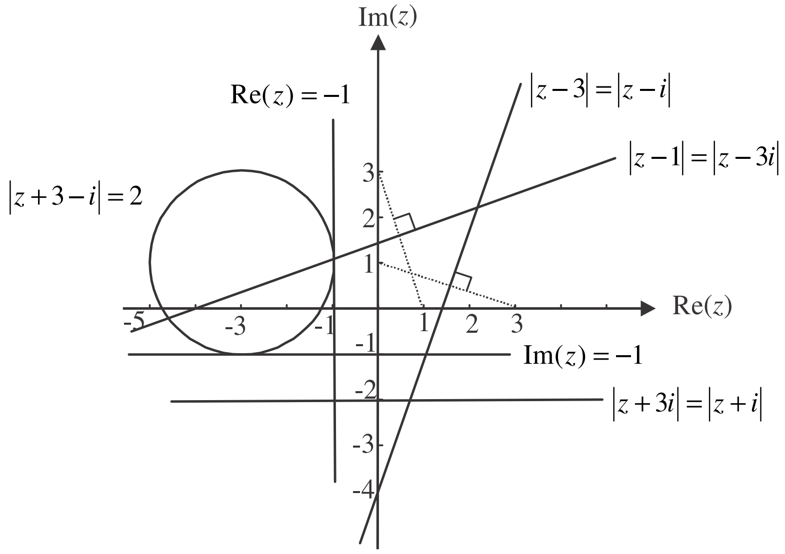
For the principle valued argument, $-\pi<\theta\le\pi$. (formula sheet)

So $z=2\text{cis}\left(\frac{\pi}{6}\right)$ or $z=2\text{cis}\left(-\frac{5\pi}{6}\right)$.

The answer is B.

Question 9
Do a quick sketch.

$$|z+3-i|=2$$
$$|z-(-3+i)|=2$$



The lines $\text{Re}(z)=-1$ and $\text{Im}(z)=-1$ are both tangents to the circle.
The lines given in options B and D don't intersect with the circle at all.
The line $|z-1|=|z-3i|$ intersects twice.
The answer is E.

Question 10

$$V=5\pi h$$
$$\frac{dV}{dh}=5\pi$$
$$\begin{aligned}\frac{dh}{dt}&=\frac{dh}{dV}\cdot\frac{dV}{dt}\\&=\frac{1}{5\pi}\left(0.1-0.02\sqrt{h}\right)\\&=\frac{10-2\sqrt{h}}{500\pi}\\&=\frac{5-\sqrt{h}}{250\pi}\end{aligned}$$

The answer is B.

Question 11
Method 1
At $x=0$, $\frac{dy}{dx}$ is undefined, so eliminate options C and E.
For $x\in R\backslash\{0\}$, $\frac{dy}{dx}>0$, so eliminate options A and D.
The answer is B.
Method 2
The slopes $\left(\frac{dy}{dx}\right)$ are influenced by x-values only, so A and B are the only possibilities.
The slopes are non-negative so it must be B.
The answer is B.

Question 12

$\frac{dy}{dx} = x^2y + 3x, \qquad y = 1 \text{ when } x = 2$

So, $x_0 = 2 \qquad y_0 = 1$

$x_1 = 2.1 \qquad y_1 = 1 + 0.1(2^2 \times 1 + 3 \times 2)$

$\qquad\qquad = 2$

$x_2 = 2.2 \qquad y_2 = 2 + 0.1(2.1^2 \times 2 + 3 \times 2.1)$

$\qquad\qquad = 3.512$

The answer is C.

Question 13

$$\int_1^2 \frac{x-1}{\sqrt{1+2x}}dx$$

Let $u = 1 + 2x$

$\frac{du}{dx} = 2$

Since $u = 1 + 2x$ Also, $x = 2,\ u = 5$

$2x = u - 1$ $x = 1,\ u = 3$

$x = \frac{u-1}{2}$

$x - 1 = \frac{u-3}{2}$

We have $\int_3^5 \frac{u-3}{2} \times u^{-\frac{1}{2}} \times \frac{1}{2}\frac{du}{dx}dx$

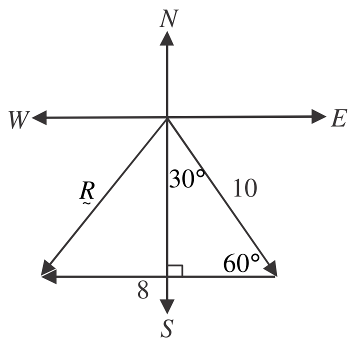
$= \frac{1}{4}\int_3^5 \frac{u-3}{\sqrt{u}}du$

The answer is A.

Question 14

Draw a diagram.

Add tip to tail to get the resultant vector $\underset{\sim}{R}$.



$|\underset{\sim}{R}|^2 = 8^2 + 10^2 - 2 \times 10 \times 8\cos(60^\circ)$ (cosine rule)

$= 84$

$|\underset{\sim}{R}| = 2\sqrt{21}$

The answer is D.

Question 15

$\underset{\sim}{r}(t) = (2t^3 + 3)\underset{\sim}{i} + t^2\underset{\sim}{j} - (1 - t^3)\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = 6t^2\underset{\sim}{i} + 2t\underset{\sim}{j} + 3t^2\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(t) = 12t\underset{\sim}{i} + 2\underset{\sim}{j} + 6t\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(2) = 24\underset{\sim}{i} + 2\underset{\sim}{j} + 12\underset{\sim}{k}$

$$\begin{aligned}\left|\ddot{\underset{\sim}{r}}(2)\right| &= \sqrt{576 + 4 + 144}\\ &= \sqrt{724}\\ &= 2\sqrt{181}\,\text{m/s}^2\end{aligned}$$

The answer is D.

Question 16

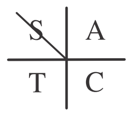
$\underset{\sim}{a} \bullet \underset{\sim}{b} = -4 + 2\sqrt{2} - 2\sqrt{2} = -4$

Also, $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$

$$\begin{aligned}&= \sqrt{4+2+4}\sqrt{4+4+2}\cos(\theta)\\ &= 10\cos(\theta)\end{aligned}$$

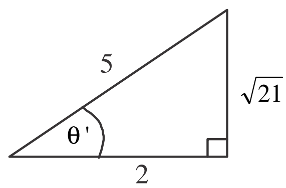
So $10\cos(\theta) = -4$

$\cos(\theta) = \dfrac{-2}{5}$



θ must be a second quadrant angle.

So $\sin(\theta) = \dfrac{\sqrt{21}}{5}$ (sin is positive in the second quadrant)



The answer is C.

Question 17

$\underset{\sim}{v}(t) = e^{3t}\underset{\sim}{i} - 4\underset{\sim}{j} + \sin(t)\underset{\sim}{k}, \ t \ge 0$

$\underset{\sim}{r}(t) = \dfrac{e^{3t}}{3}\underset{\sim}{i} - 4t\underset{\sim}{j} - \cos(t)\underset{\sim}{k} + \underset{\sim}{c}$

$\underset{\sim}{r}(0) = \dfrac{1}{3}\underset{\sim}{i} - \underset{\sim}{k} + \underset{\sim}{c} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$

So $\underset{\sim}{c} = -\dfrac{1}{3}\underset{\sim}{i} + \underset{\sim}{k}$

So $\underset{\sim}{r}(t) = \dfrac{e^{3t} - 1}{3}\underset{\sim}{i} - 4t\underset{\sim}{j} + (1 - \cos(t))\underset{\sim}{k}$

The answer is D.

Question 18

Draw in the forces.
Find T_1.
Resolving horizontally:

$T_1 = T_2 \sin(30^\circ)$

$= \frac{T_2}{2} \qquad -(1)$

Resolving vertically:

$T_2 \cos(30^\circ) = 3g$

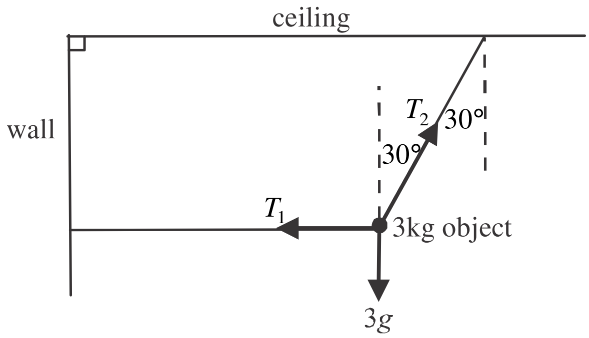
$\frac{\sqrt{3}}{2} T_2 = 3g$

$T_2 = \frac{6g}{\sqrt{3}}$

In (1) $T_1 = \frac{6g}{2\sqrt{3}}$

$= \frac{3\sqrt{3}g}{3}$

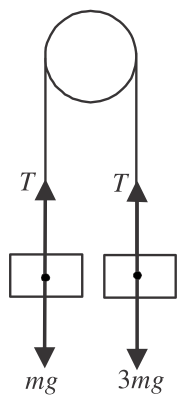
$= \sqrt{3}g$



The answer is D.

Question 19

Since $3m > m$, the $3m$ kg particle will accelerate downwards and so the m kg particle will accelerate upwards.



For the *3m kg* particle,

$3mg - T = 3ma - (1)$

For the *m kg* particle,

$T - mg = ma \quad - (2)$

(1)+(2) $2mg = 4ma$

$a = \frac{2mg}{4m}$

$= \frac{g}{2}$

$a = 4.9$

$v = 24.5 \qquad v = u + at$

$u = 0 \qquad 24.5 = 0 + 4.9 \times t$

$t = ? \qquad t = 5$

The answer is E.

Question 20

$\underset{\sim}{R} = m\underset{\sim}{a}$

$-9 = 4a$ (the direction of the $9N$ force is opposite to the direction of the particle's motion)

$a = -\frac{9}{4}$

$u = 6,\qquad v^2 = u^2 + 2as$

$v = 0\qquad 0 = 36 + 2\times -\frac{9}{4}s$

$s = ?\qquad s = 8\text{m}$

The answer is A.

Question 21

$$\text{distance travelled} = 2\times 7 + \int_2^3 v(t)dt - \int_3^9 v(t)dt + \int_9^{10} v(t)dt$$

$$= \frac{170}{3}\text{ metres}$$

The answer is D.

Question 22

$$a = 3 - x^2$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - x^2 \quad \text{(formula sheet)}$$

$$\frac{1}{2}v^2 = \int (3 - x^2)dx$$

$$\frac{1}{2}v^2 = 3x - \frac{x^3}{3} + c$$

When $x = 0,\ v = 0.$

$$0 = 0 - 0 + c$$

$$c = 0$$

So $\frac{1}{2}v^2 = 3x - \frac{x^3}{3}$

When $v = 0$

$$0 = \frac{9x - x^3}{3}$$

$$0 = \frac{x(9 - x^2)}{3}$$

The particle will also be at rest where $x = \pm 3$.

The answer is E.

SECTION 2

Question 1 (11 marks)

a. $x = t + \frac{2}{t} + 1 \qquad y = t - \frac{2}{t}$

$$(x-1)^2 = \left(\frac{t^2+2}{t}\right)^2 \qquad y^2 = \left(\frac{t^2-2}{t}\right)^2$$

$$(x-1)^2 - y^2 = \frac{t^4 + 4t^2 + 4 - \left(t^4 - 4t^2 + 4\right)}{t^2}$$ **(1 mark)**

$$(x-1)^2 - y^2 = \frac{8t^2}{t^2}$$

$$(x-1)^2 - y^2 = 8$$

$$\frac{(x-1)^2}{8} - \frac{y^2}{8} = 1$$ as required.

(1 mark)

b. $x = t + 2t^{-1} + 1 \qquad y = t - 2t^{-1}$

$$\frac{dx}{dt} = 1 - 2t^{-2} \qquad \frac{dy}{dt} = 1 + 2t^{-2}$$ **(1 mark)**

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \left(1 + \frac{2}{t^2}\right) \times \frac{t^2}{t^2 - 2}$$

$$= \frac{t^2+2}{t^2} \times \frac{t^2}{t^2-2}$$

$$= \frac{t^2+2}{t^2-2}$$ **(1 mark)**

When $\frac{dy}{dx} = 2$,

$$2 = \frac{t^2+2}{t^2-2}$$

Method 1 - solve for t using CAS

$t = \pm\sqrt{6}$

(1 mark)

Method 2 – solve for t by hand

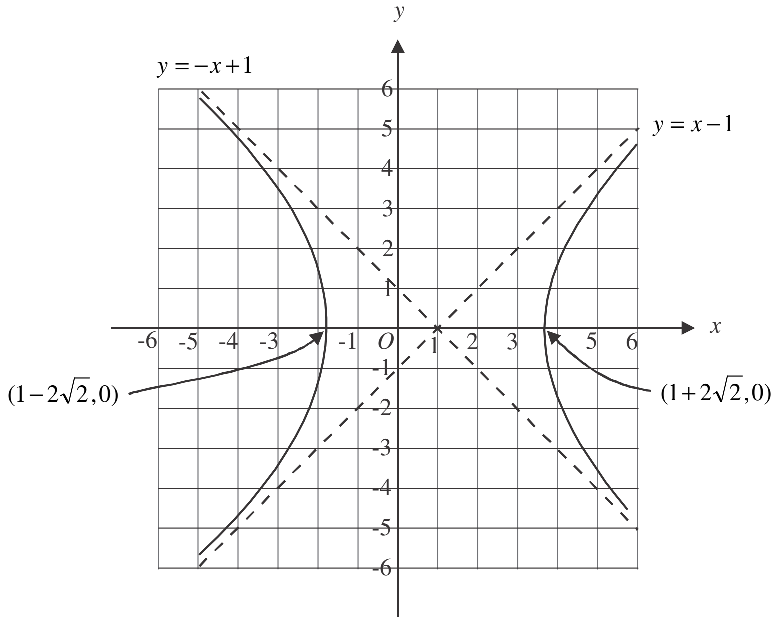
$$2(t^2 - 2) = t^2 + 2$$

$$t^2 = 6$$

$$t = \pm\sqrt{6}$$

(1 mark)

c.



<u>Asymptotes</u>: $y = \pm\frac{\sqrt{8}}{\sqrt{8}}(x-1)$

$y = \pm(x-1)$

<u>x-intercepts</u> occur when $y = 0$

$$\frac{(x-1)^2}{8} = 1$$

$$x = 1 - 2\sqrt{2} \text{ or } x = 1 + 2\sqrt{2}$$

(1 mark) – correct asymptotes
(1 mark) – correct x-intercepts
(1 mark) – correct shape

d.

$$\text{volume} = \pi \int_{1+2\sqrt{2}}^{5} y^2\,dx$$

$$= \pi \int_{1+2\sqrt{2}}^{5} ((x-1)^2 - 8)\,dx$$

$$= \frac{32\pi}{3}\left(\sqrt{2} - 1\right) \text{cubic units}$$

$$\frac{(x-1)^2}{8} - \frac{y^2}{8} = 1$$

$$\frac{y^2}{8} = \frac{(x-1)^2}{8} - 1$$

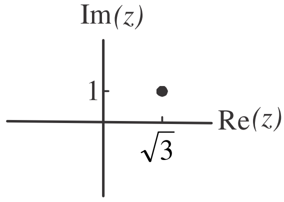
$$y^2 = (x-1)^2 - 8$$

(1 mark) – correct integrand
(1 mark) – correct terminals
(1 mark) – correct answer

Question 2 (13 marks)

a.

$$z_1 = \sqrt{3} + i$$
$$r = \sqrt{3+1} = 2$$
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{\pi}{6} \quad \text{(since } z_1 \text{ is in the first quadrant)}$$
$$z_1 = 2\text{cis}\left(\frac{\pi}{6}\right)$$



(1 mark) – correct modulus
(1 mark) – correct argument

b.

$$z_1^{\ 4} = -8 + 8\sqrt{3}i$$
$$LS = z_1^{\ 4}$$
$$= \left(2\text{cis}\left(\frac{\pi}{6}\right)\right)^4 \quad \text{(from part a.)}$$
$$= 2^4\,\text{cis}\left(\frac{4\pi}{6}\right)$$
$$= 16\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$
$$= 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= -8 + 8\sqrt{3}i$$
$$= RS$$

So z_1 satisfies the equation. **(1 mark)**

c.

$$\text{Arg}(z_1^{\ p}) = \frac{\pi}{2}, \quad p \in R$$

So, $\arg(z_1^{\ p}) = \ldots\frac{-3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}\ldots$ or $\arg(z_1^{\ p}) = \frac{\pi}{2} + 2k\pi$ where $k \in Z$

Also $\arg(z_1^{\ p}) = p \times \arg(z_1)$ **(1 mark)**

$$= p \times \frac{\pi}{6} \quad \text{(from part a.)}$$

So $p \times \frac{\pi}{6} = \ldots.\frac{-3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}\ldots$ or $\frac{p\pi}{6} = \frac{\pi}{2} + 2k\pi$

$p = \ldots -9, 3, 15, \ldots$ $\quad$ $\frac{p}{6} = \frac{1}{2} + 2k$

$p = 12k + 3$ **(1 mark)**

d. **i.**

$$|z-2z_1|=|z|$$

$$\left|x+iy-2\left(\sqrt{3}+i\right)\right|=|x+iy|$$ **(1 mark)**

$$\left|(x-2\sqrt{3})+(y-2)i\right|=|x+iy|$$

$$\sqrt{\left(x-2\sqrt{3}\right)^2+(y-2)^2}=\sqrt{x^2+y^2}$$

$$x^2-4\sqrt{3}x+12+y^2-4y+4=x^2+y^2$$

$$-4\sqrt{3}x-4y+16=0$$

$$\sqrt{3}x+y=4 \quad \text{as required}$$

(1 mark)

ii. Method 1

$$\bar{z}_1+\frac{2}{\sqrt{3}}=\sqrt{3}-i+\frac{2}{\sqrt{3}}$$

$$=\frac{5}{\sqrt{3}}-i$$

So $\bar{z}_1+\frac{2}{\sqrt{3}}$ corresponds to the point $\left(\frac{5}{\sqrt{3}},-1\right)$ on the Cartesian plane.

(1 mark)

From part **i.**, the Cartesian equation of the relation $|z-2z_1|=|z|$ is $\sqrt{3}x+y=4$.
Substituting the point into this relation gives

$$LS=\sqrt{3}\times\frac{5}{\sqrt{3}}-1$$

$$=4$$

$$=RS$$

(1 mark)

Method 2

To Show $\left|\bar{z}_1+\frac{2}{\sqrt{3}}-2z_1\right|=\left|\bar{z}_1+\frac{2}{\sqrt{3}}\right|$

$$LS=\left|\bar{z}_1+\frac{2}{\sqrt{3}}-2z_1\right|$$

$$=\left|\sqrt{3}-i+\frac{2}{\sqrt{3}}-2\left(\sqrt{3}+i\right)\right|$$

$$=\left|\sqrt{3}+\frac{2}{\sqrt{3}}-2\sqrt{3}-3i\right|$$

$$=\left|\frac{3+2-6}{\sqrt{3}}-3i\right|$$

$$=\left|\frac{-1}{\sqrt{3}}-3i\right|$$

$$=\sqrt{\frac{1}{3}+9}$$

$$=\sqrt{\frac{28}{3}}$$

(1 mark)

$$RS=\left|\bar{z}_1+\frac{2}{\sqrt{3}}\right|$$

$$=\left|\sqrt{3}-i+\frac{2}{\sqrt{3}}\right|$$

$$=\left|\frac{3+2}{\sqrt{3}}-i\right|$$

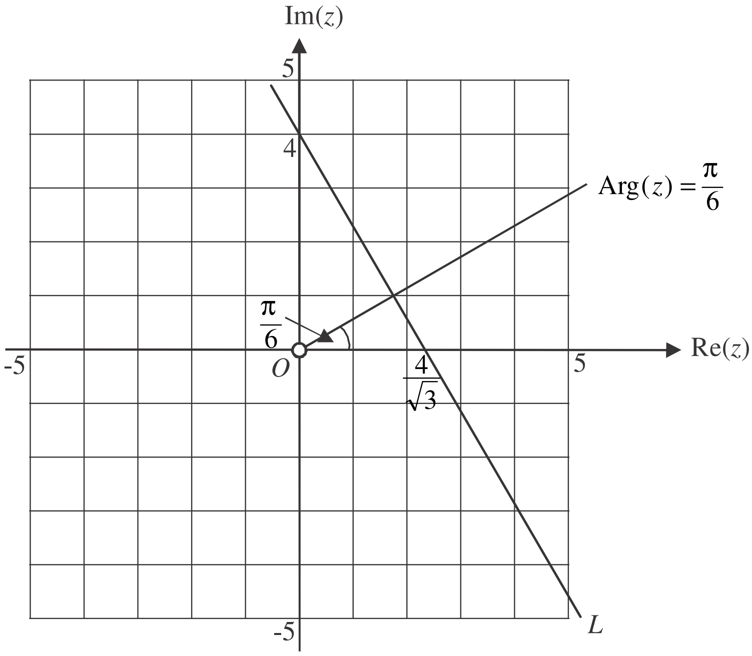
$$=\sqrt{\frac{25}{3}+1}$$

$$=\sqrt{\frac{28}{3}}$$

$$=LS$$

(1 mark)

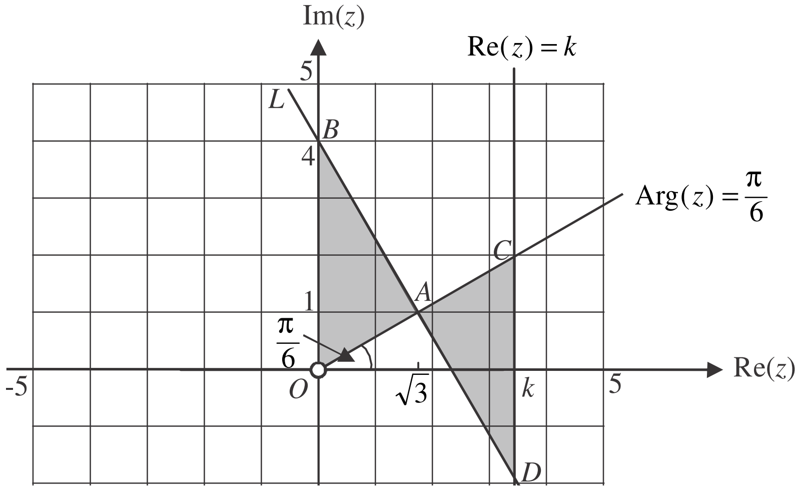
e. **i.**



(1 mark) for L including axes intercepts

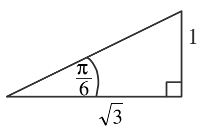
(1 mark) for $\operatorname{Arg}(z)=\frac{\pi}{6}$ including excluded origin

ii.



Start by finding the point of intersection of L and $\operatorname{Arg}(z)=\frac{\pi}{6}$.

$\operatorname{Arg}(z)=\frac{\pi}{6}$ can be expressed in Cartesian form as $y=\frac{1}{\sqrt{3}}x$.



L is $\sqrt{3}x+y=4$ or $y=-\sqrt{3}x+4$

So $$\frac{1}{\sqrt{3}}x=-\sqrt{3}x+4$$

Solve using CAS, $x=\sqrt{3}$ so $y=1$.

A is the point $\left(\sqrt{3},1\right)$. **(1 mark)**

ΔABO and ΔACD are similar because $\angle BAO=\angle CAD$ (vertically opposite angles), $\angle ABO=\angle ADC$ (alternate angles) and $\angle AOB=\angle ACD$ (alternate angles). They will be congruent, and therefore have the same area, when $k=2\sqrt{3}$.

(1 mark)

Question 3 (12 marks)

a. $\underset{\sim}{s}(t) = \cos(\pi t)\underset{\sim}{i} + \sqrt{3}\sin(\pi t)\underset{\sim}{j}$

$x = \cos(\pi t) \qquad y = \sqrt{3}\sin(\pi t)$ **(1 mark)**

$x^2 = \cos^2(\pi t) \qquad y^2 = 3\sin^2(\pi t)$

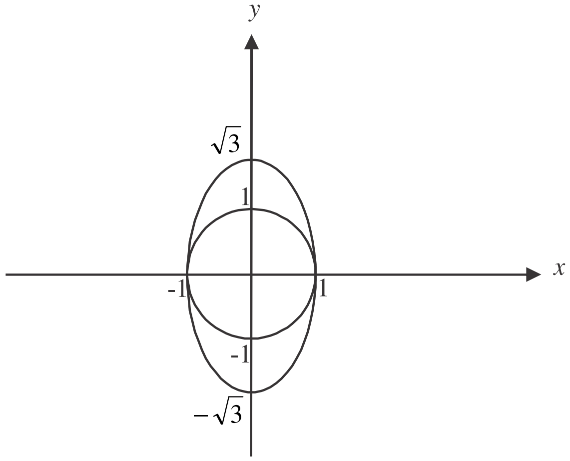
$\frac{y^2}{3} = \sin^2(\pi t)$

$x^2 + \frac{y^2}{3} = \cos^2(\pi t) + \sin^2(\pi t)$

$x^2 + \frac{y^2}{3} = 1$

(1 mark)

b.



(1 mark) – correct axes intercepts
(1 mark) – correct shape

c. $\underset{\sim}{r}(0) = \sin(0)\underset{\sim}{i} + \cos(0)\underset{\sim}{j}$

$= 0\underset{\sim}{i} + \underset{\sim}{j}$

For drone *R*, the starting position is $(0,1)$. **(1 mark)**

$\underset{\sim}{s}(0) = \cos(0)\underset{\sim}{i} + \sqrt{3}\sin(0)\underset{\sim}{j}$

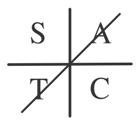
$= \underset{\sim}{i} + 0\underset{\sim}{j}$

For drone *S*, the starting position is $(1,0)$. **(1 mark)**

d. For the drones to meet,

$\sin(\pi t) = \cos(\pi t)$ <u>AND</u> $\cos(\pi t) = \sqrt{3}\sin(\pi t)$

$\tan(\pi t) = 1 \qquad \tan(\pi t) = \frac{1}{\sqrt{3}}$



(1 mark)

$\pi t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \ldots \qquad \pi t = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6} \ldots$

$t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \ldots \qquad t = \frac{1}{6}, \frac{7}{6}, \frac{13}{6}, \ldots$

Since the displacement components are not equal at the same time, the drones never meet.

(1 mark)

e. $\underset{\sim}{r}(t) = \sin(\pi t)\,\underset{\sim}{i} + \cos(\pi t)\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \pi\cos(\pi t)\,\underset{\sim}{i} - \pi\sin(\pi t)\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(3) = \pi\cos(3\pi)\,\underset{\sim}{i} - \pi\sin(3\pi)\,\underset{\sim}{j}$

$= -\pi\,\underset{\sim}{i} + 0\,\underset{\sim}{j}$

$\left|\dot{\underset{\sim}{r}}(3)\right| = \sqrt{(-\pi)^2} = \pi$ km/hr

(1 mark)

f. $\underset{\sim}{r}(t) = \sin(\pi t)\,\underset{\sim}{i} + \cos(\pi t)\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \pi\cos(\pi t)\,\underset{\sim}{i} - \pi\sin(\pi t)\,\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}}(t) = -\pi^2\sin(\pi t)\,\underset{\sim}{i} - \pi^2\cos(\pi t)\,\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}}(2) = -\pi^2\sin(2\pi)\,\underset{\sim}{i} - \pi^2\cos(2\pi)\,\underset{\sim}{j}$

$= 0\,\underset{\sim}{i} - \pi^2\,\underset{\sim}{j}$

$\left|\ddot{\underset{\sim}{r}}(2)\right| = \sqrt{(-\pi^2)^2} = \pi^2\text{km/hr}^2$

(1 mark)

g. $\dot{\underset{\sim}{r}}(t) = \pi\cos(\pi t)\,\underset{\sim}{i} - \pi\sin(\pi t)\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}\left(\frac{1}{2}\right) = 0\,\underset{\sim}{i} - \pi\,\underset{\sim}{j}$

$\dot{\underset{\sim}{s}}(t) = -\pi\sin(\pi t)\,\underset{\sim}{i} + \sqrt{3}\pi\cos(\pi t)\,\underset{\sim}{j}$

$\dot{\underset{\sim}{s}}\left(\frac{1}{2}\right) = -\pi\,\underset{\sim}{i} + 0\,\underset{\sim}{j}$ **(1 mark)**

$\dot{\underset{\sim}{r}}\left(\frac{1}{2}\right) \bullet \dot{\underset{\sim}{s}}\left(\frac{1}{2}\right) = 0 \times {}^{-}\pi - \pi \times 0 = 0$ **(1 mark)**

Since $\dot{\underset{\sim}{r}}\left(\frac{1}{2}\right) \bullet \dot{\underset{\sim}{s}}\left(\frac{1}{2}\right) = 0$, the drones must be travelling in directions that are perpendicular to each other.

Question 4 (10 marks)

a. When $t = 0$, $\tan\left(\dfrac{N - 50\pi}{100}\right) = -4$

Solve for N using CAS

$N = 100\left(k\pi + \tan^{-1}\left(\dfrac{1}{4}\right)\right)$

$N = 24.4978...$ for $k = 0$

The number of pre-sold apartments is 24 (to the nearest integer).

(1 mark)

b. As $t \to \infty$, $\tan\left(\dfrac{N - 50\pi}{100}\right) \to \infty$

so, $\left(\dfrac{N - 50\pi}{100}\right) \to \dfrac{\pi}{2}$

and so $N \to 100\pi$

The limiting number is 314 (to the nearest integer).

(1 mark)

c. Differentiate the equation

$$\tan\left(\frac{N - 50\pi}{100}\right) = \frac{t}{30} - 4 \quad \text{with respect to } t$$

$$\frac{d}{dN}\left(\tan\left(\frac{N - 50\pi}{100}\right)\right)\frac{dN}{dt} = \frac{1}{30} \quad \text{(Chain rule)}$$

$$\frac{1}{100}\sec^2\left(\frac{N - 50\pi}{100}\right)\frac{dN}{dt} = \frac{1}{30}$$

$$\frac{1}{\cos^2\left(\frac{N - 50\pi}{100}\right)}\frac{dN}{dt} = \frac{100}{30}$$

$$\frac{dN}{dt} = \frac{10}{3}\cos^2\left(\frac{N - 50\pi}{100}\right)$$

(1 mark)

Now, $\cos^2\left(\dfrac{N - 50\pi}{100}\right) - 0.3\dfrac{dN}{dt} = 0$

$$\begin{aligned}
LS &= \cos^2\left(\frac{N - 50\pi}{100}\right) - 0.3\frac{dN}{dt} \\
&= \cos^2\left(\frac{N - 50\pi}{100}\right) - \frac{3}{10} \times \frac{10}{3}\cos^2\left(\frac{N - 50\pi}{100}\right) \\
&= \cos^2\left(\frac{N - 50\pi}{100}\right) - \cos^2\left(\frac{N - 50\pi}{100}\right) \\
&= 0 \\
&= RS
\end{aligned}$$

(1 mark)

d. Method 1 – using the given expression

$$\frac{d^2N}{dt^2} = \frac{d}{dt}\left(\frac{dN}{dt}\right)$$

$$= \frac{d}{dN}\left(\frac{dN}{dt}\right)\times\frac{dN}{dt}$$

$$= \frac{\sin\left(\frac{N}{100}\right)\cos\left(\frac{N}{100}\right)}{15}\times\frac{10}{3\left(\tan^2\left(\frac{N-50\pi}{100}\right)+1\right)}$$

$$= \frac{2\sin\left(\frac{N}{100}\right)\cos\left(\frac{N}{100}\right)}{9\left(\tan^2\left(\frac{N-50\pi}{100}\right)+1\right)}$$ **(1 mark)**

$$= \frac{\sin\left(\frac{N}{50}\right)}{9\left(\tan^2\left(\frac{N-50\pi}{100}\right)+1\right)}$$

We are told that the graph has a point of inflection so this occurs when $\frac{d^2N}{dt^2} = 0$.

Solve $\sin\left(\frac{N}{50}\right) = 0$ for N. **(1 mark)**

$\frac{N}{50} = 0, \pi, 2\pi, ...$ (note that N is positive)

$N = 0, 50\pi, 100\pi, ...$

$N = 50\pi$ is the only answer within the range of the function N.

So $N = 50\pi$

Substitute this into

$\tan\left(\frac{N-50\pi}{100}\right) = \frac{t}{30} - 4$

$t = 120$

So $a = 120$ and $b = 50\pi$

(1 mark) for a

(1 mark) for b

Method 2 – otherwise

From part c., $\cos^2\left(\frac{N-50\pi}{100}\right)-0.3\frac{dN}{dt}=0$

So $$\frac{dN}{dt}=\frac{10}{3}\cos^2\left(\frac{N-50\pi}{100}\right)$$

so, $$\frac{d^2N}{dt^2}=\frac{d}{dt}\left(\frac{dN}{dt}\right)$$
$$=\frac{d}{dN}\left(\frac{dN}{dt}\right)\times\frac{dN}{dt}$$
$$=\frac{10}{3}\times\frac{\sin\left(\frac{N}{100}\right)\cos\left(\frac{N}{100}\right)}{50}\times\frac{dN}{dt}$$
$$=\frac{\sin\left(\frac{N}{100}\right)\cos\left(\frac{N}{100}\right)}{15}\times\frac{10\cos^2\left(\frac{N-50\pi}{100}\right)}{3}$$
$$=\frac{2\sin\left(\frac{N}{100}\right)\cos\left(\frac{N}{100}\right)\cos^2\left(\frac{N-50\pi}{100}\right)}{9}$$ **(1 mark)**

We are told that the graph has a point of inflection. This occurs when $\frac{d^2N}{dt^2}=0$.

Solve $\sin\left(\frac{N}{100}\right)\cos\left(\frac{N}{100}\right)\cos^2\left(\frac{N-50\pi}{100}\right)=0$ for N **(1 mark)**

$N=50(2k-1)\pi$ or $N=100k\pi$

For $k=0$, $N=-50\pi$ or $N=0$

For $k=1$, $N=50\pi$ or $N=100\pi$

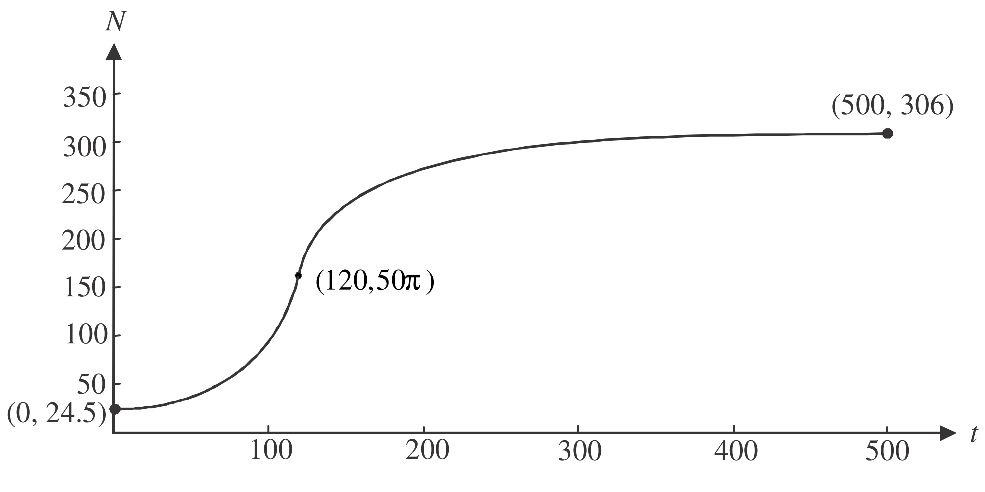
Note that N is positive and $N=50\pi$ is the only answer within the range of the function N.

Substitute $N=50\pi$ into $\tan\left(\frac{N-50\pi}{100}\right)=\frac{t}{30}-4$, so $t=120$.

So $a=120$ and $b=50\pi$ as required.

(1 mark) for a
(1 mark) for b

e.

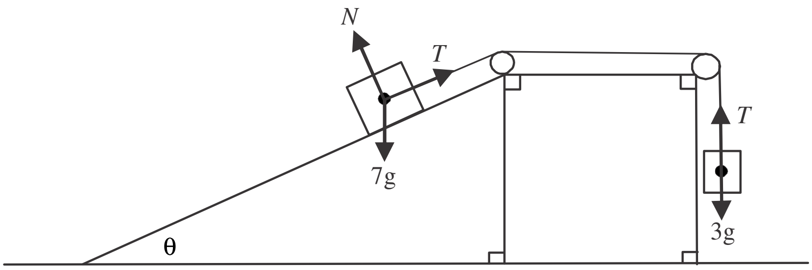


(1 mark) – correct endpoints i.e. $(0,24.5)$ and $(500,306)$

(1 mark) – correct shape including point of inflection located at $(120,50\pi)$

Question 5 (12 marks)

a.



i. Around the 7kg mass:
$7g\sin(\theta) - T = 7a \qquad -(1)$ **(1 mark)**

ii. Around the 3kg mass:
$T - 3g = 3a \qquad -(2)$ **(1 mark)**

iii. (1) gives $T = 7g\sin(\theta) - 7a$
(2) gives $T = 3g + 3a$
So $7g\sin(\theta) - 7a = 3g + 3a$

$$-10a = 3g - 7g\sin(\theta)$$
$$a = \frac{g(3 - 7\sin(\theta))}{-10}$$
$$= \frac{g(7\sin(\theta) - 3)}{10}$$

as required.

(1 mark)

iv. If the system is in equilibrium then $a = 0$.
So $\frac{g(7\sin(\theta) - 3)}{10} = 0$

$$\sin(\theta) = \frac{3}{7}$$
$$\theta = 25.4^\circ$$

(correct to 1 decimal place)

(1 mark)

v. When the 2kg mass is moving vertically upwards, $a > 0$.
So $\frac{g(7\sin(\theta) - 3)}{10} > 0$

$$7\sin(\theta) - 3 > 0$$
$$\sin(\theta) > \frac{3}{7}$$
$$25.4^\circ < \theta < 90^\circ$$

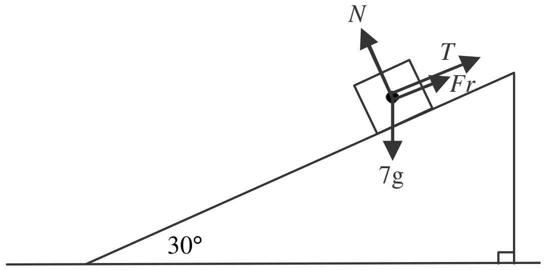
(1 mark)

b. Around the 7kg mass:

$7g\sin(30^\circ) = T + Fr$

$Fr = 3.5g - T \qquad -(1)$

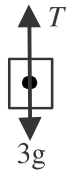
and $N = 7g\cos(30^\circ) = 59.4093...$



Around the 3kg mass:

$T = 3g \qquad -(2)$

In (1) gives $Fr = 3.5g - 3g = 4.9$



(1 mark)

If the 7kg mass is on the point of moving then $Fr = \mu N$.

Now $\mu N = 0.106 \times 59.4093...$

$= 6.2973...$

Since $4.9 < 6.2973...$, then $Fr < \mu N$ and the mass is not at the point of moving.

(1 mark)

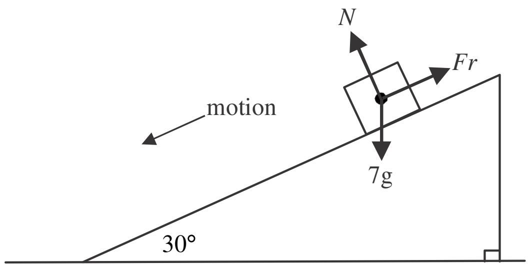
c. **i.** $7g\sin(30^\circ) - Fr = 7a$

$3.5g - Fr = 7a$

$a = \dfrac{3.5g - Fr}{7}$

Also $N = 7g\cos(30^\circ)$

$= \dfrac{7\sqrt{3}g}{2}$



So $a = \dfrac{3.5\text{g} - \mu N}{7}$

(1 mark)

$= 4.00037...$

$= 4.00\text{m/s}^2$ (to the nearest 0.01m/s^2)

(1 mark)

ii. $3g = 3a$

$a = g\text{m/s}^2$

The 3kg mass is subject only to gravitational force so its acceleration is g m/s^{-2} or 9.8 m/s^{-2}.

(1 mark)

d. Both masses are travelling with constant acceleration.

So, using $v^2 = u^2 + 2as$

for the 3kg mass, $v^2 = 0 + 2 \times 9.8 \times s$

$= 19.6s$

(1 mark)

for the 7kg mass, $v^2 = 0 + 2 \times 4.00 \times 3s$

$= 24s$

So ratio of speeds is $\sqrt{19.6\text{s}} : \sqrt{24s}$

$\sqrt{196} : \sqrt{240}$

$\sqrt{49} : \sqrt{60}$

$7 : 2\sqrt{15}$

(1 mark)