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SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2015

Question 1 (3 marks)

$$\int_0^{\sqrt{6}} \frac{x-5}{x^2+2}dx$$

$$= \int_0^{\sqrt{6}} \frac{x}{x^2+2}dx - \int_0^{\sqrt{6}} \frac{5}{x^2+2}dx$$

$$= \left[\frac{1}{2}\log_e(x^2+2)\right]_0^{\sqrt{6}} - \left[\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{6}}$$

(1 mark) **(1 mark)**

$$= \frac{1}{2}(\log_e(8) - \log_e(2)) - \frac{5}{\sqrt{2}}\left(\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(0)\right)$$

$$= \frac{1}{2}\log_e(4) - \frac{5}{\sqrt{2}}\left(\frac{\pi}{3} - 0\right)$$

$$= \log_e(2) - \frac{5\sqrt{2}\pi}{6}$$

(1 mark)

Question 2 (5 marks)

a. **i.** $\overrightarrow{CM} = \alpha\overrightarrow{CA}$ (given in question)

$$= \alpha\left(-\frac{1}{2}\underset{\sim}{b} + \underset{\sim}{a}\right)$$

$$= \alpha\underset{\sim}{a} - \frac{1}{2}\alpha\underset{\sim}{b}$$

(1 mark)

ii. $\overrightarrow{CM} = \overrightarrow{CO} + \overrightarrow{OM}$

$$= -\frac{1}{2}\underset{\sim}{b} + \beta\overrightarrow{OD}$$

$$= -\frac{1}{2}\underset{\sim}{b} + \beta\left(\overrightarrow{OA} + \overrightarrow{AD}\right)$$

(1 mark)

$$= -\frac{1}{2}\underset{\sim}{b} + \beta\left(\overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}\right)$$

$$= -\frac{1}{2}\underset{\sim}{b} + \beta\left(\underset{\sim}{a} + \frac{1}{3}(-\underset{\sim}{a} + \underset{\sim}{b})\right)$$

$$= \frac{2}{3}\beta\underset{\sim}{a} + \left(\frac{1}{3}\beta - \frac{1}{2}\right)\underset{\sim}{b}$$

(1 mark)

b. From part a.,

$$\alpha \underset{\sim}{a} - \frac{1}{2}\alpha \underset{\sim}{b} = \frac{2}{3}\beta \underset{\sim}{a} + \left(\frac{1}{3}\beta - \frac{1}{2}\right)\underset{\sim}{b}$$

So $\alpha = \frac{2}{3}\beta$ ___(1) and $-\frac{1}{2}\alpha = \frac{1}{3}\beta - \frac{1}{2}$ ___(2)

Substitute (1) into (2)

$$-\frac{1}{2} \times \frac{2}{3}\beta = \frac{1}{3}\beta - \frac{1}{2}$$

$$-\frac{1}{3}\beta = \frac{1}{3}\beta - \frac{1}{2}$$

$$\frac{2}{3}\beta = \frac{1}{2}$$

$$\beta = \frac{3}{4}$$

(1 mark)

So $\alpha = \frac{2}{3} \times \frac{3}{4}$

$$= \frac{1}{2}$$

(1 mark)

Question 3 (3 marks)

The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$

$$= \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$$

$$= \left(\left(\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \frac{1}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)\right)\frac{1}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

(1 mark)

$$= \frac{1}{9} \times (2 - 4 - 4)\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= -\frac{2}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

(1 mark)

The vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$

$$= \underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$$

$$= \underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k} - -\frac{2}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= \frac{7}{3}\underset{\sim}{i} - \frac{10}{3}\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$$

(1 mark)

If you have time, add the two resolutes together to make sure they give $\underset{\sim}{a}$.

$$-\frac{4}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} + \frac{4}{3}\underset{\sim}{k} + \frac{7}{3}\underset{\sim}{i} - \frac{10}{3}\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$$

$$= \underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$$

$$= \underset{\sim}{a}$$

Question 4 (4 marks)

Since $z=2i$ is a solution then $z=-2i$ is also a solution because the coefficients of the terms in the equation are all real (conjugate root theorem).

So $(z-2i)(z+2i)$

$=z^2+4$ is a factor

(1 mark)

Method 1 – long division

$$
\begin{array}{r}
z^2-2z+4 \\
z^2+4\overline{\smash{\big)}\,z^4-2z^3+8z^2-8z+16} \\
z^4 \qquad +4z^2 \qquad\qquad \\
\hline
-2z^3+4z^2-8z \qquad \\
-2z^3 \qquad -8z \qquad \\
\hline
4z^2 \qquad +16 \\
4z^2 \qquad +16
\end{array}
$$

Method 2 – inspection

$$
\begin{aligned}
&z^4-2z^3+8z^2-8z+16 \\
&=(z^2+4)(\_z^2\_z\_) \\
&=(z^2+4)(1z^2\_z\_) \\
&=(z^2+4)(z^2-2z\_) \\
&=(z^2+4)(z^2-2z+4)
\end{aligned}
$$

So $f(z)=(z^2+4)(z^2-2z+4)$ **(1 mark)**

$$
\begin{aligned}
&=(z^2+4)((z^2-2z+1)-1+4) \\
&=(z^2+4)((z-1)^2+3) \\
&=(z^2+4)((z-1)^2-3i^2) \\
&=(z^2+4)\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)
\end{aligned}
$$

(1 mark)

The other solutions to $f(z)=0$ are $z=-2i, 1\pm\sqrt{3}i$. **(1 mark)**

Question 5 (5 marks)

a. $u=10 \quad v=u+at$ **(1 mark)**

$v=0 \quad 0=10+4a$

$t=4 \quad a=-2.5$

$a=?$

The acceleration is -2.5ms^{-2}. **(1 mark)**

b. Method 1

$u=10 \quad s=ut+\frac{1}{2}at^2$

$v=0 \quad =10\times4+\frac{1}{2}\times{-2.5}\times16$

$t=4 \quad =40-20$

$a=-2.5 \quad =20$

$s=?$

Method 2

$$v^2=u^2+2as$$

$$0=100+2\times{-2.5}s$$

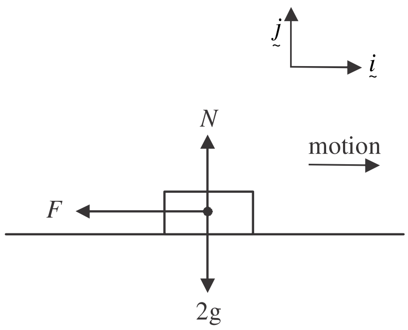
$$s=\frac{100}{5}$$

$$=20$$

The box travels 20m.

(1 mark)

c. Draw the forces on the diagram.



$$\underset{\sim}{R}=m\underset{\sim}{a}$$

$$-F\underset{\sim}{i}+(N-2g)\underset{\sim}{j}=2a\underset{\sim}{i}$$ **(1 mark)**

$$-F=2a \quad \text{and} \quad N=2g$$

$$-\mu N=2\times{-2.5}$$

$$-\mu\times2g=-5$$

$$\mu=\frac{5}{2g}$$

So $r=5$ and $s=2$. **(1 mark)**

Question 6 (4 marks)

$$2xy-\arctan\left(\frac{x}{2}\right)+y^2=5-\frac{\pi}{4}$$

$$2y+2x\frac{dy}{dx}-\frac{2}{4+x^2}+2y\frac{dy}{dx}=0$$

(1 mark) - $\frac{2}{4+x^2}$ term

(1 mark) - other terms

$$(2x+2y)\frac{dy}{dx}=-2y+\frac{2}{4+x^2}$$

$$=\frac{-2y(4+x^2)+2}{4+x^2}$$

$$\frac{dy}{dx}=\frac{-8y-2yx^2+2}{(4+x^2)\times2(x+y)}$$

$$=\frac{-4y-yx^2+1}{(4+x^2)(x+y)}$$ **(1 mark)**

At $(2,1)$ $\frac{dy}{dx}=\frac{-4-4+1}{8\times3}$

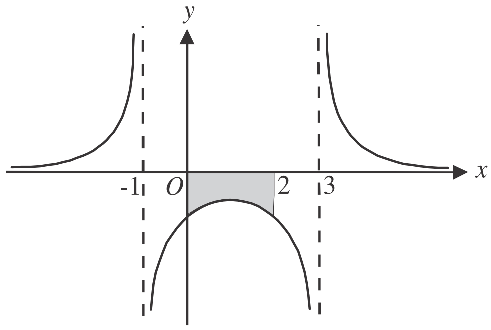
$$=\frac{-7}{24}$$

(1 mark)

Question 7 (4 marks)

Do a quick sketch.

$$y=\frac{1}{x^2-2x-3}$$
$$=\frac{1}{(x-3)(x+1)}$$



$$\text{area}=-\int_0^2 \frac{1}{(x-3)(x+1)}dx$$ **(1 mark)**

$$\text{Let } \frac{1}{(x-3)(x+1)} \equiv \frac{A}{(x-3)}+\frac{B}{(x+1)}$$
$$\equiv \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

True iff $1 \equiv A(x+1)+B(x-3)$

Put $x=-1$, true iff $1=-4B$, $B=-\frac{1}{4}$

Put $x=3$, true iff $1=4A$, $A=\frac{1}{4}$

$$\text{So area}=-\int_0^2 \left(\frac{1}{4(x-3)}-\frac{1}{4(x+1)}\right)dx$$ **(1 mark)**

$$=-\frac{1}{4}\left[\log_e|x-3|-\log_e|x+1|\right]_0^2$$ **(1 mark)**

$$=-\frac{1}{4}\left[\log_e\frac{|x-3|}{|x+1|}\right]_0^2$$
$$=-\frac{1}{4}\left(\log_e\frac{|-1|}{|3|}-\log_e\frac{|-3|}{|1|}\right)$$
$$=-\frac{1}{4}\left(\log_e\left(\frac{1}{3}\right)-\log_e\left(\frac{3}{1}\right)\right)$$
$$=-\frac{1}{4}\log_e\left(\frac{1}{9}\right)$$
$$=-\frac{1}{4}\log_e\left(3^{-2}\right)$$
$$=\frac{1}{2}\log_e(3) \text{ square units}$$

(1 mark)

Question 8 (5 marks)

a. Method 1

$v=\sqrt{2x+4}$

$a=v\frac{dv}{dx}$ (from formula sheet) **(1 mark)**

Now $\frac{dv}{dx}=\frac{1}{2}(2x+4)^{-\frac{1}{2}}\times 2$

$=\frac{1}{\sqrt{2x+4}}$

So $a=\sqrt{2x+4}\times\frac{1}{\sqrt{2x+4}}$

$a=1$

Hence acceleration is constant. **(1 mark)**

Method 2

$v=\sqrt{2x+4}$

$a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ (from formula sheet) **(1 mark)**

$=\frac{1}{2}\frac{d}{dx}(2x+4)$

$=\frac{1}{2}\times 2$

So $a=1$

Hence acceleration is constant. **(1 mark)**

b. $v=\sqrt{2x+4}$

$\frac{dx}{dt}=\sqrt{2x+4}$

$\frac{dt}{dx}=\frac{1}{\sqrt{2x+4}}$

$t=\int\frac{1}{\sqrt{2x+4}}dx$ **(1 mark)** let $u=2x+4$

$=\int u^{-\frac{1}{2}}\times\frac{1}{2}\frac{du}{dx}dx$ $\frac{du}{dx}=2$

$=\frac{1}{2}\int u^{-\frac{1}{2}}du$

$t=\frac{1}{2}u^{\frac{1}{2}}\times 2+c$

$t=\sqrt{2x+4}+c$ **(1 mark)**

When $t=0$, $x=0$, so $0=\sqrt{4}+c$ and $c=-2$.

So $t=\sqrt{2x+4}-2$

When $t=3$, $3=\sqrt{2x+4}-2$

$5=\sqrt{2x+4}$

$25=2x+4$

$x=\frac{21}{2}$

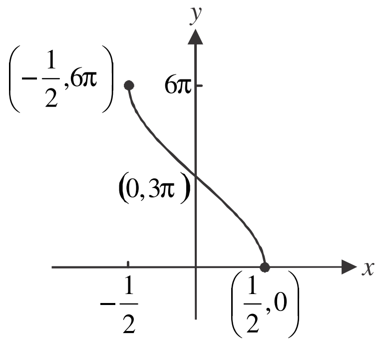
(1 mark)

Question 9 (7 marks)

a. $d_f = \left[-\frac{1}{2}, \frac{1}{2}\right]$ **(1 mark)**

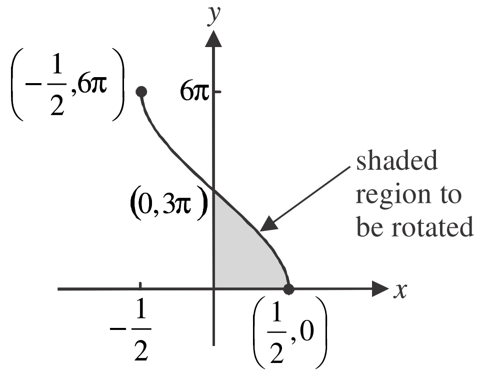
$r_f = [0, 6\pi]$ **(1 mark)**

b.



(1 mark) – correct endpoints and *y*-intercept
(1 mark) – correct shape

c. $\text{volume} = \pi \int_0^{3\pi} x^2 \, dy$ **(1 mark)**



since $y = 6\arccos(2x)$

$$\frac{y}{6} = \arccos(2x)$$

$$\cos\left(\frac{y}{6}\right) = 2x$$

$$x = \frac{1}{2}\cos\left(\frac{y}{6}\right)$$

$$x^2 = \frac{1}{4}\cos^2\left(\frac{y}{6}\right)$$

Since $\cos(2\theta) = 2\cos^2(\theta) - 1$

$$2\cos^2(\theta) = \cos(2\theta) + 1$$

$$\cos^2(\theta) = \frac{1}{2}(\cos(2\theta) + 1)$$

$$x^2 = \frac{1}{8}\left(\cos\left(\frac{y}{3}\right) + 1\right)$$

$$\text{volume} = \frac{\pi}{8}\int_0^{3\pi}\left(\cos\left(\frac{y}{3}\right) + 1\right)dy$$ **(1 mark)**

$$= \frac{\pi}{8}\left[3\sin\left(\frac{y}{3}\right) + y\right]_0^{3\pi}$$

$$= \frac{\pi}{8}\{(3\sin(\pi) + 3\pi) - (3\sin(0) + 0)\}$$

$$= \frac{3\pi^2}{8} \text{ units}^3$$

1 mark