ZSTEVE Specialist Mathematics Exam 1

Reading time: 15 minutes

Writing time: 60 minutes

No notes or calculator of any type are allowed.

40 marks are available

N.B. this exam aims to extend the knowledge and test the adaptability of students to new concepts, and thus does not purely target its questions within the VCE study design scope. However, all questions should be doable with Specialist level skills. For further investigation, notes have been included in some extension questions.

Exam written by Stephen Zhang, VCE Student in 2015.

1. Find the resolute of the vector $\vec{a} = 2\vec{\imath} + 2\vec{\jmath} + 9\vec{k}$ in the direction of $\vec{b} = \vec{\imath} - \vec{\jmath} + \vec{k}$

[2 marks]

2. [**further differential equations]**

   (a) Show that $y = e^{kx}$ can be solution to $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ (*this type of differential equation is called a second order linear homogenous equation)*

   (b) Hence, for the general differential equation $a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$ find an expression $F(k) = 0$ which can be solved for values of *k*. Give your answer as $F(k) = \cdots$

   (c) Let these solutions be k_1, k_2, in any order. Show that $y = Ae^{k_1x} + Be^{k_2x}$ must also be a solution, for arbitrary constants $A, B \in \mathbb{R}$. (*This solution is the **general solution** to this differential equation!)*

[2+2+2 = 6 marks]

3. Find the complex linear factors of $P(z) = z^3 + z^2 + z + 1$ [*hint – evaluate for z=-1*], expressing each factor in the form $(z - r\,\mathrm{cis}(\theta))$ where θ is the principal Argument.

[4 marks]

4. [**trigonometric substitutions]**
   (a) By using the substitution $x = f(u)$ where $f(u)$ is a standard trigonometric function [**not an inverse trig function]**, find the antiderivative of $\sqrt{1 - x^2}$. By restricting the values of u, justify your mathematical reasoning.

[3 marks]

(b) Hence evaluate $\int_0^1 \sqrt{1-x^2}\,dx$

[1 mark]

5. An object of mass 1 kg is moving with initial velocity $\overrightarrow{v_0} = 2\vec{\imath} + 2\vec{\jmath} + 2\vec{k}$ at time *t=0*, and is subjected to a net force $\Sigma\vec{F} = e^{-t}(\vec{\imath} + \vec{\jmath} + \vec{k})$.

(a) Find the terminal velocity of the object, i.e. $\lim_{t\to\infty} \vec{v}(t)$.

[3 marks]

(b) Find the displacement from its initial position (let this be O) at any time *t*.

[2 marks]

6. For the function $4x^2 + 2y = 9e^x$, evaluate the value of $\frac{dy}{dx}$ at $(1, \frac{9e-4}{2})$ and thus give the equation of the tangent and normal at that point.

[4 marks]

7. Using calculus, find an expression for the volume of the truncated cone with minimal and maximal radii respectively q, r and height h. Give your answer in the form $\frac{a\pi}{b}$ where a, b are real expressions.

[5 marks]

8. Express $11\vec{\imath} + 18\vec{\jmath} + 10\vec{k}$ in terms of the vectors $\vec{\imath} + 6\vec{\jmath} + 2\vec{k}$ and $\vec{3} + 2\vec{\jmath} + 2\vec{k}$.

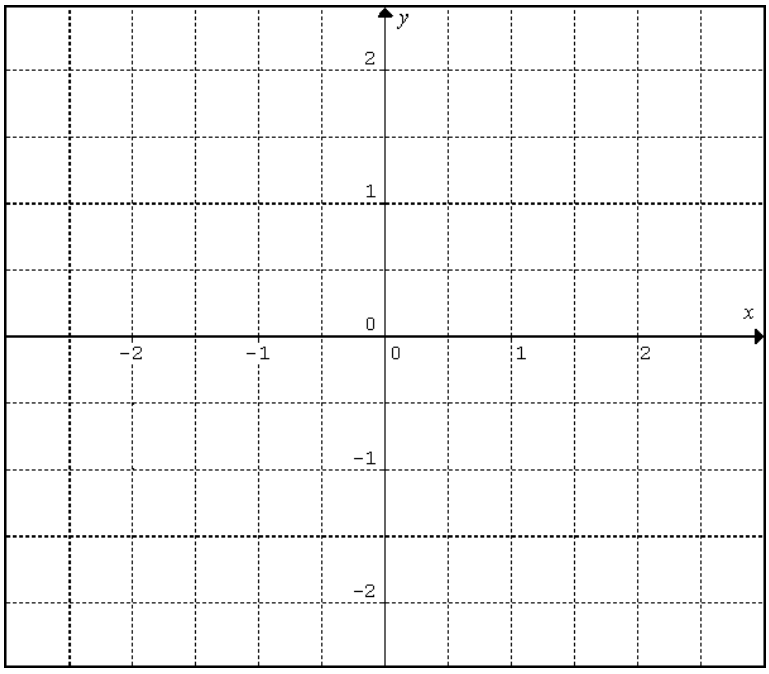
[2 marks]

9. Find the area of the triangle ΔOAB for O(0,0), A(2,1), and B(1,4) using a vector method.

[3 marks]

10.

(a) Sketch the direction field $\frac{dy}{dx} = xy$ for $x = 0, \pm 1, \pm 2$ and $y = 0, \pm 1, \pm 2$.



[2 marks]

(b) Given that $\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{1}{g(y)}\frac{dy}{dx} = f(x)$, and using the change of variable rule $\int f(u)\frac{du}{dx}dx = \int f(u)du$, find the general solution curve for this differential equation. (*These are called separable equations, and from what I hear are due to be introduced to the 2016 VCE Specialist study design)*

[4 marks]

(c) Using calculus if necessary, sketch the member of this family of curves which passes through $(0, 1)$ on the diagram above.

[2 marks]

END OF EXAMINATION

Total marks: 40